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Additive functional inequalities in Banach spaces

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Abstract

In this paper, we prove the Hyers-Ulam stability of the following function inequalities:

$$\|f(x)+f(y)+f(z)\| \le \left\|Kf\left(\frac{x+y+z}{K}\right)\right\| \quad (0<|K|<3),$$

$$\|f(x)+f(y)+Kf(z)\| \le \left\|Kf\left(\frac{x+y}{K}+z\right)\right\| \quad (0<K\neq 2)$$

in Banach spaces.

MSC: Primary 39B62; 39B52; 46B25

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1 Introduction and preliminaries

The stability problem of functional equations originated from the question of Ulam [1] in 1940 concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta 0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta$$

for all $x,y\in E$ and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta$$

for all $x\in E$. Moreover, if $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in E$, then T is $\mathbb{R}$-linear. In 1978, Th.M. Rassias [3] proved the following theorem.

Theorem 1.1 *Let $f: E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \epsilon\left(\|x\|^p+\|y\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T: E \to E'$ such that

$$\left\|f(x)-T(x)\right\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then inequality (1.1) *holds for all $x, y \neq 0$, and* (1.2) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.*

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. On the other hand, J.M. Rassias [5] generalized the Hyers-Ulam stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.2 [6, 7] *If it is assumed that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f: E \to E'$ is a mapping from a norm space E into a Banach space E' such that the inequality*

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \epsilon\|x\|^{p_1}\|y\|^{p_2}$$

holds for all $x, y \in E$, then there exists a unique additive mapping $T: E \to E'$ such that

$$\left\|f(x)-T(x)\right\| \leq \frac{\Theta}{2-2^p}\|x\|^p$$

for all $x \in E$. If, in addition, for every $x \in E$, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [8–11].

In [12], Park *et al.* investigated the following inequalities:

$$\begin{aligned}
&\left\|f(x)+f(y)+f(z)\right\| \leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \\
&\left\|f(x)+f(y)+f(z)\right\| \leq \left\|f(x+y+z)\right\|, \\
&\left\|f(x)+f(y)+2f(z)\right\| \leq \left\|2f\left(\frac{x+y}{2}+z\right)\right\|
\end{aligned}$$

in Banach spaces. Recently, Cho *et al.* [13] investigated the following functional inequality:

$$\left\|f(x)+f(y)+f(z)\right\| \leq \left\|Kf\left(\frac{x+y+z}{K}\right)\right\| \quad \left(0 < |K| < |3|\right)$$

in non-Archimedean Banach spaces. Lu and Park [14] investigated the following functional inequality:

$$\left\| \sum_{i=1}^{N} f(x_i) \right\| \leq \left\| Kf\left(\frac{\sum_{i=1}^{N}(x_i)}{K} \right) \right\| \quad \bigl(0 < |K| \leq N\bigr)$$

in Fréchet spaces.

In this paper, we investigate the following functional inequalities:

$$\bigl\| f(x) + f(y) + f(z) \bigr\| \leq \left\| Kf\left(\frac{x+y+z}{K} \right) \right\| \quad \bigl(0 < |K| < 3\bigr), \tag{1.3}$$

$$\bigl\| f(x) + f(y) + Kf(z) \bigr\| \leq \left\| Kf\left(\frac{x+y}{K} + z \right) \right\| \quad (0 < K \neq 2) \tag{1.4}$$

and prove the Hyers-Ulam stability of functional inequalities (1.3) and (1.4) in Banach spaces.

Throughout this paper, assume that X is a normed vector space and that $(Y, \|\cdot\|)$ is a Banach space.

2 Hyers-Ulam stability of functional inequality (1.3)

Throughout this section, assume that K is a real number with $0 < |K| < 3$.

Proposition 2.1 *Let $f : X \to Y$ be a mapping such that*

$$\bigl\| f(x) + f(y) + f(z) \bigr\| \leq \left\| Kf\left(\frac{x+y+z}{K} \right) \right\| \tag{2.1}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is additive.

Proof Letting $x = y = z = 0$ in (2.1), we get

$$\bigl\| 3f(0) \bigr\| \leq \bigl\| Kf(0) \bigr\|.$$

So, $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.1), we get

$$\bigl\| f(x) + f(-x) \bigr\| \leq \bigl\| Kf(0) \bigr\| = 0$$

for all $x \in X$. So, $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (2.1), we get

$$\bigl\| f(x) + f(y) - f(x+y) \bigr\| = \bigl\| f(x) + f(y) + f(-x-y) \bigr\| \leq \bigl\| Kf(0) \bigr\| = 0$$

for all $x, y \in X$. Thus,

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

Theorem 2.2 *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x) + f(y) + f(z)\| \le \left\| Kf\left(\frac{x+y+z}{K}\right)\right\| + \phi(x, y, z), \tag{2.2}$$

where $\phi : X^3 \to [0, \infty)$ satisfies

$$\widetilde{\phi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \le \frac{1}{2}\widetilde{\phi}(-x, -x, 2x) + \widetilde{\phi}(x, -x, 0) \tag{2.4}$$

for all $x \in X$.

Proof It follows from (2.3) that $\phi(0, 0, 0) = 0$. Letting $x = y = z = 0$ in (2.2), we get $\|3f(0)\| \le \|Kf(0)\| + \phi(0, 0, 0) = \|Kf(0)\|$. So, $f(0) = 0$.

Letting $y = x$, $z = -2x$ in (2.2), we get

$$\|2f(x) + f(-2x)\| \le \phi(x, x, -2x)$$

for all $x \in X$. So,

$$\left\|2f\left(\frac{x}{2}\right) + f(-x)\right\| \le \phi\left(\frac{x}{2}, \frac{x}{2}, -x\right) \tag{2.5}$$

for all $x \in X$.

Letting $y = -x$ and $z = 0$ in (2.2), we get

$$\|f(x) + f(-x)\| \le \phi(x, -x, 0) \tag{2.6}$$

for all $x \in X$. It follows from (2.5) and (2.6) that

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\
&\quad \le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\quad \le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) + 2^{j+1} f\left(\frac{-x}{2^{j+1}}\right) - 2^{j+1} f\left(\frac{-x}{2^{j+1}}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\quad \le \sum_{j=l}^{m-1} \left[\left\|2^j f\left(\frac{x}{2^j}\right) + 2^{j+1} f\left(\frac{-x}{2^{j+1}}\right)\right\| + \left\|2^{j+1} f\left(\frac{-x}{2^{j+1}}\right) + 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|\right] \\
&\quad \le \sum_{j=l}^{m-1} \frac{1}{2} 2^{j+1} \phi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}, \frac{2x}{2^{j+1}}\right) + \sum_{j=l}^{m-1} 2^{j+1} \phi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}, 0\right)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. We define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.4).

Next, we show that $A(x)$ is an additive mapping.

$$\begin{aligned}\|A(x) + A(-x)\| &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} \frac{1}{2} 2^{n+1} \phi\left(\frac{2x}{2^{n+1}}, \frac{-2x}{2^{n+1}}, 0\right) = 0\end{aligned}$$

and so $A(-x) = -A(x)$ for all $x \in X$.

$$\begin{aligned}\|A(x) + A(y) - A(x+y)\| &= \|A(x) + A(y) + A(-x-y)\| \\ &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + f\left(\frac{-x-y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} \frac{1}{2} 2^{n+1} \phi\left(\frac{2x}{2^{n+1}}, \frac{2y}{2^{n+1}}, \frac{2(x+y)}{2^{n+1}}\right) = 0\end{aligned}$$

for all $x, y \in X$. Thus, the mapping $A : X \to Y$ is additive.

Now, we prove the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (2.4). Then we obtain

$$\begin{aligned}\|A(x) - T(x)\| &= \lim_{n\to\infty} 2^n \left\| A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 2^n \left[\left\| A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right] \\ &\le \lim_{n\to\infty} \left[\widetilde{\phi}\left(\frac{x}{2^n}, \frac{-x}{2^n}, \frac{2x}{2^n}\right) + 2\widetilde{\phi}\left(\frac{x}{2^n}, \frac{-x}{2^n}, 0\right)\right] = 0\end{aligned}$$

for all $x \in X$. Then we can conclude that $A(x) = T(x)$ for all $x \in X$. This completes the proof. □

Corollary 2.3 *Let p and θ be positive real numbers with $p > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(x) + f(y) + f(z)\| \le \left\| Kf\left(\frac{x+y+z}{K}\right)\right\| + \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^p + 6}{2^p - 2}\theta\|x\|^p$$

for all $x \in X$.

3 Hyers-Ulam stability of functional inequality (1.4)

Throughout this section, assume that K is a real number with $0 < K \ne 2$.

Proposition 3.1 *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + Kf(z)\| \le \left\|Kf\left(\frac{x+y}{K} + z\right)\right\| \tag{3.1}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is additive.

Proof Letting $x = y = z = 0$ in (3.1), we get

$$\|(K+2)f(0)\| \le \|Kf(0)\|.$$

So, $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (3.1), we get

$$\|f(x) + f(-x)\| \le \|Kf(0)\| = 0$$

for all $x \in X$. So, $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = \frac{-x-y}{K}$ in (3.1), we get

$$\left\|f(x) + f(y) - Kf\left(\frac{x+y}{K}\right)\right\| = \left\|f(x) + f(y) + Kf\left(\frac{-x-y}{K}\right)\right\| \le \|Kf(0)\| = 0$$

for all $x, y \in X$. Thus,

$$Kf\left(\frac{x+y}{K}\right) = f(x) + f(y) \tag{3.2}$$

for all $x, y \in X$. Letting $y = 0$ in (3.2), we get $Kf(\frac{x}{K}) = f(x)$ for all $x \in X$. So,

$$f(x+y) = Kf\left(\frac{x+y}{K}\right) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

Theorem 3.2 *Let K be a positive real number with $K < 2$. Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x) + f(y) + Kf(z)\| \le \left\|Kf\left(\frac{x+y}{K} + z\right)\right\| + \phi(x, y, z), \tag{3.3}$$

where $\phi : X^3 \to [0, \infty)$ satisfies

$$\widetilde{\phi}(x, y, z) := \sum_{j=1}^{\infty} \left(\frac{2}{K}\right)^j \phi\left(\left(\frac{K}{2}\right)^j x, \left(\frac{K}{2}\right)^j y, \left(\frac{K}{2}\right)^j z\right) < \infty \tag{3.4}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \le \frac{1}{2}\widetilde{\phi}\left(-x, -x, \frac{2}{K}x\right) + \widetilde{\phi}(x, -x, 0) \tag{3.5}$$

for all $x \in X$.

Proof It follows from (3.4) that $\phi(0,0,0) = 0$. Letting $x = y = z = 0$ in (3.3), we get $\|(K+2)f(0)\| \leq \|Kf(0)\| + \phi(0,0,0) = \|Kf(0)\|$. So, $f(0) = 0$.

Letting $y = -x$, $z = 0$ in (3.3), we get

$$\|f(x) + f(-x)\| \leq \phi(x,-x,0) \tag{3.6}$$

for all $x \in X$. Letting $x = y = Kx$, $z = -2x$ in (3.3), we obtain

$$\|f(Kx) + f(Kx) + f(-2x)\| \leq \phi(Kx, Kx, -2x)$$

for all $x \in X$. So,

$$\left\|\frac{2}{K}f\left(\frac{K}{2}x\right) + f(-x)\right\| \leq \frac{1}{K}\phi\left(\frac{Kx}{2}, \frac{Kx}{2}, -x\right) \tag{3.7}$$

for all $x \in X$. It follows from (3.6) and (3.7) that

$$\begin{aligned}
&\left\|\left(\frac{2}{K}\right)^l f\left(\left(\frac{K}{2}\right)^l x\right) - \left(\frac{2}{K}\right)^m f\left(\left(\frac{K}{2}\right)^m x\right)\right\| \\
&\quad\leq \sum_{j=l}^{m-1}\left\|\left(\frac{2}{K}\right)^j f\left(\left(\frac{K}{2}\right)^j x\right) - \left(\frac{2}{K}\right)^{j+1} f\left(\left(\frac{K}{2}\right)^{j+1} x\right)\right\| \\
&\quad\leq \sum_{j=l}^{m-1}\left[\left\|\left(\frac{2}{K}\right)^j f\left[\left(\frac{K}{2}\right)^j x\right] + \left(\frac{2}{K}\right)^{j+1} f\left[\left(\frac{K}{2}\right)^{j+1}(-x)\right]\right\|\right. \\
&\qquad\left. + \left\|\left(\frac{2}{K}\right)^{j+1} f\left[\left(\frac{K}{2}\right)^{j+1}(-x)\right] + \left(\frac{2}{K}\right)^{j+1} f\left[\left(\frac{K}{2}\right)^{j+1} x\right]\right\|\right] \\
&\quad\leq \sum_{j=l}^{m-1}\left[\frac{1}{K}\left(\frac{2}{K}\right)^j \phi\left(-\left(\frac{K}{2}\right)^{j+1}x, -\left(\frac{K}{2}\right)^{j+1}x, \left(\frac{K}{2}\right)^j x\right)\right. \\
&\qquad\left. + \left(\frac{2}{K}\right)^{j+1}\phi\left(\left(\frac{K}{2}\right)^{j+1}x, \left(\frac{K}{2}\right)^{j+1}(-x), 0\right)\right]
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{2}{K})^n f((\frac{K}{2})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{2}{K})^n f((\frac{K}{2})^n x)\}$ converges. So, we may define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}((\frac{2}{K})^n f((\frac{K}{2})^n x))$ for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.5).

Next, we claim that $A(x)$ is an additive mapping. It follows from (3.6) that

$$\begin{aligned}
\|A(x) + A(-x)\| &= \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \left\|f\left(\left(\frac{K}{2}\right)^n x\right) + f\left(-\left(\frac{K}{2}\right)^n x\right)\right\| \\
&\leq \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \phi\left(\left(\frac{K}{2}\right)^n x, -\left(\frac{K}{2}\right)^n x, 0\right) \\
&= \lim_{n\to\infty}\frac{K}{2}\left(\frac{2}{K}\right)^{n+1}\phi\left(\left(\frac{K}{2}\right)^{n+1}\left(\frac{2}{K}x\right), \left(\frac{K}{2}\right)^{n+1}\left(-\frac{2}{K}x\right), 0\right) \\
&= 0
\end{aligned}$$

and so $A(-x) = -A(x)$ for all $x \in X$.

It follows from (3.3) that

$$\|f(Kx) - Kf(x)\| = \|f(Kx) + f(0) + Kf(-x)\| \leq \phi(Kx, 0, -x)$$

for all $x \in X$. Hence,

$$\begin{aligned}
&\|A(x) + A(y) - A(x+y)\| \\
&\quad = \|A(x) + A(y) + A(-x-y)\| \\
&\quad = \left\|A(x) + A(y) + KA\left(\frac{-x-y}{K}\right) - KA\left(\frac{-x-y}{K}\right) + A(-x-y)\right\| \\
&\quad \leq \left\|A(x) + A(y) + KA\left(\frac{-x-y}{K}\right)\right\| + \left\|A(-x-y) - KA\left(\frac{-x-y}{K}\right)\right\| \\
&\quad = \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \left[\left\|f\left(\left(\frac{K}{2}\right)^n x\right) + f\left(\left(\frac{K}{2}\right)^n y\right) + Kf\left(\left(\frac{K}{2}\right)^n \frac{-x-y}{K}\right)\right\|\right. \\
&\qquad \left. + \left\|f\left(\left(\frac{K}{2}\right)^n (-x-y)\right) - Kf\left(\left(\frac{K}{2}\right)^n \frac{-x-y}{K}\right)\right\|\right] \\
&\quad \leq \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \phi\left(\left(\frac{K}{2}\right)^n x, \left(\frac{K}{2}\right)^n y, \left(\frac{K}{2}\right)^n \frac{-x-y}{K}\right) \\
&\qquad + \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \phi\left(\left(\frac{K}{2}\right)^n (-x-y), 0, \left(\frac{K}{2}\right)^n \frac{x+y}{K}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So, the mapping $A : X \to Y$ is an additive mapping.

Now, we show the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (3.5). Then we get

$$\begin{aligned}
&\|A(x) - T(x)\| \\
&\quad = \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \left\|A\left(\left(\frac{K}{2}\right)^n x\right) - T\left(\left(\frac{K}{2}\right)^n x\right)\right\| \\
&\quad \leq \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \left[\left\|A\left(\left(\frac{K}{2}\right)^n x\right) - f\left(\left(\frac{K}{2}\right)^n x\right)\right\|\right. \\
&\qquad \left. + \left\|T\left(\left(\frac{K}{2}\right)^n x\right) - f\left(\left(\frac{K}{2}\right)^n x\right)\right\|\right] \\
&\quad \leq \lim_{n\to\infty}\left(\frac{2}{K}\right)^n \widetilde{\phi}\left(\left(\frac{K}{2}\right)^n (-x), \left(\frac{K}{2}\right)^n (-x), \frac{2}{K}\left(\frac{K}{2}\right)^n x\right) \\
&\qquad + 2\lim_{n\to\infty}\left(\frac{2}{K}\right)^n \widetilde{\phi}\left(\left(\frac{K}{2}\right)^n x, \left(\frac{K}{2}\right)^n x, 0\right) = 0
\end{aligned}$$

for all $x \in X$. Thus, we may conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. So, the mapping $A : X \to Y$ is a unique additive mapping satisfying (3.5). □

Corollary 3.3 *Let p, θ and K be positive real numbers with $p > 1$ and $K < 2$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(x) + f(y) + Kf(z)\| \leq \left\|Kf\left(\frac{x+y}{K} + z\right)\right\| + \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{3.8}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{\frac{1}{K}(\frac{2}{K})^p + \frac{6}{K}}{(\frac{2}{K})^p - \frac{2}{K}} \theta \|x\|^p$$

for all $x \in X$.

Theorem 3.4 *Let K be a real number with $K > 2$. Assume that a mapping $f : X \to Y$ satisfies inequality* (3.3), *where $\phi : X^3 \to [0, \infty)$ satisfies*

$$\widetilde{\phi}(x, y, z) := \sum_{j=0}^{\infty} \left(\frac{K}{2}\right)^j \phi\left(\left(\frac{2}{K}\right)^j x, \left(\frac{2}{K}\right)^j y, \left(\frac{2}{K}\right)^j z\right) < \infty \tag{3.9}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \le \frac{1}{2}\widetilde{\phi}\left(x, x, -\frac{2}{K}x\right) + \frac{K}{2}\widetilde{\phi}\left(\frac{2}{K}x, -\frac{2}{K}x, 0\right) \tag{3.10}$$

for all $x \in X$.

Proof It follows from (3.9) that $\phi(0, 0, 0) = 0$. Letting $x = y = z = 0$ in (3.3), we get $\|(K + 2)f(0)\| \le \|Kf(0)\| + \phi(0, 0, 0) = \|Kf(0)\|$. So, $f(0) = 0$.

Replacing x by $\frac{2}{K}x$ in (3.7), we get

$$\left\|f(x) + \frac{K}{2}f\left(-\frac{2}{K}x\right)\right\| \le \frac{1}{2}\phi\left(x, x, -\frac{2}{K}x\right) \tag{3.11}$$

for all $x \in X$. It follows from (3.6) and (3.11) that

$$\begin{aligned}
&\left\|\left(\frac{K}{2}\right)^l f\left(\left(\frac{2}{K}\right)^l x\right) - \left(\frac{K}{2}\right)^m f\left(\left(\frac{2}{K}\right)^m x\right)\right\| \\
&\quad \le \sum_{j=l}^{m-1} \left\|\left(\frac{K}{2}\right)^j f\left(\left(\frac{2}{K}\right)^j x\right) - \left(\frac{K}{2}\right)^{j+1} f\left(\left(\frac{2}{K}\right)^{j+1} x\right)\right\| \\
&\quad \le \sum_{j=l}^{m-1} \left[\left\|\left(\frac{K}{2}\right)^j f\left[\left(\frac{2}{K}\right)^j x\right] + \left(\frac{K}{2}\right)^{j+1} f\left[\left(\frac{2}{K}\right)^{j+1} (-x)\right]\right\|\right. \\
&\qquad \left. + \left\|\left(\frac{K}{2}\right)^{j+1} f\left[\left(\frac{2}{K}\right)^{j+1} (-x)\right] + \left(\frac{K}{2}\right)^{j+1} f\left[\left(\frac{2}{K}\right)^{j+1} x\right]\right\|\right] \\
&\quad \le \sum_{j=l}^{m-1} \left[\frac{1}{2}\left(\frac{K}{2}\right)^j \phi\left(\left(\frac{2}{K}\right)^j x, \left(\frac{2}{K}\right)^j x, -\left(\frac{2}{K}\right)^{j+1} x\right)\right. \\
&\qquad \left. + \left(\frac{K}{2}\right)^{j+1} \phi\left(\left(\frac{2}{K}\right)^{j+1} x, \left(\frac{2}{K}\right)^{j+1} (-x), 0\right)\right]
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{K}{2})^n f((\frac{2}{K})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{K}{2})^n f((\frac{2}{K})^n x)\}$ converges. So, we may define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}((\frac{K}{2})^n f((\frac{2}{K})^n x))$ for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.10).

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5 *Let p, θ and K be positive real numbers with $p > 1$ and $K > 2$. Let $f : X \to Y$ be a mapping satisfying* (3.8). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{\frac{1}{K}(\frac{2}{K})^p + \frac{6}{K}}{\frac{2}{K} - (\frac{2}{K})^p}\theta\|x\|^p$$

for all $x \in X$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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