J. Korean Math. Soc. **50** (2013), No. 1, pp. 61–80
http://dx.doi.org/10.4134/JKMS.2013.50.1.061

α-COMPLETELY POSITIVE MAPS ON LOCALLY C^*-ALGEBRAS, KREIN MODULES AND RADON-NIKODÝM THEOREM

Jaeseong Heo[†], Un Cig Ji[‡], and Young Yi Kim

Abstract. In this paper, we study α-completely positive maps between locally C^*-algebras. As a generalization of a completely positive map, an α-completely positive map produces a Krein space with indefinite metric, which is useful for the study of massless or gauge fields. We construct a KSGNS type representation associated to an α-completely positive map of a locally C^*-algebra on a Krein locally C^*-module. Using this construction, we establish the Radon-Nikodým type theorem for α-completely positive maps on locally C^*-algebras. As an application, we study an extremal problem in the partially ordered cone of α-completely positive maps on a locally C^*-algebra.

1. Introduction

One of most elegant approaches to quantum field theory is the algebraic approach, which works for massive fields as well as massless or gauge fields. In massless or gauge fields, the state space may be a space with indefinite metric. Motivated by this physical fact, many people extended the GNS construction to Krein spaces. In particular, motivated by P-functional, Heo, Hong and Ji [7] introduced a notion of α-completely positive maps as a natural generalization of the notion of completely positive maps, and Heo and Ji [8] proved the Radon-Nikodým type theorem which gives the one-to-one correspondence between α-completely positive maps and their corresponding positive operators.

Received November 21, 2011; Revised March 14, 2012.

2010 *Mathematics Subject Classification.* Primary 46L08, 46K10; Secondary 47L60, 46L55.

Key words and phrases. locally C^*-algebra, Hilbert locally C^*-module, α-completely positive map, J-representation, Krein module, minimal Krein quadruple, non-commutative Radon-Nikodým theorem.

[†] This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0005070).

[‡] This research was supported by Basic Science Research Program through the NRF funded by the MEST (No. R01-2010-002-2514-0).

A *locally C^*-algebra* (or *pro-C^*-algebra*) is a complete Hausdorff (complex) topological $*$-algebra of which the topology is determined by the collection $S(\mathcal{A})$ of all continuous C^*-seminorms on it. The notion of locally C^*-algebras was first systematically studied by Inoue [9] as a generalization of C^*-algebras, and then Phillips [15, 16] studied locally C^*-algebras that are needed for representable K-theory of σ-C^*-algebras. A locally C^*-algebra is topologically $*$-isomorphic to an inverse limit of C^*-algebras. It is known that locally C^*-algebras are useful for the study of non-commutative algebraic topology, pseudodifferential operators and quantum field theory [3, 4, 15].

Main purpose of this paper is to construct a KSGNS (Kasparov-Stinespring-Gelfand-Naimark-Segal) type representation on a Krein locally C^*-module associated with an α-completely positive map on a locally C^*-algebra and establish a Radon-Nikodým type theorem for α-completely positive maps on a locally C^*-algebra. Non-commutative Radon-Nikodým theorems have attracted a great deal of attention in operator algebras and mathematical physics. There have been considerable works on non-commutative Radon-Nikodým theorems not only for C^*-algebras but also for algebras of unbounded operators [1, 6, 14, 17]. In the proofs of non-commutative Radon-Nikodým type theorems, it is essential to find the adjoint of a bounded linear operator. However, any bounded module map between Hilbert C^*-modules need not be adjointable. To overcome this difficulty for a Radon-Nikodým type theorem for α-completely positive maps on a Krein C^*-module, we use the construction [18] of a self-dual Hilbert C^*-module from a general Hilbert C^*-module.

This paper is organized as follows. In Section 2, we recall some basic notions of Hilbert modules over locally C^*-algebras and α-completely positive maps on locally C^*-algebras. In Section 3, we introduce a notion of a Krein locally C^*-module as a generalization of a Krein C^*-module and construct a KSGNS type representation of a locally C^*-algebra $\mathcal{A}$ associated with an α-completely positive map. The construction leads to a J_ρ-representation of the locally C^*-algebra $\mathcal{A}$ on a Krein locally C^*-module. In Section 4, we establish the Radon-Nikodým type theorem for α-completely positive maps on locally C^*-algebras. As an application, we study an extremal problem in the partially ordered cone of α-completely positive maps on a locally C^*-algebra. Such problems for completely positive maps on C^*-algebras were studied by Arveson [1].

2. Preliminaries and notations

Let $\mathcal{A}$ be a locally C^*-algebra of which topology is understood as following. Let $S(\mathcal{A})$ be the set of all continuous C^*-seminorms on $\mathcal{A}$, and then for each $p \in S(\mathcal{A})$, the kernel $\ker(p) = \{a \in \mathcal{A} : p(a) = 0\}$ of p becomes a closed ideal in $\mathcal{A}$. Then $\mathcal{A}_p = \mathcal{A}/\ker(p)$ is a C^*-algebra with the norm induced by p. We denote by $\mathbf{q}_p$ the canonical map from $\mathcal{A}$ onto $\mathcal{A}_p$ and by $a_p = \mathbf{q}_p(a)$ the image of a in $\mathcal{A}_p$. Since $S(\mathcal{A})$ can be considered as a directed set with the order $p \geq q$ if $p(a) \geq q(a)$ $(a \in \mathcal{A})$ for all $p \geq q$ in $S(\mathcal{A})$, there is a canonical surjective map

$\mathbf{q}_{pq} : \mathcal{A}_p \to \mathcal{A}_q$ such that $\mathbf{q}_{pq}(a_p) = a_q$ for every $a_p \in \mathcal{A}_p$. Then the set

$$\{\mathcal{A}_p, \mathbf{q}_{pq} : \mathcal{A}_p \to \mathcal{A}_q, p \geq q\}$$

becomes an inverse system of C^*-algebras and the inverse limit $\varprojlim_p \mathcal{A}_p$ is a locally C^*-algebra which is isomorphic to $\mathcal{A}$. Let $M_n(\mathcal{A})$ denote the $*$-algebra of all $n \times n$ matrices over $\mathcal{A}$ with the usual algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of $\mathcal{A}$. Then $M_n(\mathcal{A})$ is a locally C^*-algebra and it is isomorphic to

$$\varprojlim_p M_n(\mathcal{A}_p),$$

where p runs through $S(\mathcal{A})$. The topology on the locally C^*-algebra $M_n(\mathcal{A})$ is determined by the family of C^*-seminorms $\{p_n : p \in S(\mathcal{A})\}$, where $p_n([a_{ij}]) = \big\|[\mathbf{q}_p(a_{ij})]\big\|_{M_n(\mathcal{A})}$.

Example 2.1. We now give some examples of locally C^*-algebras and refer [15] and its references for more examples.

(1) Every C^*-algebra is a locally C^*-algebra. A closed $*$-subalgebra of a locally C^*-algebra is again a locally C^*-algebra.
(2) Let $C(\Omega)$ be the set of all continuous complex-valued functions on a compactly generated space Ω. If we equip $C(\Omega)$ with the topology of uniform convergence on compact subsets, then $C(\Omega)$ becomes a locally C^*-algebra. Thus, we see that if X is any nonempty subset of the complex field $\mathbb{C}$, then $C(X)$ is a locally C^*-algebra.
(3) The product of C^*-algebras with the product topology is a locally C^*-algebra.
(4) Let $\mathcal{A}$ be a locally C^*-algebra. The unitization $\mathcal{A}^1$ is the vector space $\mathcal{A} \oplus \mathbb{C}$, topologized as the direct sum and multiplication defined as for the unitization of C^*-algebras. Then $\mathcal{A}^1$ is a locally C^*-algebra since $\mathcal{A}^1 = \varprojlim_p \mathcal{A}^1_p$.

A self-adjoint $a \in \mathcal{A}$ is *positive* if there exists an element $b \in \mathcal{A}$ such that $a = b^*b$. Note that $a \geq 0$ if and only if for every $p \in \mathcal{S}(\mathcal{A})$, $a_p = \mathbf{q}_p(a) \geq 0$ in $\mathcal{A}_p$. An *approximate unit* for $\mathcal{A}$ is an increasing net $\{e_\lambda\}$ of positive elements of $\mathcal{A}$ such that

$$\|e_\lambda\|_\infty = \sup\{p(e_\lambda) : p \in S(\mathcal{A})\} \leq 1 \quad \text{for all } \lambda,$$

and $p(e_\lambda a - a) \to 0$ and $p(ae_\lambda - a) \to 0$ for all $a \in \mathcal{A}$ and $p \in S(\mathcal{A})$. The set of all bounded elements of $\mathcal{A}$ is denoted by

$$\mathbf{b}(\mathcal{A}) = \{a \in \mathcal{A} : \|a\|_\infty < \infty\},$$

which is a C^*-algebra with the norm $\|\cdot\|_\infty$ and is dense in $\mathcal{A}$. A morphism of locally C^*-algebras is a continuous morphism of $*$-algebras and an isomorphism of locally C^*-algebras is a morphism that is invertible and such that its inverse is also a morphism. We refer [5, 9, 12, 15] for more detailed information about locally C^*-algebras.

Definition 2.2. Let $\mathcal{A}$ be a locally C^*-algebra, and let $\mathcal{E}$ be a (complex) vector space which is a right $\mathcal{A}$-module, compatibly with the algebra structure. Then $\mathcal{E}$ is called a *pre-Hilbert $\mathcal{A}$-module* if it is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ which is linear in the second variable and satisfies the following properties:

(i) $\langle \xi, \xi \rangle \geq 0$, and the equality holds only if $\xi = 0$,
(ii) $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,
(iii) $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$.

We say that $\mathcal{E}$ is a *Hilbert $\mathcal{A}$-module* if $\mathcal{E}$ is complete with respect to the seminorms $\|\xi\|_p = p(\langle \xi, \xi \rangle)^{1/2}$ for $p \in S(\mathcal{A})$.

Throughout this paper, $\mathcal{A}$ and $\mathcal{E}$ denote a locally C^*-algebra and a Hilbert $\mathcal{A}$-module, respectively, unless specified otherwise.

For any $p \in S(\mathcal{A})$ and $N_p = \{\xi \in \mathcal{E} : p(\langle \xi, \xi \rangle) = 0\}$, we write $\mathcal{E}_p$ for the Hilbert $\mathcal{A}_p$-module $\mathcal{E}/N_p$ with $(\xi + N_p)\mathbf{q}_p(a) = \xi a + N_p$ and $\langle \xi + N_p, \eta + N_p \rangle = \mathbf{q}_p(\langle \xi, \eta \rangle)$. We denote by $\mathbf{Q}_p$ the canonical map from $\mathcal{E}$ onto $\mathcal{E}_p$ and ξ_p denotes the image $\mathbf{Q}_p(\xi)$. For $p \geq q$ in $S(\mathcal{A})$, there is a canonical surjective map $\mathbf{Q}_{pq} : \mathcal{E}_p \to \mathcal{E}_q$ such that $\mathbf{Q}_{pq}(\xi_p) = \xi_q$ for $\xi_p \in \mathcal{E}_p$. Then $\{\mathcal{E}_p, \mathbf{Q}_{pq} : \mathcal{E}_p \to \mathcal{E}_q, p \geq q\}$ is an inverse system of Hilbert C^*-modules in the sense that

$$\begin{aligned} \mathbf{Q}_{pq}(\xi_p a_p) &= \mathbf{Q}_{pq}(\xi_p)\mathbf{q}_{pq}(a_p) \quad \text{for } \xi_p \in \mathcal{E}_p,\ a_p \in \mathcal{A}_p, \\ \langle \mathbf{Q}_{pq}(\xi_p), \mathbf{Q}_{pq}(\eta_p) \rangle &= \mathbf{q}_{pq}(\langle \xi_p, \eta_p \rangle) \quad \text{for } \xi_p, \eta_p \in \mathcal{E}_p, \\ \mathbf{Q}_{qr} \circ \mathbf{Q}_{pq} &= \mathbf{Q}_{pr} \quad \text{for } p \geq q \geq r. \end{aligned}$$

Then the inverse limit $\varprojlim_{p \in S(\mathcal{A})} \mathcal{E}_p$ is a Hilbert $\mathcal{A}$-module with

$$\begin{aligned} &(\xi_p)_{p \in S(\mathcal{A})}(a_p)_{p \in S(\mathcal{A})} = (\xi_p a_p)_{p \in S(\mathcal{A})} \qquad \text{and} \\ &\langle (\xi_p)_{p \in S(\mathcal{A})}, (\eta_p)_{p \in S(\mathcal{A})} \rangle = (\langle \xi_p, \eta_p \rangle)_{p \in S(\mathcal{A})} \end{aligned}$$

and it is isomorphic to the Hilbert $\mathcal{A}$-module $\mathcal{E}$.

Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{E} \to \mathcal{F}$ is said to be *adjointable* if there is a map $T^* : \mathcal{F} \to \mathcal{E}$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$. We denote by $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ the set of all adjointable maps from $\mathcal{E}$ into $\mathcal{F}$ and write $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ for $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$. The *strict topology* on $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is defined by the family of seminorms $\{\| \cdot \|_{p,\xi} : p \in S(\mathcal{A}), \xi \in \mathcal{E}\}$, where

$$\|T\|_{p,\xi} = \|T\xi\|_p + \|T^*\xi\|_p.$$

Since $T(N_p) \subset N_p^{\mathcal{F}} = \{\eta \in \mathcal{F} : p(\langle \eta, \eta \rangle) = 0\}$ for all $p \in S(\mathcal{A})$ and $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, we can define a map $(\mathbf{q}_p)_* : \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p)$ by

$$(\mathbf{q}_p)_*(T)(\mathbf{Q}_p(\xi)) = \mathbf{Q}_p^{\mathcal{F}}(T(\xi)) \quad (T \in \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}), \xi \in \mathcal{E}),$$

where $\mathbf{Q}_p^{\mathcal{F}}$ is the canonical map from $\mathcal{F}$ onto $\mathcal{F}_p$. We denote by T_p the operator $(\mathbf{q}_p)_*(T)$. The topology on $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is given by the family of seminorms $\{\tilde{p}\}_{p \in S(\mathcal{A})}$, where

$$\tilde{p}(T) = \|(\mathbf{q}_p)_*(T)\| = \|T_p\|. \tag{2.1}$$

Then $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ becomes a locally C^*-algebra. The connecting maps of the inverse system $\{\mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p) : p \in S(\mathcal{A})\}$ are denoted by $(\mathbf{q}_{pq})_* : \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p) \to \mathcal{L}_{\mathcal{A}_q}(\mathcal{E}_q, \mathcal{F}_q)$ and the connecting maps are defined as follows:

$$(\mathbf{q}_{pq})_*(T_p)(\mathbf{Q}_p(\xi)) = \mathbf{Q}^{\mathcal{F}}_{pq}(T_p(\mathbf{Q}_p(\xi))) \quad \text{for } p \geq q,$$

where $\mathbf{Q}^{\mathcal{F}}_{pq} : \mathcal{F}_p \to \mathcal{F}_q$ $(p \geq q)$ are the connecting maps of a family $\{\mathcal{F}_p\}$ of Hilbert C^*-modules. Then the family $\{\mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p), (\mathbf{q}_{pq})_*, p \geq q\}$ is an inverse system of Banach spaces and the inverse limit $\varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p)$ is isomorphic to $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. See [15] for inverse limits of Hilbert C^*-modules and Banach spaces.

Let $\mathcal{A}, \mathcal{B}$ be locally C^*-algebras and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module. A continuous linear map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is *strict* if for some approximate unit $\{e_\lambda\}$ of $\mathcal{A}$, $\{\rho(e_\lambda)\}$ is strictly Cauchy in $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$. A *multiplier algebra* $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is the set of all multipliers $(\mathbf{l}, \mathbf{r})$ where $\mathbf{l}, \mathbf{r} : \mathcal{A} \to \mathcal{A}$ are morphisms of left, right $\mathcal{A}$-modules such that

$$a\mathbf{l}(b) = \mathbf{r}(a)b, \quad \mathbf{l}(ab) = \mathbf{l}(a)b \;\text{ and }\; \mathbf{r}(ab) = a\mathbf{r}(b) \quad \text{for all } a, b \in \mathcal{A}.$$

The *strict topology* on $\mathcal{M}(\mathcal{A})$ is the topology generated by the seminorms $\|\cdot\|_{p,a}$ $(p \in S(\mathcal{A}),\ a \in \mathcal{A})$ where

$$\|(\mathbf{l}, \mathbf{r})\|_{p,a} = p(\mathbf{l}(a)) + p(\mathbf{r}(a)).$$

The map $a \mapsto (\mathbf{l}_a, \mathbf{r}_a)$ is a homeomorphism of $\mathcal{A}$ onto the closed ideal of $\mathcal{M}(\mathcal{A})$ where $\mathbf{l}_a(b) = ab$ and $\mathbf{r}_a(b) = ba$, and the image of $\mathcal{A}$ under this map is dense in $\mathcal{M}(\mathcal{A})$ in the strict topology. Moreover, $\mathcal{M}(\mathcal{A})$ becomes a locally C^*-algebra (see [15, Theorem 3.14]).

Definition 2.3 (cf. [7])**.** A Hermitian map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is called α-*completely positive* (briefly, α-*CP*) if there is a continuous linear Hermitian map $\alpha : \mathcal{A} \to \mathcal{A}$ such that

(i) $\alpha^2 = \mathrm{id}_{\mathcal{A}}$, where $\mathrm{id}_{\mathcal{A}}$ is the identity map on $\mathcal{A}$,

(ii) for any approximate unit $\{e_\lambda\}_{\lambda \in \Lambda}$ for $\mathcal{A}$, $\{\alpha(e_\lambda)\}_{\lambda \in \Lambda}$ is also an approximate unit,

(iii) $\rho(ab) = \rho(\alpha(a)\alpha(b)) = \rho(\alpha(ab))$ for any $a, b \in \mathcal{A}$,

(iv) for any $n \geq 1$, $a_1, \ldots, a_n \in \mathcal{A}$ and $\xi_1, \ldots, \xi_n \in \mathcal{F}$,

$$\sum_{i,j=1}^{n} \left\langle \xi_i, \rho(\alpha(a_i)^* a_j)\xi_j \right\rangle \geq 0,$$

(v) for any $a, a_1, \ldots, a_n \in \mathcal{A}$, there exists a constant $M(a) > 0$ such that

$$\left(\rho(\alpha(aa_i)^* aa_j)\right)_{n \times n} \leq M(a) \left(\rho\big(\alpha(a_i)^* a_j\big)\right)_{n \times n},$$

where $(\cdot)_{n \times n}$ denotes an $n \times n$ operator matrix,

(vi) there exist a strictly continuous positive linear map $\rho' : \mathcal{M}(\mathcal{A}) \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ and a constant $K > 0$ such that $\rho(\alpha(a)^* a) \leq K\rho'(a^* a)$ for any $a \in \mathcal{A}$.

The following example shows the existence of an α-CP map which is not completely positive on the 2×2-matrix algebra.

Example 2.4. Let $a>1$ be a constant and let $\mathcal{D}$ be the set of 2×2 diagonal matrices over the complex field $\mathbb{C}$, i.e.,

$$\mathcal{D}=\left\{\left(\begin{array}{cc} x & 0\\ 0 & y\end{array}\right)\,\middle|\, x,y\in\mathbb{C}\right\}.$$

We define a Hermitian map $\alpha:\mathcal{D}\to\mathcal{D}$ by

$$\alpha\left(\left(\begin{array}{cc} x & 0\\ 0 & y\end{array}\right)\right)=\left(\begin{array}{cc} ax+(1-a)y & 0\\ 0 & (1+a)x-ay\end{array}\right).$$

For some $a_i, b_i\in\mathbb{R}$ with $a_i>0$ $(i=1,2)$, we define a Hermitian map $\rho:\mathcal{D}\to\mathcal{D}$ by

$$\rho\left(\left(\begin{array}{cc} x & 0\\ 0 & y\end{array}\right)\right)=\left(\begin{array}{cc} a_1x+b_1y & 0\\ 0 & a_2x+b_2y\end{array}\right).$$

Now, we assume that $a_1=a_2$ and $a_i(1-a)=b_i(1+a)$ $(i=1,2)$. Then ρ is α-completely positive, but is not completely positive. See [7] for the proof and detailed information about α-completely positive maps.

3. KSGNS type constructions for α-CP maps on locally C^*-algebra

Let $\mathcal{B}$ be a locally C^*-algebra and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module with a $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle$. Suppose that a (fundamental) symmetry J on $\mathcal{F}$, i.e., $J=J^*=J^{-1}$, is given to produce a $\mathcal{B}$-valued indefinite inner product

$$\langle\xi,\eta\rangle_J=\langle\xi,J\eta\rangle\qquad(\xi,\eta\in\mathcal{F}).$$

In this case, the pair $(\mathcal{F},J)$ is called a *Krein $\mathcal{B}$-module.* Let $\mathcal{A}$ be a locally C^*-algebra. A representation $\pi:\mathcal{A}\to\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is called a *J-representation on a Krein $\mathcal{B}$-module* $(\mathcal{F},J)$ if π is a homomorphism of $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ such that

$$\pi(a^*)=\pi(a)^J:=J\pi(a)^*J\quad\text{for all } a\in\mathcal{A}.$$

Bhatt and Karia [2] gave the Stinespring's construction for locally C^*-algebras and Joiţa [10] generalized the KSGNS construction in the context of Hilbert modules over locally C^*-algebras.

Let p be any element in $S(\mathcal{A})$. We say that a linear map β on $\mathcal{A}$ is *p-continuous* if there exists a constant $C_p>0$ such that $p(\beta(a))\le C_p\cdot p(a)$ for all $a\in\mathcal{A}$. In the following theorem, we give a representation associated with an α-CP map between locally C^*-algebras, which is a generalization of our KSGNS type representation associated to an α-CP map on a Krein C^*-module [7].

Theorem 3.1. *Let $\mathcal{A},\mathcal{B}$ be locally C^*-algebras and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module. If $\rho:\mathcal{A}\to\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is a strictly continuous α-CP linear map where α is p-continuous for each $p\in S(\mathcal{A})$, then there exist a Krein $\mathcal{B}$-module $(\mathcal{F}_\rho,J_\rho)$, a J_ρ-representation $\pi_\rho:\mathcal{A}\to\mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ and an operator $V_\rho\in\mathcal{L}_{\mathcal{B}}(\mathcal{F},\mathcal{F}_\rho)$ such that*

(i) $\rho(a) = V_\rho^* \pi_\rho(a) V_\rho$ *(so,* $\rho(a^*) = V_\rho^* \pi_\rho(a)^{J_\rho} V_\rho$*) for all* $a \in \mathcal{A}$*,*
(ii) $\pi_\rho(\mathcal{A}) V_\rho(\mathcal{F})$ *is dense in* $\mathcal{F}_\rho$*,*
(iii) $V_\rho^* \pi_\rho(a)^* \pi_\rho(b) V_\rho = V_\rho^* \pi_\rho(\alpha(a)^* b) V_\rho$ *for all* $a, b \in \mathcal{A}$*.*

Proof. First we suppose that $\mathcal{B}$ is a C^*-algebra as in [10, Theorem 4.6]. Since ρ is continuous, there exist a C^*-seminorm $p \in S(\mathcal{A})$ and a constant $C > 0$ such that $\|\rho(a)\| \leq C \cdot p(a)$ for all $a \in \mathcal{A}$. Thus, we can find a linear map $\rho_p : \mathcal{A}_p \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ with $\rho_p \circ \mathbf{q}_p = \rho$. Since α is p-continuous for each $p \in S(\mathcal{A})$, we have that $\alpha(a) \in \ker(p)$ for all $a \in \ker(p)$. We denote by α_p the induced map on $\mathcal{A}_p$ from α. Then $\alpha_p : \mathcal{A}_p \to \mathcal{A}_p$ is a hermitian map such that

$$\alpha_p \circ \mathbf{q}_p = \mathbf{q}_p \circ \alpha \quad \text{and} \quad \alpha_p^2 = \mathrm{id}_{\mathcal{A}_p}.$$

We claim that the map $\rho_p : \mathcal{A}_p \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is α_p-completely positive. Indeed, for $n \geq 1$, $a_{1,p}, \ldots, a_{n,p} \in \mathcal{A}_p$ and $\xi_1, \ldots, \xi_n \in \mathcal{F}$ we have that

$$\begin{aligned}
\sum_{i=1}^n \big\langle \xi_i, \rho_p(\alpha_p(a_{i,p})^* a_{j,p}) \xi_j \big\rangle &= \sum_{i=1}^n \big\langle \xi_i, \rho_p(\alpha_p(\mathbf{q}_p(a_i)^*) \mathbf{q}_p(a_j)) \xi_j \big\rangle \\
&= \sum_{i=1}^n \big\langle \xi_i, \rho_p(\mathbf{q}_p(\alpha(a_i)^* a_j)) \xi_j \big\rangle \\
&= \sum_{i=1}^n \big\langle \xi_i, \rho(\alpha(a_i)^* a_j) \xi_j \big\rangle \geq 0.
\end{aligned}$$

For all $a_p, b_p \in \mathcal{A}_p$, we obtain that

$$\begin{aligned}
\rho_p\big(\alpha_p(a_p)\alpha_p(b_p)\big) &= \rho_p\big(\mathbf{q}_p(\alpha(a)\alpha(b))\big) = \rho\big(\alpha(a)\alpha(b)\big) \\
&= \rho(ab) = \rho_p\big(\mathbf{q}_p(ab)\big) = \rho_p(a_p b_p).
\end{aligned}$$

We show that the property (v) in Definition 2.3 holds. Let $a_p, a_{1,p}, \ldots, a_{n,p} \in \mathcal{A}_p$. Then we have that

$$\begin{aligned}
\Big(\rho_p(\alpha_p(a_p a_{p,i})^* a_p a_{p,j})\Big)_{n \times n} &= \Big((\rho_p \circ \mathbf{q}_p)(\alpha(a a_i)^* a a_j)\Big)_{n \times n} \\
&\leq M(a) \Big(\rho(\alpha(a_i)^* a_j)\Big)_{n \times n} \\
&= M(a) \Big(\rho_p(\alpha_p(a_{p,i})^* a_{p,j})\Big)_{n \times n}.
\end{aligned}$$

Let ρ' and $K > 0$ be as in (vi) of Definition 2.3. We denote by $\mathcal{M}(\mathcal{A}_p)$ the multiplier algebra of $\mathcal{A}_p$. The map $\mathbf{q}_p : \mathcal{A} \to \mathcal{A}_p$ can be extended to a map $\bar{\mathbf{q}}_p : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A}_p)$ since $\mathcal{A}$ is dense in $\mathcal{M}(\mathcal{A})$ and

$$\mathcal{M}(\mathcal{A}_p) \cong \mathcal{M}(\mathcal{A}) / \ker(\| \cdot \|_p).$$

By the continuity of ρ', there is a map $\rho'_p : \mathcal{M}(\mathcal{A}_p) \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ such that $\rho' = \rho'_p \circ \bar{\mathbf{q}}_p$ and ρ'_p is a strictly continuous positive linear map. Hence we have that

$$\rho_p\big(\alpha_p(a_p)^* a_p\big) = \rho_p\big(\mathbf{q}_p(\alpha(a)^* a)\big) = \rho\big(\alpha(a)^* a\big)$$

$$\leq K\rho'(a^*a) = K\rho'_p(a_p^*a_p).$$

Therefore, ρ_p is an α_p-CP linear map from a C^*-algebra $\mathcal{A}_p$ into a C^*-algebra $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$.

By Theorem 4.4 in [7], there exist a Krein $\mathcal{B}$-module $(\mathcal{F}_\rho, J_\rho)$, a J_ρ-representation $\pi_p : \mathcal{A}_p \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ and an operator V_ρ in $\mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}_\rho)$ such that

(i) $\rho_p(a_p) = V_\rho^*\pi_p(a_p)V_\rho$ and so $\rho_p(a_p^*) = V_\rho^*\pi_p(a_p)^{J_\rho}V_\rho$ for all $a_p \in \mathcal{A}_p$,
(ii) $\pi_p(\mathcal{A}_p)\big[V_\rho(\mathcal{F})\big]$ is dense in $\mathcal{F}_\rho$,
(iii) $V_\rho^*\pi_p(a_p)^*\pi_p(b_p)V_\rho = V_\rho^*\pi_p\big(\alpha_p(a_p)^*b_p\big)V_\rho$ for all $a_p, b_p \in \mathcal{A}_p$.

We define a map $\pi_\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ by $\pi_\rho(a) = (\pi_p \circ \mathbf{q}_p)(a)$ $(a \in \mathcal{A})$. Then π_ρ becomes a J_ρ-representation of $\mathcal{A}$ on $\mathcal{F}_\rho$ and we have that

$$\rho(a) = (\rho_p \circ \mathbf{q}_p)(a) = V_\rho^*\pi_p(a_p)V_\rho = V_\rho^*\pi_\rho(a)V_\rho \quad \text{for } a \in \mathcal{A}.$$

Furthermore, the set $\pi_\rho(\mathcal{A})[V_\rho(\mathcal{F})] = \pi_p(\mathcal{A}_p)[V_\rho(\mathcal{F})]$ is dense in $\mathcal{F}_\rho$.

We assume that $\mathcal{B}$ is a locally C^*-algebra. Since ρ is continuous, for each $p \in S(\mathcal{B})$ there are $q_p \in S(\mathcal{A})$ and a constant $C_p > 0$ such that $\tilde{p}(\rho(a)) \leq C_p \cdot q_p(a)$ for any $a \in \mathcal{A}$, where $\tilde{p}$ is defined as in (2.1). There exists a linear map $\rho_p : \mathcal{A}_{q_p} \to \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p)$ by $\rho_p \circ \mathbf{q}_{q_p} = (\mathbf{q}_p)_* \circ \rho$, where $(\mathbf{q}_p)_* : \mathcal{L}_{\mathcal{B}}(\mathcal{F}) \to \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p)$ is defined by

$$(\mathbf{q}_p)_*(T)(\mathbf{Q}_p(\xi)) = \mathbf{Q}_p(T(\xi)) \quad \text{for any } \xi \in \mathcal{F}.$$

Denoting by α_{q_p} the induced map of $\mathcal{A}_{q_p}$ from α, we have that $\alpha_{q_p} \circ \mathbf{q}_{q_p} = \mathbf{q}_{q_p} \circ \alpha$.

We claim that the map $\rho_p \circ \mathbf{q}_{q_p}$ is α-completely positive. To do this, we will show the α_{q_p}-complete positivity of ρ_p. Indeed, for any $a_{1,q_p}, \ldots, a_{n,q_p} \in \mathcal{A}_{q_p}$ and $\xi_{1,p}, \ldots, \xi_{n,p} \in \mathcal{F}_p$, we have that

$$\begin{aligned}
\sum_{i=1}^n \langle \xi_{p,i}, \rho_p(\alpha_{q_p}(a_{q_p,i})^* a_{q_p,j})\xi_{p,j}\rangle &= \sum_{i=1}^n \langle \mathbf{Q}_p(\xi_i), (\rho_p \circ \mathbf{q}_{q_p})(\alpha(a_i)^* a_j)\mathbf{Q}_p(\xi_j)\rangle \\
&= \sum_{i=1}^n \langle \mathbf{Q}_p(\xi_i), (\mathbf{q}_p)_*(\rho(\alpha(a_i)^* a_j))\mathbf{Q}_p(\xi_j)\rangle \\
&= \sum_{i=1}^n \mathbf{q}_p(\langle \xi_i, \rho(\alpha(a_i)^* a_j)\xi_j\rangle) \geq 0.
\end{aligned}$$

For any $a_{q_p}, b_{q_p} \in \mathcal{A}_{q_p}$, we obtain that

$$\rho_p(\alpha_{q_p}(a_{q_p})\alpha_{q_p}(b_{q_p})) = (\mathbf{q}_p)_*(\rho(\alpha(a)\alpha(b))) = \rho_p(a_{q_p}b_{q_p}).$$

Let $a_{q_p}, a_{1,q_p}, \ldots, a_{n,q_p}$ be elements in $\mathcal{A}_{q_p}$. Then we have that

$$\begin{aligned}
\big(\rho_p(\alpha_{q_p}(a_{q_p}a_{i,q_p})^* a_{q_p}a_{j,q_p})\big)_{n\times n} &= \big((\rho_p \circ \mathbf{q}_{q_p})(\alpha(aa_i)^* aa_j)\big)_{n\times n} \\
&= ((\mathbf{q}_p)_*(\rho(\alpha(aa_i)^* aa_j)))_{n\times n} \\
&\leq M(a)\,((\mathbf{q}_p)_*(\rho(\alpha(a_j)^* a_i)))_{n\times n} \\
&= M(a)\,\big(\rho_p(\alpha_{q_p}(a_{i,q_p})^* a_{j,q_p})\big)_{n\times n},
\end{aligned}$$

where $M(a)$ is the positive constant in (v) of Definition 2.3. Let $\rho' : \mathcal{M}(\mathcal{A}) \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ be a strictly continuous positive map satisfying condition (vi) of Definition 2.3. Then we can find the map $\rho'_p : \mathcal{M}(\mathcal{A}_{q_p}) \to \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p)$ which is positive and satisfies $(\mathbf{q}_p)_* \circ \rho' = \rho'_p \circ \mathbf{q}_{q_p}$. Then we have that

$$\rho_p\big(\alpha_{q_p}(a_{q_p})^* a_{q_p}\big) = (\mathbf{q}_{q_p})_*\big(\rho(\alpha(a)^* a)\big) \leq K \cdot \rho'_p(a^*_{q_p} a_{q_p}),$$

where $K > 0$ is in (vi) of Definition 2.3. Therefore, the map $\rho_p \circ \mathbf{q}_{q_p}$ is α-completely positive.

By the preceding argument, there exist a Krein $\mathcal{B}_p$-module $(\mathcal{F}_{\rho_p}, J_p)$, a J_p-representation π_{ρ_p} and an operator $V_{\rho_p} \in \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p, \mathcal{F}_{\rho_p})$ such that

(i) $(\rho_p \circ \mathbf{q}_{q_p})(a) = V^*_{\rho_p} \pi_{\rho_p}(a) V_{\rho_p}$ for all $a \in \mathcal{A}$,
(ii) $\pi_{\rho_p}(\mathcal{A})\big[V_{\rho_p}(\mathcal{F}_p)\big]$ is dense in $\mathcal{F}_{\rho_p}$,
(iii) $V^*_{\rho_p} \pi_{\rho_p}(a)^* \pi_{\rho_p}(b) V_{\rho_p} = V^*_{\rho_p} \pi_{\rho_p}\big(\alpha(a)^* b\big) V_{\rho_p}$ for all $a, b \in \mathcal{A}$.

For $p, r \in S(\mathcal{B})$ with $r \leq p$, we have $\tilde{r}(\rho(a)) \leq \tilde{p}(\rho(a)) \leq C_p \cdot q_p(a)$ for all $a \in \mathcal{A}$. We may assume that $q_r \leq q_p$. In the construction of the Krein C^*-module, we know that $\mathcal{F}_{\rho_p}$ is the completion of the quotient space

$$\mathcal{A}_{q_p} \otimes_{\text{alg}} \mathcal{F}_p / \ker(\langle \cdot, \cdot \rangle_p),$$

where $\otimes_{\text{alg}}$ is the algebraic tensor product and

$$\langle a_{q_p} \otimes \xi_p, b_{q_p} \otimes \eta_p \rangle_p = \langle \xi_p, \rho_p(\alpha_{q_p}(a_{q_p})^* b_{q_p}) \eta_p \rangle.$$

We consider the linear map $\Psi_{pr} : \mathcal{A}_{q_p} \otimes_{\text{alg}} \mathcal{F}_p \to \mathcal{A}_{q_r} \otimes_{\text{alg}} \mathcal{F}_r$ defined by

$$\Psi_{pr}(a_{q_p} \otimes \xi_p) := \mathbf{q}_{q_p q_r}(a_{q_p}) \otimes \mathbf{Q}_{pr}(\xi_p) = a_{q_r} \otimes \xi_r.$$

Let $a_{q_p}, b_{q_p} \in \mathcal{A}_{q_p}$ and $\xi_p, \eta_p \in \mathcal{F}_p$. Then we obtain that

$$\begin{aligned}
&\big\langle \Psi_{pr}(a_{q_p} \otimes \xi_p), \Psi_{pr}(b_{q_p} \otimes \eta_p) \big\rangle \\
&= \big\langle \mathbf{q}_{q_p q_r}(a_{q_p}) \otimes \mathbf{Q}_{pr}(\xi_p), \mathbf{q}_{q_p q_r}(b_{q_p}) \otimes \mathbf{Q}_{pr}(\eta_p) \big\rangle \\
&= \big\langle \mathbf{Q}_{pr}(\xi_p), \rho_r(\alpha_{q_r}(\mathbf{q}_{q_p q_r}(a_{q_p})^*) \mathbf{q}_{q_p q_r}(b_{q_p})) \mathbf{Q}_{pr}(\eta_p) \big\rangle \\
&= \big\langle \mathbf{Q}_{pr}(\xi_p), (\mathbf{q}_r)_*(\rho(\alpha(a)^* b)) \mathbf{Q}_{pr}(\eta_p) \big\rangle \\
&= \big\langle \mathbf{Q}_{pr}(\xi_p), \mathbf{Q}_{pr}(\rho_p(\alpha_{q_p}(a_{q_p})^* b_{q_p}) \eta_p) \big\rangle \\
&= \mathbf{q}_{pr}\big(\langle a_{q_p} \otimes \xi_p, b_{q_p} \otimes \eta_p \rangle\big).
\end{aligned}$$

Hence Ψ_{pr} induces a linear map from $\mathcal{A}_{q_p} \otimes_{\text{alg}} \mathcal{F}_p / \ker(\langle \cdot, \cdot \rangle_p)$ into $\mathcal{A}_{q_r} \otimes_{\text{alg}} \mathcal{F}_r / \ker(\langle \cdot, \cdot \rangle_r)$ that can be extended to a linear map, still denoted by Ψ_{pr}, from $\mathcal{F}_{\rho_p}$ into $\mathcal{F}_{\rho_r}$. Therefore, the set $\{\mathcal{F}_{\rho_p}, \mathcal{B}_p, \Psi_{pr} : \mathcal{F}_{\rho_p} \to \mathcal{F}_{\rho_r}, p \geq r\}$ is an inverse system of Hilbert C^*-modules.

From the proof of Theorem 4.6 in [10], we obtain the following isomorphisms

$$\mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}_\rho) = \varprojlim_p \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p, \mathcal{F}_{\rho_p}) \quad \text{and} \quad \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho) = \varprojlim_p \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_{\rho_p}),$$

where $\mathcal{F}_\rho = \varprojlim_p \mathcal{F}_{\rho_p}$. Since $\Psi_{pr} \circ V_{\rho_p} = V_{\rho_r} \circ \mathbf{Q}_{pr}$ holds for every $p, r \in S(\mathcal{B})$ with $p \geq r$, we have that

$$\big(V_{\rho_p}\big)_{p \in S(\mathcal{B})} \in \varprojlim_p \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p, \mathcal{F}_{\rho_p}).$$

Since $\Psi_{pr} \circ \pi_{\rho_p}(a) = \pi_{\rho_r}(a) \circ \Psi_{pr}$ for all $a \in \mathcal{A}$, we obtain that $(\pi_{\rho_p}(a))_{p \in S(\mathcal{B})} \in \varprojlim_p \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_{\rho_p})$. The map $\pi_\rho : \mathcal{A} \to \varprojlim_p \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_{\rho_p})$ given by

$$\pi_\rho(a) = \big(\pi_{\rho_p}(a)\big)_{p \in S(\mathcal{B})}$$

is a continuous representation of $\mathcal{A}$ on $\mathcal{F}_\rho$. From the equality

$$\big((\mathbf{q}_p)_* \circ \rho\big)(a) = (\rho_p \circ \mathbf{q}_{q_p})(a) = V_{\rho_p}^* \pi_{\rho_p}(a) V_{\rho_p},$$

we obtain that $\rho(a) = V_\rho^* \pi_\rho(a) V_\rho$, where $V_\rho = \big(V_{\rho_p}\big)_{p \in S(\mathcal{B})}$. Moreover, it follows from the density of $\pi_{\rho_p}(\mathcal{A})\big[V_{\rho_p}(\mathcal{F}_p)\big]$ in $\mathcal{F}_{\rho_p}$ that the closure of $\pi_\rho(\mathcal{A})\big[V_\rho(\mathcal{F})\big]$ is $\mathcal{F}_\rho$.

By the relation $V_{\rho_p}^* \pi_{\rho_p}(a)^* \pi_{\rho_p}(b) V_{\rho_p} = V_{\rho_p}^* \pi_{\rho_p}(\alpha(a)^* b) V_{\rho_p}$ $(a, b \in \mathcal{A})$, we have that

$$V_\rho^* \pi_\rho(a)^* \pi_\rho(b) V_\rho = \big(V_{\rho_p}^* \pi_{\rho_p}(\alpha(a)^* b) V_{\rho_p}\big)_{p \in S(\mathcal{B})} = V_\rho^* \pi_\rho(\alpha(a)^* b) V_\rho.$$

Since π_{ρ_p} is a J_p-representation of $\mathcal{A}$, we also have that

$$\pi_\rho(a^*) = \big(\pi_{\rho_p}(a^*)\big)_{p \in S(\mathcal{B})} = \big(\pi_{\rho_p}(a)^{J_p}\big)_{p \in S(\mathcal{B})} = \pi_\rho(a)^{J_\rho},$$

which implies that π_ρ is a J_ρ-representation. $\square$

The quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ satisfying (i) and (iii) in Theorem 3.1 is called the *Krein quadruple associated with an α-CP map ρ*. If, in addition, (ii) is satisfied, then such a quadruple is said to be *minimal*. The following theorem says that such a minimal quadruple is unique up to unitary equivalence.

Theorem 3.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and ρ be as in Theorem 3.1 and let $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ be the minimal Krein quadruple. Suppose that $\mathcal{F}'$ is a Hilbert $\mathcal{B}$-module and $W \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}')$. If $\pi : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}')$ is a continuous J_ρ-representation such that*

(i) *$\rho(a) = W^* \pi(a) W$ for all $a \in \mathcal{A}$,*
(ii) *$\pi(\mathcal{A})\big[W(\mathcal{F})\big]$ is dense in $\mathcal{F}'$,*
(iii) *$W^* \pi(a)^* \pi(b) W = W^* \pi\big(\alpha(a)^* b\big) W$ for all $a, b \in \mathcal{A}$,*

then there exists a unitary operator $U \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho, \mathcal{F}')$ such that

$$W = U V_\rho \ \textit{and}\ \pi(a) = U \pi_\rho(a) U^* \quad \textit{for all } a \in \mathcal{A}.$$

Proof. The proof is the same as that of [10, Theorem 4.6], so we omit it. $\square$

4. Radon-Nikodým type theorem for α-completely positive maps

In this section we prove the Radon-Nikodým type theorem for α-CP maps on locally C^*-algebras, which may regarded as a generalization of the results in [8]. In the proof of non-commutative Radon-Nikodým type theorems concerned with Hilbert space structure, it is essential that a bounded linear operator on a Hilbert space has an adjoint. However, any bounded module map between Hilbert C^*-modules need not be adjointable. To overcome this difficulty for a Radon-Nikodým type theorem, we use a construction of a self-dual Hilbert C^*-module from a Hilbert C^*-module, which is similar to that in [18]. A Radon-Nikodým type theorem for completely positive maps on locally C^*-algebras can be found in [11], whose proof is similar to ours for α-CP maps.

Throughout this section, we assume that $\mathcal{A}$ is a locally C^*-algebra and $\mathcal{F}$ is a Hilbert C^*-module over $\mathcal{B}$. We denote by $\mathcal{F}^\# = \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, \mathcal{B})$ the set of all bounded $\mathcal{B}$-module maps of $\mathcal{F}$ into $\mathcal{B}$. Then $\mathcal{F}^\#$ naturally becomes a right Banach $\mathcal{B}$-module with the (canonical) action of $\mathcal{B}$ on $\mathcal{F}^\#$ given by $(T \cdot b)(\xi) = b^* T(\xi)$. Note that each $\xi \in \mathcal{F}$ gives rise to the map $\xi^\# \in \mathcal{F}^\#$ defined by $\xi^\#(\eta) = \langle \xi, \eta \rangle$ for $\eta \in \mathcal{F}$. Since the map $\iota : \mathcal{F} \to \mathcal{F}^\#$ given by $\iota(\xi) = \xi^\#$ is an isometric $\mathcal{B}$-module map, we can regard $\mathcal{F}$ as a submodule of $\mathcal{F}^\#$ by identifying with $\iota(\mathcal{F})$. We call $\mathcal{F}$ *self-dual* if $\mathcal{F} = \mathcal{F}^\#$, that is, every bounded $\mathcal{B}$-module map $T : \mathcal{F} \to \mathcal{B}$ is of the form $\langle \xi_T, \cdot \rangle$ for some element $\xi_T \in \mathcal{F}$.

For reader's convenience, we review some results about self-dual Hilbert C^*-modules, see [13] for more details. Let $\mathcal{E}$ be a Hilbert $\mathcal{B}$-module. In the case of a von Neumann algebra $\mathcal{B}$, the $\mathcal{B}$-valued inner product on $\mathcal{E}$ extends to $\mathcal{E}^\# \times \mathcal{E}^\#$ in such a way as to make $\mathcal{E}^\#$ into a self-dual Hilbert $\mathcal{B}$-module. Furthermore, any bounded $\mathcal{B}$-module map $T : \mathcal{E} \to \mathcal{F}$ extends uniquely to a bounded $\mathcal{B}$-module map $\widetilde{T} : \mathcal{E}^\# \to \mathcal{F}^\#$. If $\mathcal{E}$ is self-dual, then $\mathcal{B}_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) = \mathcal{L}_{\mathcal{B}}(\mathcal{E}, \mathcal{F})$, where $\mathcal{B}_{\mathcal{B}}(\mathcal{E}, \mathcal{F})$ is the space of all bounded $\mathcal{B}$-module maps of $\mathcal{E}$ into $\mathcal{F}$. In particular, we have that $\mathcal{B}_{\mathcal{B}}(\mathcal{E}) = \mathcal{L}_{\mathcal{B}}(\mathcal{E})$. We briefly review a construction of a self-dual Hilbert C^*-module from given Hilbert C^*-module. We refer [18] for a detailed information of the construction.

Let $\mathcal{B}^{**}$ be the enveloping von Neumann algebra of $\mathcal{B}$ and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module. The algebraic tensor product $\mathcal{F} \otimes_{\mathrm{alg}} \mathcal{B}^{**}$ becomes a right $\mathcal{B}^{**}$-module with the multiplication $(\xi \otimes a)b = \xi \otimes ab$. If we define a $\mathcal{B}^{**}$-valued inner product $[\cdot, \cdot]$ on $\mathcal{F} \otimes \mathcal{B}^{**}$ by

$$\left[\sum_{i=1}^{n} \xi_i \otimes a_i, \sum_{j=1}^{m} \eta_j \otimes b_j \right] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i^* \langle \xi_i, \eta_j \rangle b_j,$$

then $(\mathcal{F} \otimes_{\mathrm{alg}} \mathcal{B}^{**}) / \ker[\cdot, \cdot]$ becomes a pre-Hilbert $\mathcal{B}^{**}$-module containing $\mathcal{F}$ as a $\mathcal{B}$-submodule, since the map $\xi \mapsto \xi \otimes 1 + \ker[\cdot, \cdot]$ is isometric. Let $\widehat{\mathcal{F}}$ be the Hilbert C^*-module completion of $(\mathcal{F} \otimes_{\mathrm{alg}} \mathcal{B}^{**}) / \ker[\cdot, \cdot]$ with respect to the norm induced by $[\cdot, \cdot]$. We denote by $\widetilde{\mathcal{F}}$ the self-dual Hilbert $\mathcal{B}^{**}$-module $(\widehat{\mathcal{F}})^\#$ (see

[13, Theorem 3.2] for the self-duality of $\widetilde{\mathcal{F}}$). We will consider $\mathcal{F}$ as embedded in $\widetilde{\mathcal{F}}$ without making distinction.

Remark 4.1. Now, we recall the unique extension of operators in $\mathcal{B}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$ or $\mathcal{B}_{\mathcal{B}}(\mathcal{F},\mathcal{F}^{\#})$ to operators on a self-dual Hilbert module as follows;

(1) For each T in $\mathcal{B}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$, we have the inequality
$$\left[\sum_{i=1}^{n} T\xi_i\otimes a_i, \sum_{i=1}^{n} T\xi_i\otimes a_i\right]_{\widehat{\mathcal{F}}} \le \|T\|^2 \left[\sum_{i=1}^{n} \xi_i\otimes a_i, \sum_{i=1}^{n} \xi_i\otimes a_i\right]_{\widehat{\mathcal{E}}},$$
so that T can be extended uniquely to a bounded $\mathcal{B}^{**}$-module map $\widehat{T}$ from $\widehat{\mathcal{E}}$ into $\widehat{\mathcal{F}}$. Then by [13, Proposition 3.6], $\widehat{T}$ extends uniquely to a bounded $\mathcal{B}^{**}$-module map $\widetilde{T}$ from $\widetilde{\mathcal{E}}$ to $\widetilde{\mathcal{F}}$ with $\|T\| = \|\widetilde{T}\|$. Indeed, we first consider the map $\left(\widehat{T}\right)^{\natural} : \widehat{\mathcal{F}} \to \widetilde{\mathcal{E}}$ defined by $\left(\left(\widehat{T}\right)^{\natural}(\zeta)\right)(\xi) = \left[\zeta, \widehat{T}\xi\right]_{\widehat{\mathcal{F}}}$, $(\zeta\in\widehat{\mathcal{F}},\ \xi\in\widehat{\mathcal{E}})$. We define a map $\widetilde{T} = \left(\left(\widehat{T}\right)^{\natural}\right)^{\natural} : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{F}}$ by
$$\left(\left(\left(\widehat{T}\right)^{\natural}\right)^{\natural}(\tau)\right)(\zeta) = \left[\tau, \left(\widehat{T}\right)^{\natural}\zeta\right]_{\widetilde{\mathcal{E}}} \quad (\tau\in\widetilde{\mathcal{E}},\ \zeta\in\widehat{\mathcal{F}}).$$
Let $\mathcal{E},\mathcal{F}$ and $\mathcal{H}$ be Hilbert C^*-modules over $\mathcal{B}$. Then we have that $\widetilde{ST} = \widetilde{S}\widetilde{T}$ for all $T\in\mathcal{B}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$ and $S\in\mathcal{B}_{\mathcal{B}}(\mathcal{F},\mathcal{H})$ and that $\left(\widetilde{T}\right)^* = \widetilde{T^*}$ if $T\in\mathcal{L}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$.

(2) For any $T\in\mathcal{B}_{\mathcal{B}}(\mathcal{F},\mathcal{F}^{\#})$, we can naturally extend to an element $\overline{T}\in\mathcal{B}_{\mathcal{B}^{**}}(\widehat{\mathcal{F}},\widetilde{\mathcal{F}})$ as follows:
$$\left[\overline{T}\left(\sum_{i=1}^{n}\xi_i\otimes a_i\right)\right]\left(\sum_{j=1}^{m}\eta_j\otimes b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i^*\left[T(\xi_i)\right](\eta_j)b_j.$$
Hence we extend it again to an element $\widetilde{T} = \left(\overline{T}\right)^{\natural}\in\mathcal{B}_{\mathcal{B}^{**}}(\widetilde{\mathcal{F}})$.

We refer [18] for more details of these two extensions.

Let $(\mathcal{F},J)$ be a Krein C^*-module over $\mathcal{B}$. We extend J to $\widehat{J}$ on the algebraic tensor product $\mathcal{F}\otimes_{\mathrm{alg}}\mathcal{B}^{**}$ as follows:
$$\widehat{J}\left(\sum_{i=1}^{n}\xi_i\otimes a_i\right) = \sum_{i=1}^{n} J(\xi_i)\otimes a_i.$$
If the pair $(\mathcal{F},J)$ is a Krein $\mathcal{B}$-module, then the indefinite inner product $[\cdot,\cdot]_{\widehat{J}}$ on $\mathcal{F}\otimes_{\mathrm{alg}}\mathcal{B}^{**}$ defined by
$$[F,G]_{\widehat{J}} = \left[F, \widehat{J}(G)\right], \qquad F,G\in\mathcal{F}\otimes\mathcal{B}^{**}$$
gives the Krein $\mathcal{B}^{**}$-module structure. Then the pair $(\widehat{\mathcal{F}},\widehat{J})$ becomes a Krein $\mathcal{B}^{**}$-module. Moreover, we can easily see that the induced map $\widetilde{J}$ by $\widehat{J}$ is again a (fundamental) symmetry on $\widetilde{\mathcal{F}}$ and so $(\widetilde{\mathcal{F}},\widetilde{J})$ becomes a Krein $\mathcal{B}^{**}$-module. From the construction of above extensions, we obtain that any J-representation

π of $\mathcal{A}$ on a Hilbert $\mathcal{B}$-module $\mathcal{F}$ induces a $\widetilde{J}$-representation $\widetilde{\pi}$ of $\mathcal{A}$ on a self-dual Hilbert $\mathcal{B}^{**}$-module $\widetilde{\mathcal{F}}$ given by $\widetilde{\pi}(a) = \widetilde{\pi(a)}$. Hence an α-CP linear map ρ from a C^*-algebra $\mathcal{A}$ to a $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ induces a $\widetilde{J_\rho}$-representation $\widetilde{\pi_\rho}$ of $\mathcal{A}$ on a self-dual Hilbert $\mathcal{B}^{**}$-module $\widetilde{\mathcal{F}_\rho}$.

Now we extend the results in [8] and [18] for a strictly continuous α-CP map from a locally C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$. In the remaining of this section, we assume that α is p-continuous for each $p \in S(\mathcal{A})$.

We denote by α-CP$(\mathcal{A}, \mathcal{F})$ the set of all strictly continuous α-CP maps of $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$. We define a partial order $\leq$ on α-CP$(\mathcal{A}, \mathcal{F})$ as follows:

$$\phi_1 \leq \phi_2 \Longleftrightarrow \phi_2 - \phi_1 \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F}) \quad \text{for } \phi_1, \phi_2 \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F}).$$

Let $(\mathcal{F}_i, J_i, \pi_i, V_i)$ $(i = 1, 2)$ be minimal Krein quadruples associated with $\phi_i \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$, which is constructed in Theorem 3.1.

Theorem 4.2. *If $\phi_1, \phi_2 \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$ with $\phi_1 \leq \phi_2$, then there is a bounded $\mathcal{B}$-module map S from $\mathcal{F}_2$ into $\mathcal{F}_1$ such that $SV_2 = V_1$, $SJ_2 = J_1 S$ and*

$$S\pi_2(a) = \pi_1(a)S \quad \textit{for all } a \in \mathcal{A}.$$

Moreover, S extends uniquely to a bounded $\mathcal{B}^{**}$-module map $\widetilde{S}$ from $\widetilde{\mathcal{F}_2}$ into $\widetilde{\mathcal{F}_1}$ with $\|\widetilde{S}\| = \|S\|$.

Proof. By Theorem 3.1, there exist a Krein $\mathcal{B}$-module $(\mathcal{F}_i, J_i)$, a J_i-representation π_i of $\mathcal{A}$ on $\mathcal{F}_i$ and an operator $V_i \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}_i)$ such that

$$\phi_i(a) = V_i^* \pi_i(a) V_i \quad \text{and} \quad V_i^* \pi_i(a)^* \pi_i(b) V_i = V_i^* \pi_i(\alpha(a)^* b) V_i \quad (i = 1, 2).$$

We define a bounded $\mathcal{B}$-module map $S : \mathcal{F}_2 \to \mathcal{F}_1$ by

$$S\big(\pi_2(a) V_2 \xi\big) = \pi_1(a) V_1 \xi \quad (a \in \mathcal{A},\ \xi \in \mathcal{F}). \tag{4.2}$$

Since $\phi_1 \leq \phi_2$, it immediately follows from the definition that $\|S\| \leq 1$. It is easy to see that $SV_2 = V_1$, $SJ_2 = J_1 S$ and $S\pi_2(a) = \pi_1(a)S$ for all $a \in \mathcal{A}$ (see the proof of [8, Lemma 3.3] for more details). Hence, we can extend S to a bounded $\mathcal{B}^{**}$-module map $\widehat{S}$ from $\mathcal{F}_2 \otimes_{\text{alg}} \mathcal{B}^{**}$ into $\mathcal{F}_1 \otimes_{\text{alg}} \mathcal{B}^{**}$. Moreover, $\widehat{S}$ maps $\ker[\cdot,\cdot]_2$ into $\ker[\cdot,\cdot]_1$ since $\phi_1 \leq \phi_2$. By continuity, we extend $\widehat{S}$ again to a bounded $\mathcal{B}^{**}$-module map, still denoted by $\widehat{S}$, from $\widehat{\mathcal{F}_2}$ into $\widehat{\mathcal{F}_1}$. By [13, Proposition 3.6], $\widehat{S}$ is uniquely extended to a $\mathcal{B}^{**}$-module map $\widetilde{S}$ from $\widetilde{\mathcal{F}_2}$ into $\widetilde{\mathcal{F}_1}$. □

Let $T = (\widetilde{S})^* \widetilde{S}$. If we extend π_2 to $\widetilde{\pi_2}$ as a $\widetilde{J_2}$-representation of $\mathcal{A}$ on $\widetilde{\mathcal{F}_2}$, then

$$T\widetilde{\pi_2}(a) = \widetilde{\pi_2}(a)T \quad \text{and} \quad \widetilde{\phi_1}(a) = \widetilde{V_2}^* T \widetilde{\pi_2}(a) \widetilde{V_2} \quad \text{for all } a \in \mathcal{A}.$$

Indeed, we obtain from the equation (4.2) that $\widetilde{S}\widetilde{\pi_2}(a) = \widetilde{\pi_1}(a)\widetilde{S}$ for all $a \in \mathcal{A}$, which also implies $\big(\widetilde{S}\big)^* \widetilde{\pi_1}(a) = \widetilde{\pi_2}(a)\big(\widetilde{S}\big)^*$ for all $a \in \mathcal{A}$. Thus, we can easily get $T\widetilde{\pi_2}(a) = \widetilde{\pi_2}(a)T$. Moreover, we have that for all $a \in \mathcal{A}$

$$\widetilde{V_2}^* T \widetilde{\pi_2}(a) \widetilde{V_2} = \widetilde{V_2}^* (\widetilde{S})^* \widetilde{S} \widetilde{\pi_2}(a) \widetilde{V_2} = \widetilde{V_2}^* (\widetilde{S})^* \widetilde{\pi_1}(a) \widetilde{S} \widetilde{V_2} = \widetilde{\phi_1}(a).$$

For any $\phi \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$, we denote by $\widetilde{\pi_\phi}(\mathcal{A})'$ the commutant of $\widetilde{\pi_\phi}(\mathcal{A})$ in $\mathcal{L}_{\mathcal{B}^{**}}(\widetilde{\mathcal{F}_\phi})$. If a self-adjoint operator $T \in \widetilde{\pi_\phi}(\mathcal{A})'$ commutes with $\widetilde{J_\phi}$ and if

$$\widetilde{V_\phi}^* T \widetilde{\pi_\phi}(a) \widetilde{V_\phi}\big|_{\mathcal{F}} \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}) \quad \text{for all } a \in \mathcal{A},$$

then the restriction of T to $\mathcal{F}_\phi$ is a $\mathcal{B}$-module map $T\big|_{\mathcal{F}_\phi}$ from $\mathcal{F}_\phi$ into $\mathcal{F}_\phi^{\#}$. Indeed, for all $a, b \in \mathcal{A}$ and $\xi, \eta \in \mathcal{F}$, we have that

$$\begin{aligned} \langle T\pi_\phi(a)V_\phi\xi, \pi_\phi(b)V_\phi\eta\rangle &= \left\langle T\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\xi, \widetilde{\pi_\phi}(b)\widetilde{V_\phi}\eta\right\rangle \\ &= \left\langle \xi, \widetilde{V_\phi}^* T\widetilde{\pi_\phi}(\alpha(a)^*b)\widetilde{V_\phi}\eta\right\rangle, \end{aligned}$$

which implies that $\langle T\pi_\phi(a)V_\phi\xi, \pi_\phi(b)V_\phi\eta\rangle \in \mathcal{B}$ since the set $\pi_\phi(\mathcal{A})\big[V_\phi(\mathcal{F})\big]$ spans a dense submodule of $\mathcal{F}_\phi$. Therefore, the range of the restriction $T\big|_{\mathcal{F}_\phi}$ is in $\mathcal{F}_\phi^{\#}$.

Let $C^*(\phi, \mathcal{F})$ be the C^*-subalgebra of $\mathcal{L}_{\mathcal{B}^{**}}(\widetilde{\mathcal{F}_\phi})$ generated by

$$\left\{T \in \widetilde{\pi_\phi}(\mathcal{A})' : \ T\widetilde{J_\phi} = \widetilde{J_\phi}T \text{ and } \widetilde{V_\phi}^* T\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}} \in \mathcal{L}_{\mathcal{B}}(\mathcal{F})\right\}.$$

Let $(\mathcal{F}_\phi, J_\phi, \pi_\phi, V_\phi)$ be a minimal Krein quadruple associated with $\phi \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$ which is constructed in Theorem 3.1.

Proposition 4.3. *If $T \in C^*(\phi, \mathcal{F})$ is a positive operator, then the linear map $\phi_T : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ defined by*

$$\phi_T(a) = \widetilde{V_\phi}^* T\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}}$$

is a strictly continuous α-CP map from $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F}_\phi)$. If $T \le I_{\widetilde{\mathcal{F}_\phi}}$, then $\phi_T \le \phi$.

Proof. In the same way as in the proof of [8, Proposition 3.1], we can prove that ϕ_T is α-CP, so that we omit its proof. Also, we can easily see that $T \le I_{\widetilde{\mathcal{F}_\phi}}$ implies $\phi_T \le \phi$. We only need to show that ϕ_T is strictly continuous. Let $\{e_\lambda\}$ be an approximate unit for $\mathcal{A}$. For any $\xi \in \mathcal{F}$ we have that

$$\begin{aligned} \|\phi_T(e_\lambda)\xi - \phi_T(e_\mu)\xi\| &= \left\|\widetilde{V_\phi}^* T\big[\widetilde{\pi_\phi}(e_\lambda) - \widetilde{\pi_\phi}(e_\mu)\big]\widetilde{V_\phi}\xi\right\| \\ &\le \left\|\widetilde{V_\phi}^* T\right\| \left\|\big[\pi_\phi(e_\lambda) - \pi_\phi(e_\mu)\big]V_\phi\xi\right\|. \end{aligned}$$

Since the net $\{\pi_\phi(e_\lambda)V_\phi\xi\}$ is Cauchy in $\mathcal{F}_\phi$, $\{\phi_T(e_\lambda)\}$ is a strictly Cauchy net. This completes the proof. □

For any $\phi \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$, we introduce two sets as follows:

$$\begin{aligned} [0, \phi]_\alpha &= \{\psi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F}) : \psi \le \phi\}, \\ [0, I]_\phi &= \{T \in C^*(\phi, \mathcal{F}) : 0 \le T \le I_{\widetilde{\mathcal{F}_\phi}}\}, \end{aligned}$$

where $(\mathcal{F}_\phi, J_\phi, \pi_\phi, V_\phi)$ be a minimal Krein quadruple associated with ϕ. We now consider the map $T \mapsto \phi_T$ from $[0, I]_\phi$ to $[0, \phi]_\alpha$. Then it is not hard to show that the map is affine. To show the injectivity, let $T \in [0, I]_\phi$ such that $\phi_T = 0$. Then we have that

$$\widetilde{V_\phi}^* T \widetilde{\pi_\phi}(a) \widetilde{V_\phi}\big|_{\mathcal{F}} = 0 \quad \text{for all } a \in \mathcal{A},$$

which implies that $0 = \langle T\pi_\phi(a)V_\phi\xi, \pi_\phi(b)V_\phi\eta\rangle$ for any $a, b \in \mathcal{A}$ and $\xi, \eta \in \mathcal{F}$. Since the set $\{\pi_\phi(b)V_\phi\eta : b \in \mathcal{A}, \eta \in \mathcal{F}\}$ spans a dense submodule of $\mathcal{F}_\phi$, we have that $T\big|_{\mathcal{F}_\phi} = 0$ and so $T = 0$. In fact, since $T\big|_{\mathcal{F}_\phi} \in \mathcal{B}_{\mathcal{B}}(\mathcal{F}_\phi, \mathcal{F}_\phi^{\#})$, by (2) in Remark 4.1, it can be extended to an element $\overline{T\big|_{\mathcal{F}_\phi}} \in \mathcal{B}_{\mathcal{B}^{**}}(\widehat{\mathcal{F}_\phi}, \widetilde{\mathcal{F}_\phi})$ with

$$\left[\overline{T\big|_{\mathcal{F}_\phi}}\left(\sum_{i=1}^{n} x_i \otimes a_i\right)\right]\left(\sum_{j=1}^{m} y_j \otimes b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i^* \left[T\big|_{\mathcal{F}_\phi}(x_i)\right](y_j) b_j$$

for any $x_i, y_j \in \mathcal{F}_\phi$ and $a_i, b_j \in \mathcal{B}^{**}$ with $1 \le i \le n, 1 \le j \le m$. From the equality

$$\begin{aligned}\left[T\big|_{\mathcal{F}_\phi}(x_i)\right](y_j) &= \left(T((x_i \otimes I_{\mathcal{B}^{**}})^{\#})\right)(y_j \otimes I_{\mathcal{B}^{**}}) \\ &= \left\langle T\big((x_i \otimes I_{\mathcal{B}^{**}})^{\#}\big),\, (y_j \otimes I_{\mathcal{B}^{**}})^{\#}\right\rangle,\end{aligned}$$

we obtain that

$$\overline{T\big|_{\mathcal{F}_\phi}}\left(\sum_{i=1}^{n} x_i \otimes a_i\right) = T\left(\left(\sum_{i=1}^{n} x_i \otimes a_i\right)^{\#}\right).$$

Hence the map $\overline{T\big|_{\mathcal{F}_\phi}}$ can be extended to $\widetilde{T\big|_{\mathcal{F}_\phi}} = \left(\overline{T\big|_{\mathcal{F}_\phi}}\right)^{\natural}$ in $\mathcal{B}_{\mathcal{B}^{**}}(\widetilde{\mathcal{F}_\phi})$. For any $\tau \in \widetilde{\mathcal{F}_\phi}$ and $\zeta \in \widehat{\mathcal{F}_\phi}$, we have that

$$\begin{aligned}\left(\left(\overline{T\big|_{\mathcal{F}_\phi}}\right)^{\natural}(\tau)\right)(\zeta) &= \left\langle \tau, \overline{T\big|_{\mathcal{F}_\phi}}(\zeta)\right\rangle = \left\langle \tau, T(\zeta^{\#})\right\rangle \\ &= \left\langle T^*(\tau), \zeta^{\#}\right\rangle = (T^*(\tau))(\zeta),\end{aligned}$$

which means that $\widetilde{T\big|_{\mathcal{F}_\phi}} = T^*$. Hence, if $T\big|_{\mathcal{F}_\phi} = 0$, then $T^* = 0$ and so $T = 0$.

Theorem 4.4. *Let $\phi \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$. Then the map $T \mapsto \phi_T$ is an affine order isomorphism of $[0, I]_\phi$ onto $[0, \phi]_\alpha$.*

Proof. We have already proved that the map $T \mapsto \phi_T$ is affine and injective. Now, we prove that the map is onto, i.e., each element $\psi \in [0, \phi]_\alpha$ is of the form $\psi = \phi_T$ for some $T \in [0, I]_\phi$. Let $(\mathcal{F}_\psi, J_\psi, \pi_\psi, V_\psi)$ be a minimal Krein quadruple associated with ψ. By Theorem 4.2, there exists a bounded $\mathcal{B}$-module map S from $\mathcal{F}_\phi$ to $\mathcal{F}_\psi$ such that

$$\|S\| \le 1, \quad SV_\phi = V_\psi \quad \text{and} \quad SJ_\phi = J_\psi S.$$

Moreover, S extends uniquely to a bounded $\mathcal{B}^{**}$-module map $\widetilde{S}$ from $\widetilde{\mathcal{F}_\phi}$ to $\widetilde{\mathcal{F}_\psi}$ with $\|S\| = \|\widetilde{S}\|$. By defining $T := \widetilde{S}^*\widetilde{S}$, we have that $0 \le T \le I_{\widetilde{\mathcal{F}_\phi}}$. The preceding argument before Proposition 4.3 shows that the operator T commutes with $\widetilde{J_\phi}$ and $\widetilde{\pi_\phi}(a)$ for all $a \in \mathcal{A}$. Therefore, we have that

$$\widetilde{V_\phi}^* T\widetilde{\pi_\phi}(a)\widetilde{V_\phi} = \widetilde{V_\phi}^* \widetilde{S}^*\widetilde{S}\widetilde{\pi_\phi}(a)\widetilde{V_\phi} = \widetilde{V_\phi}^* \widetilde{S}^*\widetilde{\pi_\psi}(a)\widetilde{S}\widetilde{V_\phi} = \widetilde{V_\psi}^* \widetilde{\pi_\psi}(a)\widetilde{V_\psi},$$

this implies that

$$\phi_T(a) = \widetilde{V_\phi}^* T\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}} = \widetilde{\psi}(a)\big|_{\mathcal{F}} = \psi(a) \in \mathcal{L}_{\mathcal{B}}(\mathcal{F})$$

for all $a \in \mathcal{A}$ and so $\psi = \phi_T$. Thus, the map $T \mapsto \phi_T$ from $[0, I]_\phi$ to $[0, \phi]_\alpha$ is surjective. We can also see that $\phi_{T_1} \le \phi_{T_2}$ whenever $T_1 \le T_2$ in $C^*(\phi, \mathcal{F})$, so that the map preserves the order relation. □

An element $\phi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ is said to be *pure* if for every $\psi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$, $\psi \le \phi$ implies that ψ is a scalar multiple of ϕ. Equivalently, $\phi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ is pure if the only possible decompositions of ϕ are of the form $\phi = \phi_1 + \phi_2$ ($\phi_i \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$) when each ϕ_i is a scalar multiple of ϕ.

Corollary 4.5. *Let $\phi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ be unital. Then ϕ is pure if and only if $[0, I]_\phi$ only consists of scalar multiples of $I_{\mathcal{F}_\phi}$.*

Proof. We give the proof by a modification of the proof of Corollary 3.5 in [8]. We first assume that $\phi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ is not pure, that is,

$$\phi = \lambda\phi_1 + (1 - \lambda)\phi_2 \quad \text{for some } 0 < \lambda < 1,$$

where $\phi_1, \phi_2 \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ with $\phi_1 \ne \phi_2$. Let $(\mathcal{F}_\phi, J_\phi, \pi_\phi, V_\phi)$ be a minimal Krein quadruple associated with $\phi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$. Since $\phi - \lambda\phi_1 = (1 - \lambda)\phi_2$ is in $\alpha\text{-CP}(\mathcal{A}, \mathcal{F})$, we have the inequality $\lambda\phi_1 \le \phi$, that is, $\lambda\phi_1 \in [0, \phi]_\alpha$. Hence, by Theorem 4.4, there is $T \in [0, I]_\phi$ such that $\lambda\phi_1 = \phi_T$. Moreover, since ϕ is not pure, $\lambda\phi_1$ is not scalar multiple of ϕ. Thus T is not scalar multiple of $I_{\widetilde{\mathcal{F}_\phi}}$.

Conversely, assume that $[0, I]_\phi$ contains $0 < T < I_{\widetilde{\mathcal{F}_\phi}}$ which is not a scalar multiple of $I_{\widetilde{\mathcal{F}_\phi}}$. For any $0 < \lambda < 1$, we define two linear maps ϕ_1 and ϕ_2 from $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ by

$$\begin{aligned}\phi_1(a) &= \lambda^{-1} \cdot \widetilde{V_\phi}^* T\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}},\\ \phi_2(a) &= (1 - \lambda)^{-1} \cdot \widetilde{V_\phi}^* (I_{\widetilde{\mathcal{F}_\phi}} - T)\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}}.\end{aligned}$$

Then it follows that $\phi_1, \phi_2 \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ and $\phi = \lambda\phi_1 + (1 - \lambda)\phi_2$. Since T is not a scalar multiple of $I_{\widetilde{\mathcal{F}_\phi}}$, the map ϕ is not pure. □

From now on, let $\mathcal{A}$ be a unital locally C^*-algebra. For a fixed positive operator $P \in \mathcal{L}_{\mathcal{B}}(\mathcal{F})$, we denote by $\alpha\text{-CP}(\mathcal{A}, \mathcal{F}; P)$ the set of ϕ in $\alpha\text{-CP}(\mathcal{A}, \mathcal{F})$ such that $\phi(1_{\mathcal{A}}) = P$. We can get the following theorem about extreme points by modifying the proof of Theorem 3.8 in [18] and we also refer [1, Theorem 1.4.6] for a similar result.

Theorem 4.6. *Let $\phi \in \alpha\text{-CP}(\mathcal{A}, \mathcal{F}; P)$. The followings are equivalent*:

(i) *the map ϕ is an extreme point in $\alpha\text{-CP}(\mathcal{A}, \mathcal{F}; P)$;*

(ii) *$T \mapsto \widetilde{V_\phi}^* T\widetilde{V_\phi}|_{\mathcal{F}}$ is a one-to-one mapping from $[0, I]_\phi$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$;*

(iii) *$T \mapsto \widetilde{V_\phi}^* T\widetilde{V_\phi}$ is a one-to-one mapping from $C^*(\phi, \mathcal{F})$ into $\mathcal{L}_{\mathcal{B}^{**}}(\widetilde{\mathcal{F}})$.*

Proof. (i) $\Rightarrow$ (iii) We assume that ϕ is an extreme point in α-CP$(\mathcal{A}, \mathcal{F}; P)$ and $\widetilde{V_\phi}^* T\widetilde{V_\phi} = 0$ for some $T \in C^*(\phi, \mathcal{F})$. Since the map $T \mapsto \widetilde{V_\phi}^* T\widetilde{V_\phi}$ preserves adjoint, we can assume that T is self-adjoint. Choose positive real numbers μ, λ such that

$$\frac{1}{4} I_{\widetilde{\mathcal{F}_\phi}} \leq \mu T + \lambda I_{\widetilde{\mathcal{F}_\phi}} \leq \frac{3}{4} I_{\widetilde{\mathcal{F}_\phi}}.$$

If we put $W = \mu T + \lambda I_{\widetilde{\mathcal{F}_\phi}}$, then we have that

$$\widetilde{V_\phi}^* W\widetilde{V_\phi} = \mu\widetilde{V_\phi}^* T\widetilde{V_\phi} + \lambda\widetilde{V_\phi}^*\widetilde{V_\phi} = \lambda\widetilde{P},$$

so that $\frac{1}{4}P \leq \lambda P \leq \frac{3}{4}P$. This implies that $0 < \lambda < 1$. If ϕ_1 and ϕ_2 on $\mathcal{A}$ are defined by

$$\phi_1(a) = \widetilde{V_\phi}^* W\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}} \quad \text{and} \quad \phi_2(a) = \widetilde{V_\phi}^*(I - W)\widetilde{\pi_\phi}(a)\widetilde{V_\phi}\big|_{\mathcal{F}},$$

then we have that $\phi = \phi_1 + \phi_2$ and that $\lambda^{-1}\phi_1$, $(1-\lambda)^{-1}\phi_2 \in \alpha$-CP$(\mathcal{A}, \mathcal{F}; P)$. Since ϕ is an extreme point in α-CP$(\mathcal{A}, \mathcal{F}; P)$, we have that

$$\lambda^{-1}\phi_1 = (1-\lambda)^{-1}\phi_2 = \phi.$$

The equality $\phi_1 = \lambda\phi$ implies that $\widetilde{V_\phi}^* W\widetilde{\pi_\phi}(a)\widetilde{V_\phi} = \lambda\widetilde{V_\phi}^*\widetilde{\pi_\phi}(a)\widetilde{V_\phi}$ for all $a \in \mathcal{A}$. Hence we obtain from Theorem 4.4 that $W = \lambda I_{\widetilde{\mathcal{F}_\phi}}$, so that $T = 0$.

(iii) $\Rightarrow$ (ii) Suppose that the map $T \mapsto \widetilde{V_\phi}^* T\widetilde{V_\phi}$ is injective on $C^*(\phi, \mathcal{F})$. To show the injectivity of the map $T \mapsto \widetilde{V_\phi}^* T\widetilde{V_\phi}|_{\mathcal{F}}$, we assume that

$$\widetilde{V_\phi}^* T\widetilde{V_\phi}|_{\mathcal{F}} = 0 \quad \text{for some } T \in [0, I]_\phi.$$

Since $\widetilde{V_\phi}^* T\widetilde{V_\phi}|_{\mathcal{F}}$ is an operator in $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$, by the construction (1) in Remark 4.1, we can easily see that $\widetilde{V_\phi}^* T\widetilde{V_\phi}$ is the unique extension of $\widetilde{V_\phi}^* T\widetilde{V_\phi}|_{\mathcal{F}}$. Therefore, we have that $\widetilde{V_\phi}^* T\widetilde{V_\phi} = 0$, which implies that $T = 0$.

(ii) $\Rightarrow$ (i) Suppose that the map $T \mapsto \widetilde{V_\phi}^* T\widetilde{V_\phi}|_{\mathcal{F}}$ is injective on $[0, I]_\phi$. For any $\phi_1, \phi_2 \in \alpha$-CP$(\mathcal{A}, \mathcal{F}; P)$, let $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ with $0 \leq \lambda \leq 1$. Since $\phi \geq \lambda\phi_1$ in α-CP$(\mathcal{A}, \mathcal{F})$, there exists an operator $T_1 \in [0, I]_\phi$ such that

$$\lambda\widetilde{\phi_1}(a) = \widetilde{V_\phi}^* T_1\widetilde{\pi_\phi}(a)\widetilde{V_\phi} \quad \text{for all } a \in \mathcal{A}.$$

Hence we have that $\lambda\widetilde{V_\phi}^*\widetilde{V_\phi} = \lambda\widetilde{P} = \lambda\widetilde{\phi_1}(1_{\mathcal{A}}) = \widetilde{V_\phi}^* T_1\widetilde{V_\phi}$, which implies that $T_1 = \lambda I_{\widetilde{\mathcal{F}_\phi}}$. Therefore, we have that $\phi_1 = \phi$ and $\phi_2 = \phi$. $\square$

Lemma 4.7. *Let $\phi \in \alpha$-CP$(\mathcal{A}, \mathcal{F})$ with the minimal Krein quadruple $(\mathcal{F}_\phi, J_\phi, \pi_\phi, V_\phi)$. If S is in the commutant $\phi(\mathcal{A})'$, then there exists an operator $T \in \pi_\phi(\mathcal{A})'$ such that*

$$TJ_\phi = J_\phi T, \quad TV_\phi = V_\phi S, \quad V_\phi^* T = SV_\phi^*.$$

Proof. The proof is a slight modification of the proof of [1, Theorem 1.3.1], but we sketch a proof for the reader's convenience.

Let $a_i \in \mathcal{A}$, $\xi_i \in \mathcal{F}$ with $i = 1, \dots, n$. We see that

$$\left\| \sum_{i=1}^{n} \pi_\phi(a_i) V_\phi S \xi_i \right\| \leq \| S \| \left\| \sum_{i=1}^{n} \pi_\phi(a_i) V_\phi \xi_i \right\|.$$

Now let $\mathcal{F}_n = \mathbb{C}^n \otimes \mathcal{F}$, $S_n = I_n \otimes S \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}_n)$ and $V_{\phi,n} = I_n \otimes V_\phi$ with the identity I_n in $\mathbb{C}^n$. We denote by $\pi_{\phi,n}$

$$\pi_{\phi,n} = \begin{pmatrix} \pi_\phi(a_1) & \pi_\phi(a_2) & \cdots & \pi_\phi(a_n) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $V_{\phi,n}^* \pi_{\phi,n}^* \pi_{\phi,n} V_{\phi,n} \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}_n)$ has the operator matrix $(V_\phi^* \pi_\phi(a_i)^* \pi_\phi(a_j) V_\phi)$ which commutes with S'. If we put $\Xi_n = \xi_1 \oplus \cdots \oplus \xi_n \in \mathcal{F}_n$, then we have that

$$\left\| \sum_{i=1}^{n} \pi_\phi(a_i) V_\phi S \xi_i \right\|^2 \leq \| S \|^2 \left\| \sum_{i=1}^{n} \pi_\phi(a_i) V_\phi \xi_i \right\|^2.$$

The operator $T : \sum_{i=1}^{n} \pi_\phi(a_i) V_\phi \xi_i \mapsto \sum_{i=1}^{n} \pi_\phi(a_i) V_\phi S \xi_i$ extends uniquely to an adjointable operator on $\mathcal{F}_\phi$ and $TJ_\phi = J_\phi T$. By taking $a = 1_{\mathcal{A}}$, we obtain that $TV_\phi \xi = V_\phi S \xi$ and $V_\phi^* T \pi_\phi(a) V_\phi \xi = SV_\phi^* \pi_\phi(a) V_\phi \xi$, which implies that $V_\phi^* T = SV_\phi^*$. Moreover, T commute with $\pi_\phi(\mathcal{A})$. □

The following corollary is a generalization of Corollary 1.4.7 in [1].

Corollary 4.8. *Let ϕ be an extreme point of α-$CP(\mathcal{A}, \mathcal{F}; I_{\mathcal{F}})$ with the minimal Krein quadruple $(\mathcal{F}_\phi, J_\phi, \pi_\phi, V_\phi)$ and let $\mathcal{Z}$ be the center of $\mathcal{A}$. Assume that $\phi(\mathcal{Z}) \subseteq \phi(\mathcal{A})'$ and $J_\phi \in \pi_\phi(\mathcal{Z})'$. Then $\phi(az) = \phi(a)\phi(z)$ for any $a \in \mathcal{A}, z \in \mathcal{Z}$.*

Proof. Since $\phi \in \alpha$-$CP(\mathcal{A}, \mathcal{F}; I_{\mathcal{F}})$, we have that $V_\phi^* V_\phi = \phi(1_{\mathcal{A}}) = I_{\mathcal{F}}$. Thus, V_ϕ is an isometry and $V_\phi V_\phi^*$ is a projection in $\mathcal{L}_{\mathcal{B}}(\mathcal{F}_\phi)$. For any $z \in \mathcal{Z}$, we have $\phi(z) \in \phi(\mathcal{A})'$. By Lemma 4.7, there exists $T \in \pi_\phi(\mathcal{A})'$ such that $TJ_\phi = J_\phi T$, $TV_\phi = V_\phi \phi(z)$ and $V_\phi^* T = \phi(z) V_\phi^*$. Hence we have that

$$TV_\phi V_\phi^* = V_\phi \phi(z) V_\phi^* = V_\phi V_\phi^* T. \tag{4.3}$$

On the other hand, it follows from assumption that $\pi_\phi(z) \in \pi_\phi(\mathcal{A})'$ and $\pi_\phi(z) J_\phi = J_\phi \pi_\phi(z)$. Hence, we can easily see that $\widetilde{\pi_\phi}(z) \in C^*(\phi, \mathcal{F})$. Moreover, by Theorem 4.6 we have that $V_\phi^* \pi_\phi(z) V_\phi = \phi(z) = V_\phi^* T V_\phi$, which implies that $\widetilde{\pi_\phi}(z) = \widetilde{T}$. Then, by (4.3), $\widetilde{\pi_\phi}(z)$ commutes with $\widetilde{V_\phi}\widetilde{V_\phi}^*$, and then

$$\widetilde{\phi}(az) = \widetilde{V_\phi}^* \widetilde{\pi_\phi}(a) \widetilde{\pi_\phi}(z) \widetilde{V_\phi} = \widetilde{V_\phi}^* \widetilde{\pi_\phi}(a) \widetilde{V_\phi} \widetilde{V_\phi}^* \widetilde{\pi_\phi}(z) \widetilde{V_\phi} = \widetilde{\phi}(a) \widetilde{\phi}(z).$$

Hence we get the equality $\phi(az) = \phi(a)\phi(z)$ for $a \in \mathcal{A}$ and $z \in \mathcal{Z}$. □

Acknowledgment. A part of this research was carried out while the first author was visiting the KIAS. He wishes to thank all members in the KIAS for their hospitality during his visit. We would also like to thank the anonymous referee for the careful reading and pointing out some mistake.

References

[1] W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224.

[2] S. J. Bhatt and D. J. Karia, *Complete positivity, tensor products and C^*-nuclearity for inverse limits of C^*-algebras*, Proc. Indian Acad. Sci. Math. Sci. **101** (1991), no. 3, 149–167.

[3] H. O. Cordes and E. Schrohe, *On the symbol homomorphism of a certain Frechet algebra of singular integral operators*, Integral Equations Operator Theory **8** (1985), no. 5, 641–649.

[4] M. Dubois-Violette, *A generalization of the classical moment problem on $*$-algebras with applications to relativistic quantum theory I, II*, Commun. Math. Phys. **43** (1975), no. 3, 225–254; Comm. Math. Phys. **54** (1977), no. 2, 151–172.

[5] M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Mathematics Studies **200**, Elsevier Science B.V., Amsterdam, 2005.

[6] S. Gudder, *A Radon-Nikodým theorem for $*$-algebras*, Pacific J. Math. **80** (1979), no. 1, 141–149.

[7] J. Heo, J. P. Hong, and U. C. Ji, *On KSGNS representations on Krein C^*-modules*, J. Math. Phys. **51** (2010), no. 5, 053504, 13 pp.

[8] J. Heo and U. C. Ji, *Radon-Nikodým type theorem for α-completely positive maps*, J. Math. Phys. **51** (2010), no. 10, 103505, 10 pp.

[9] A. Inoue, *Locally C^*-algebra*, Mem. Fac. Sci. Kyushu Univ. Ser. A **25** (1971), 197–235.

[10] M. Joiţa, *Strict completely positive maps between locally C^*-algebras and representations on Hilbert modules*, J. London Math. Soc. **66** (2002), no. 2, 421–432.

[11] ———, *A Radon-Nikodým theorem for completely multi-positive linear maps and its applications*, in: "Topological Algebras and Applications", pp. 235–245, Contemp. Math. **427**, Amer. Math. Soc., Providence, RI, 2007.

[12] ———, *Frames of multipliers in tensor products of Hilbert modules over pro-C^*-algebras*, J. Math. Anal. Appl. **367** (2010), no. 2, 522–534.

[13] W. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

[14] G. Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras*, Acta Math. **130** (1973), 53–87.

[15] N. C. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory **19** (1988), no. 1, 159–195.

[16] ———, *Representable K-theory for σ-C^*-algebras*, K-Theory **3** (1989), no. 5, 441–478.

[17] S. Sakai, *A Radon-Nikodým theorem in W^*-algebras*, Bull. Amer. Math. Soc. **71** (1965), 149–151.

[18] S.-K. Tsui, *Completely positive module maps and completely positive extreme maps*, Proc. Amer. Math. Soc. **124** (1996), no. 2, 437–445.

Jaeseong Heo
Department of Mathematics
Research Institute for Natural Sciences
Hanyang University
Seoul 133-791, Korea
E-mail address: hjs@hanyang.ac.kr

Un Cig Ji
Department of Mathematics
Research Institute of Mathematical Finance
Chungbuk National University
Cheongju 361-763, Korea
E-mail address: uncigji@chungbuk.ac.kr

Young Yi Kim
Department of Mathematics
Chungbuk National University
Cheongju 361-763, Korea
E-mail address: kimyy@chungbuk.ac.kr