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Comment on 'Approximate $*$-derivations and approximate quadratic $*$-derivations on C^*-algebras' [Jang, Park, J. Inequal. Appl. **2011** (2011), Article ID 55]

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Abstract

In (J. Inequal. Appl. 2011:Article ID 55, Section 4, 2011), Jang and Park proved the Hyers-Ulam stability of quadratic $*$-derivations on Banach $*$-algebras. One can easily show that all the quadratic $*$-derivations δ in Section 4 must be trivial. So the results are trivial. In this paper, we correct the statements and prove the corrected results.

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1 Introduction and preliminaries

Suppose that $\mathcal{A}$ is a complex Banach $*$-algebra. A $\mathbb{C}$-linear mapping $\delta : D(\delta) \to \mathcal{A}$ is said to be a *derivation* on $\mathcal{A}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$, where $D(\delta)$ is a domain of δ and $D(\delta)$ is dense in $\mathcal{A}$. If δ satisfies the additional condition $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{A}$, then δ is called a $*$-*derivation* on $\mathcal{A}$. It is well known that if $\mathcal{A}$ is a C^*-algebra and $D(\delta)$ is $\mathcal{A}$, then the derivation δ is bounded.

A C^*-dynamical system is a triple $(\mathcal{A}, G, \alpha)$ consisting of a C^*-algebra $\mathcal{A}$, a locally compact group G, and a pointwise norm continuous homomorphism α of G into the group $\mathrm{Aut}(\mathcal{A})$ of $*$-automorphisms of $\mathcal{A}$. Every bounded $*$-derivation δ arises as an infinitesimal generator of a dynamical system for $\mathbb{R}$. In fact, if δ is a bounded $*$-derivation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, then there exists an element h in the enveloping von Neumann algebra $\mathcal{A}''$ such that

$$\delta(x) = ad_{ih}(x)$$

for all $x \in \mathcal{A}$. The theory of bounded derivations of C^*-algebras is important in the quantum mechanics (see [2–4]).

A functional equation is called *stable* if any function satisfying the functional equation 'approximately' is near to a true solution of the functional equation.

In 1940, Ulam [5] proposed the following question concerning the stability of group homomorphisms: *Under what condition does there exist an additive mapping near an approximately additive mapping?* Hyers [6] answered the problem of Ulam for the case where G_1 and G_2 are Banach spaces. A generalized version of the theorem of Hyers for an approximately linear mapping was given by Rassias [7]. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors (see [8–20]).

Jang and Park [1, Section 4] proved the Hyers-Ulam stability of quadratic $*$-derivations on Banach $*$-algebras.

Theorem 1.1 ([1, Theorem 4.2]) *Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\tilde{\varphi}(a,b,c,d) := \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi\bigl(2^k a, 2^k b, 2^k c, 2^k d\bigr) < \infty,$$

$$\begin{aligned} &\bigl\| f(\lambda a + \lambda b + cd) + f(\lambda a - \lambda b + cd) - 2\lambda^2 f(a) - 2\lambda^2 f(b) - 2f(c)d^2 - 2c^2 f(d) \bigr\| \\ &\quad \le \varphi(a,b,c,d), \end{aligned} \tag{1.1}$$

$$\bigl\| f\bigl(a^*\bigr) - f(a)^* \bigr\| \le \varphi(a,a,a,a)$$

for all $a,b,c,d \in \mathcal{A}$ and all $\lambda \in \mathbb{T} := \{\mu \in \mathbb{C} : |\mu| = 1\}$. Also, if for each fixed $a \in \mathcal{A}$ the mapping $t \to f(ta)$ from $\mathbb{R}$ to $\mathcal{A}$ is continuous, then there exists a unique quadratic $*$-derivation δ on $\mathcal{A}$ satisfying

$$\bigl\| f(a) - \delta(a) \bigr\| \le \frac{1}{4} \tilde{\varphi}(a,a,0,0)$$

for all $a \in \mathcal{A}$.

Letting $\lambda = 1$, $b = 0$ and $d = I$ (identity) in (1.1) of Theorem 1.1, we get

$$\bigl\| f(a+c) + f(a+c) - 2f(a) - 2f(c) - 2c^2 f(I) \bigr\| \le \varphi(a,0,c,I)$$

and

$$\begin{aligned} &\frac{1}{4^n} \bigl\| f\bigl(2^n(a+c)\bigr) + f\bigl(2^n(a+c)\bigr) - 2f\bigl(2^n a\bigr) - 2f\bigl(2^n c\bigr) - 2 \cdot 4^n c^2 f\bigl(2^n I\bigr) \bigr\| \\ &\quad \le \frac{1}{4^n} \varphi\bigl(2^n a, 0, 2^n c, 2^n I\bigr) \end{aligned}$$

for all $a, c \in \mathcal{A}$. Thus $2\delta(a+c) = 2\delta(a) + 2\delta(c) + 2c^2 d'$ for some $d' \in \mathcal{A}$. Since δ is quadratic, $2\delta(a) + 2\delta(-c) + 2(-c)^2 d' = 2\delta(a) + 2\delta(c) + 2c^2 d'$ and so $2\delta(a+c) = 2\delta(a-c)$. Letting $c = a$ in the last equality, we get $2\delta(2a) = 2\delta(0) = 0$. So δ must be zero. Thus the results are trivial.

In this paper, we correct the wrong statements in [1] and prove the corrected results.

2 Hyers-Ulam stability of quadratic $*$-derivations on Banach $*$-algebras

In this section, we correct the statements of [1, Section 4] and prove the Hyers-Ulam stability of the corrected results.

Definition 2.1 Let $\mathcal{A}$ be a $*$-normed algebra. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is a quadratic $*$-derivation on $\mathcal{A}$ if δ satisfies the following properties:

(1) δ is a quadratic mapping,

(2) δ is quadratic homogeneous, that is, $\delta(\lambda a) = \lambda^2\delta(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$,
(3) $\delta(ab) = \delta(a)b^2 + a^2\delta(b)$ for all $a, b \in \mathcal{A}$,
(4) $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{A}$.

Example 2.2 Let $\mathcal{A}$ be a commutative $*$-normed algebra. For a given self-adjoint element $x \in \mathcal{A}$, let $\delta : \mathcal{A} \to \mathcal{A}$ be given by

$$\delta(a) = i(xa^2 - a^2x)$$

for all $x \in \mathcal{A}$. Then it is easy to show that $\delta : \mathcal{A} \to \mathcal{A}$ is a quadratic $*$-derivation on $\mathcal{A}$.

Theorem 2.3 *Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^2 \to [0,\infty)$ such that*

$$\tilde{\varphi}(a,b) := \sum_{k=0}^{\infty} \frac{1}{4^k}\varphi(2^k a, 2^k b) < \infty,$$

$$\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \le \varphi(a,b), \tag{2.1}$$

$$\|f(cd) - f(c)d^2 - c^2 f(d)\| \le \varphi(c,d), \tag{2.2}$$

$$\|f(a^*) - f(a)^*\| \le \varphi(a,a) \tag{2.3}$$

for all $a, b, c, d \in \mathcal{A}$ and all $\lambda \in \mathbb{T}$. Also, if for each fixed $a \in \mathcal{A}$ the mapping $t \to f(ta)$ from $\mathbb{R}$ to $\mathcal{A}$ is continuous, then there exists a unique quadratic $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \le \frac{1}{4}\tilde{\varphi}(a,a) \tag{2.4}$$

for all $a \in \mathcal{A}$.

Proof Putting $a = b$ and $\lambda = 1$ in (2.1), we have

$$\|f(2a) - 4f(a)\| \le \varphi(a,a)$$

for all $a \in \mathcal{A}$. One can use induction to show that

$$\left\|\frac{f(2^n a)}{4^n} - \frac{f(2^m a)}{4^m}\right\| \le \frac{1}{4}\sum_{k=m}^{n-1} \frac{\varphi(2^k a, 2^k a)}{4^k} \tag{2.5}$$

for all $n > m \ge 0$ and all $a \in \mathcal{A}$. It follows from (2.5) that the sequence $\{\frac{f(2^n a)}{4^n}\}$ is Cauchy. Since $\mathcal{A}$ is complete, this sequence is convergent. Define

$$\delta(a) := \lim_{n\to\infty} \frac{f(2^n a)}{4^n}.$$

Since $f(0) = 0$, we have $\delta(0) = 0$. Replacing a and b by $2^n a$ and $2^n b$, respectively, in (2.1), we get

$$\left\|\frac{f(2^n(\lambda a + \lambda b))}{4^n} + \frac{f(2^n(\lambda a - \lambda b))}{4^n} - 2\lambda^2\frac{f(2^n a)}{4^n} - 2\lambda^2\frac{f(2^n b)}{4^n}\right\| \le \frac{\varphi(2^n a, 2^n b)}{4^n}.$$

Taking the limit as $n \to \infty$, we obtain

$$\delta(\lambda a + \lambda b) + \delta(\lambda a - \lambda b) = 2\lambda^2\delta(a) + 2\lambda^2\delta(b) \tag{2.6}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{T}$. Putting $\lambda = 1$ in (2.6), we obtain that δ is a quadratic mapping. It is well known that the quadratic mapping δ satisfying (2.4) is unique (see [4] or [20]).

Setting $b := a$ in (2.6), we get

$$\delta(2\lambda a) = 4\lambda^2\delta(a)$$

for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{T}$. Hence

$$\delta(\lambda a) = \lambda^2\delta(a)$$

for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{T}$. Under the assumption that $f(ta)$ is continuous in $t \in \mathbb{R}$ for each fixed $a \in \mathcal{A}$, by the same reasoning as in the proof of [9], we obtain that $\delta(\lambda a) = \lambda^2\delta(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Hence

$$\delta(\lambda a) = \delta\left(\frac{\lambda}{|\lambda|}|\lambda|a\right) = \frac{\lambda^2}{|\lambda|^2}\delta\big(|\lambda|a\big) = \frac{\lambda^2}{|\lambda|^2}|\lambda|^2\delta(a) = \lambda^2\delta(a)$$

for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$ ($\lambda \neq 0$). This means that δ is quadratic homogeneous.

Replacing c and d by $2^n c$ and $2^n d$, respectively, in (2.2), we get

$$\begin{aligned}
&\left\| \frac{f(2^n c \cdot 2^n d)}{4^{2n}} - \frac{2^{2n}c^2 f(2^n d)}{4^{2n}} - \frac{f(2^n c)2^{2n}d^2}{4^{2n}} \right\| \\
&\quad = \left\| \frac{f(2^{2n}cd)}{4^{2n}} - \frac{2^{2n}c^2}{2^{2n}}\frac{f(2^n d)}{4^n} - \frac{f(2^n c)}{4^n}\frac{2^{2n}d^2}{2^{2n}} \right\| \\
&\quad \le \frac{\varphi(2^n c, 2^n d)}{4^{2n}} \le \frac{\varphi(2^n c, 2^n d)}{4^n}
\end{aligned}$$

for all $c, d \in \mathcal{A}$.

Thus we have

$$\left\| \delta(cd) - c^2\delta(d) - \delta(c)d^2 \right\| \le \lim_{n\to\infty} \frac{\varphi(2^n c, 2^n d)}{4^n} = 0.$$

Replacing a and a^* by $2^n a$ and $2^n a^*$, respectively, in (2.3), we get

$$\left\| \frac{1}{4^n} f\big(2^n a^*\big) - \frac{1}{2^n} f\big(4^n a\big)^* \right\| \le \frac{1}{4^n}\varphi\big(2^n a, 2^n a\big).$$

Passing to the limit as $n \to \infty$, we get the $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{A}$. So δ is a quadratic $*$-derivation on $\mathcal{A}$, as desired. □

Corollary 2.4 *Let ε, p be positive real numbers with $p < 2$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\left\| f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b) \right\| \le \varepsilon\big(\|a\|^p + \|b\|^p\big), \tag{2.7}$$

$$\left\|f(cd) - c^2 f(d) - f(c)d^2\right\| \le \varepsilon\left(\|c\|^p + \|d\|^p\right), \tag{2.8}$$

$$\left\|f\left(a^*\right) - f(a)^*\right\| \le 2\varepsilon\|a\|^p \tag{2.9}$$

for all $a, b, c, d \in \mathcal{A}$ *and all* $\lambda \in \mathbb{T}$. *Also, if for each fixed* $a \in \mathcal{A}$ *the mapping* $t \to f(ta)$ *is continuous, then there exists a unique quadratic* $*$*-derivation* δ *on* $\mathcal{A}$ *satisfying*

$$\left\|f(a) - \delta(a)\right\| \le \frac{2\varepsilon}{4 - 2^p}\|a\|^p$$

for all $a \in \mathcal{A}$.

Proof Putting $\varphi(a, b) = \varepsilon(\|a\|^p + \|b\|^p)$ in Theorem 2.3, we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 2.5 *Suppose that* $f : \mathcal{A} \to \mathcal{A}$ *is a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^2 \to [0, \infty)$ *satisfying (2.1), (2.2), (2.3) and*

$$\sum_{k=1}^{\infty} 4^{2k}\varphi\left(\frac{a}{2^k}, \frac{b}{2^k}\right) < \infty$$

for all $a, b \in \mathcal{A}$. *Also, if for each fixed* $a \in \mathcal{A}$ *the mapping* $t \to f(ta)$ *from* $\mathbb{R}$ *to* $\mathcal{A}$ *is continuous, then there exists a unique quadratic* $*$*-derivation* δ *on* $\mathcal{A}$ *satisfying*

$$\left\|f(a) - \delta(a)\right\| \le \frac{1}{4}\tilde{\varphi}(a, a)$$

for all $a \in \mathcal{A}$, *where*

$$\tilde{\varphi}(a, b) := \sum_{k=1}^{\infty} 4^k \varphi\left(\frac{a}{2^k}, \frac{b}{2^k}\right).$$

Corollary 2.6 *Let* ε, p *be positive real numbers with* $p > 4$. *Suppose that* $f : \mathcal{A} \to \mathcal{A}$ *is a mapping satisfying (2.7), (2.8) and (2.9). Also, if for each fixed* $a \in \mathcal{A}$ *the mapping* $t \to f(ta)$ *is continuous, then there exists a unique quadratic* $*$*-derivation* δ *on* $\mathcal{A}$ *satisfying*

$$\left\|f(a) - \delta(a)\right\| \le \frac{2\varepsilon}{2^p - 4}\|a\|^p$$

for all $a \in \mathcal{A}$.

Proof Putting $\varphi(a, b) = \varepsilon(\|a\|^p + \|b\|^p)$ in Theorem 2.5, we get the desired result. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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References

1. Jang, S, Park, C: Approximate $*$-derivations and approximate quadratic $*$-derivations on C^*-algebras. J. Inequal. Appl. **2011**, Article ID 55 (2011)
2. Bratteli, O: Derivation, Dissipation and Group Actions on C^*-Algebras. Lecture Notes in Mathematics, vol. 1229. Springer, Berlin (1986)
3. Bratteli, O, Goodman, FM, Jørgensen, PET: Unbounded derivations tangential to compact groups of automorphisms II. J. Funct. Anal. **61**, 247-289 (1985)
4. Lee, S, Jang, S: Unbounded derivations on compact actions of C^*-algebras. Commun. Korean Math. Soc. **5**, 79-86 (1990)
5. Ulam, SM: Problems in Modern Mathematics. Science Edn., Chapter VI. Wiley, New York (1940)
6. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
7. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
8. Aczl, J, Dhombres, J: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
9. Czerwik, S: On the stability of the quadratic mapping in normed spaces. Abh. Math. Semin. Univ. Hamb. **62**, 59-64 (1992)
10. Gajda, Z: On stability of additive mappings. Int. J. Math. Math. Sci. **14**, 431-434 (1991)
11. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Basel (1998)
12. Jun, K, Kim, H: Approximate derivations mapping into the radicals of Banach algebras. Taiwan. J. Math. **11**, 277-288 (2007)
13. Kannappan, P: Quadratic functional equation and inner product spaces. Results Math. **27**, 368-372 (1995)
14. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
15. Jung, S: Hyers-Ulam-Rassias Stability of Functional Equations in Non-Linear Analysis. Springer Optimization and Its Applications, vol. 48. Springer, New York (2011)
16. Skof, F: Propriet locali e approssimazione di operatori. Rend. Semin. Mat. Fis. Milano **53**, 113-129 (1983)
17. Lee, J, An, J, Park, C: On the stability of quadratic functional equations. Abstr. Appl. Anal. **2008**, Article ID 628178 (2008)
18. Gharetapeh, SK, Eshaghi Gordji, M, Ghaemi, MB, Rashidi, E: Ternary Jordan homomorphisms in C^*-ternary algebras. J. Nonlinear Sci. Appl. **4**, 1-10 (2011)
19. Park, C, Boo, D: Isomorphisms and generalized derivations in proper CQ^*-algebras. J. Nonlinear Sci. Appl. **4**, 19-36 (2011)
20. Javadian, A, Eshaghi Gordji, M, Savadkouhi, MB: Approximately partial ternary quadratic derivations on Banach ternary algebras. J. Nonlinear Sci. Appl. **4**, 60-69 (2011)

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