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Hyers-Ulam-Rassias stability of the additive-quadratic mappings in non-Archimedean Banach spaces

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Abstract
Using the fixed point and direct methods, we prove the generalized Hyers-Ulam stability of the following additive-quadratic functional equation in non-Archimedean normed spaces

$$r\left[f\left(\frac{x+y+z}{s}\right)+f\left(\frac{x-y+z}{s}\right)+f\left(\frac{x+y-z}{s}\right)+f\left(\frac{-x+y+z}{s}\right)\right]$$
$$=\gamma f(x)+\gamma f(y)+\gamma f(z),$$

where r, s, γ are positive real numbers.
MSC: Primary 39B55; 46S10

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1 Introduction and preliminaries

A classical question in the theory of functional equations is the following: 'When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?' If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [44] in 1940. In the next year, Hyers [23] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [39] proved a generalization of Hyers' theorem for additive mappings. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta [21] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [43] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [13] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The reader is referred to [1–42] and references therein for detailed information on stability of functional equations.

In 1897, Hensel [22] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [16, 25–27, 33]).

Definition 1.1 By a *non-Archimedean field,* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|:\mathbb{K}\to[0,\infty)$ such that for all $r,s\in\mathbb{K}$, the following conditions hold:
(1) $|r|=0$ if and only if $r=0$;
(2) $|rs|=|r||s|$;
(3) $|r+s|\le\max\{|r|,|s|\}$.

Definition 1.2 Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:X\to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(1) $\|x\|=0$ if and only if $x=0$;
(2) $\|rx\|=|r|\|x\|$ ($r\in\mathbb{K}$, $x\in X$);
(3) The strong triangle inequality (ultrametric); namely,

$$\|x+y\|\le\max\bigl\{\|x\|,\|y\|\bigr\},\quad x,y\in X.$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean space.

Due to the fact that

$$\|x_n-x_m\|\le\max\bigl\{\|x_{j+1}-x_j\|:m\le j\le n-1\bigr\}\quad(n>m).$$

Definition 1.3 A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Definition 1.4 Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.5 ([13, 17]) *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either*

$$d\bigl(J^nx,J^{n+1}x\bigr)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^nx,J^{n+1}x)<\infty$, $\forall n\ge n_0$;
(2) the sequence $\{J^nx\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y=\{y\in X\mid d(J^{n_0}x,y)<\infty\}$;
(4) $d(y,y^*)\le\frac{1}{1-\alpha}d(y,Jy)$ for all $y\in Y$.

In 1996, G. Isac and Th. M. Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed-point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [14, 15, 35, 36, 40]).

This paper is organized as follows: In Section 2, using the fixed-point method, we prove the Hyers-Ulam stability of the following additive-quadratic functional equation:

$$rf\left(\frac{x+y+z}{s}\right)+rf\left(\frac{x-y+z}{s}\right)+rf\left(\frac{x+y-z}{s}\right)+rf\left(\frac{-x+y+z}{s}\right)$$
$$=\gamma f(x)+\gamma f(y)+\gamma f(z), \tag{1.1}$$

where $x, y, z \in X$, in non-Archimedean normed space. In Section 3, using direct methods, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.1) in non-Archimedean normed spaces.

It is easy to see that a mapping f with $f(0) = 0$ is a solution of equation (1.1) if and only if f is of the form $f(x) = A(x) + Q(x)$ for all $x \in X$.

2 Stability of functional equation (1.1): a fixed point method

In this section, we deal with the stability problem for the additive-quadratic functional equation (1.1). In the rest of the present article, let $|2| \neq 1$.

Theorem 2.1 *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(2x,2y,2z) \leq |2|\alpha\varphi(x,y,z) \tag{2.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\left\| rf\left(\frac{x+y+z}{s}\right)+rf\left(\frac{x-y+z}{s}\right)+rf\left(\frac{x+y-z}{s}\right)\right.$$
$$\left.+rf\left(\frac{-x+y+z}{s}\right)-\gamma f(x)-\gamma f(y)-\gamma f(z)\right\|_Y \leq \varphi(x,y,z) \tag{2.2}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\|_Y \leq \frac{\max\{\varphi(2x,0,0),\varphi(x,x,0)\}}{|2\gamma|(1-\alpha)} \tag{2.3}$$

for all $x \in X$.

Proof Putting $y = z = 0$ in (2.2) and replacing x by $2x$, we get

$$\left\| rf\left(\frac{2x}{s}\right)-\frac{\gamma}{2}f(2x)\right\|_Y \leq \frac{1}{|2|}\varphi(2x,0,0) \tag{2.4}$$

for all $x \in X$. Putting $y = x$ and $z = 0$ in (2.2), we have

$$\left\| rf\left(\frac{2x}{s}\right)-\gamma f(x)\right\|_Y \leq \frac{1}{|2|}\varphi(x,x,0) \tag{2.5}$$

for all $x \in X$. By (2.4) and (2.5), we get

$$\begin{aligned}
\left\| \frac{f(2x)}{2} - f(x) \right\|_Y &= \frac{1}{|\gamma|} \left\| \frac{\gamma}{2} f(2x) \pm rf\left(\frac{2x}{s}\right) - \gamma f(x) \right\|_Y \\
&\leq \frac{1}{|\gamma|} \max\left\{ \left\| rf\left(\frac{2x}{s}\right) - \frac{\gamma}{2} f(2x) \right\|_Y, \left\| rf\left(\frac{2x}{s}\right) - \gamma f(x) \right\|_Y \right\} \\
&\leq \frac{1}{|2\gamma|} \max\left\{ \varphi(2x,0,0), \varphi(x,x,0) \right\}. \qquad (2.6)
\end{aligned}$$

Consider the set $S := \{h : X \to Y\}$ and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{ \mu \in (0,+\infty) : \left\| g(x) - h(x) \right\|_Y \leq \mu \max\left\{ \varphi(2x,0,0), \varphi(x,x,0) \right\}, \forall x \in X \right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [30]). Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{1}{2} g(2x)$ for all $x \in X$. Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\left\| g(x) - h(x) \right\|_Y \leq \epsilon \max\left\{ \varphi(2x,0,0), \varphi(x,x,0) \right\}$$

for all $x \in X$. Hence,

$$\begin{aligned}
\left\| Jg(x) - Jh(x) \right\|_Y &= \left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\|_Y \\
&= \frac{\|g(2x) - h(2x)\|_Y}{|2|} \\
&\leq \frac{\epsilon}{|2|} \max\left\{ \varphi(4x,0,0), \varphi(2x,2x,0) \right\} \\
&\leq \alpha \cdot \epsilon \max\left\{ \varphi(2x,0,0), \varphi(x,x,0) \right\}
\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq \alpha\varepsilon$. This means that $d(Jg,Jh) \leq \alpha d(g,h)$ for all $g,h \in S$.

It follows from (2.6) that $d(f,Jf) \leq \frac{1}{|2\gamma|}$. By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$2A(x) = A(2x) \qquad (2.7)$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $M = \{g \in S : d(h,g) < \infty\}$. This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0,\infty)$ satisfying $\|f(x) - A(x)\|_Y \leq \mu \max\{\varphi(2x,0,0), \varphi(x,x,0)\}$ for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x) \quad \text{for all } x \in X;$$

(3) $d(f,A) \leq \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality $d(f,A) \leq \frac{1}{|2\gamma|(1-\alpha)}$. This implies that the inequalities (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\left\| rA\left(\frac{x+y+z}{s}\right) + rA\left(\frac{x-y+z}{s}\right) + rA\left(\frac{x+y-z}{s}\right)\right. \\
&\quad \left. + rA\left(\frac{-x+y+z}{s}\right) - \gamma A(x) - \gamma A(y) - \gamma A(z)\right\|_Y \\
&= \lim_{n\to\infty} \frac{1}{|2|^n} \left\| rf\left(\frac{2^n(x+y+z)}{s}\right) + rf\left(\frac{2^n(x-y+z)}{s}\right) + rf\left(\frac{2^n(x+y-z)}{s}\right)\right. \\
&\quad \left. + rf\left(\frac{2^n(-x+y+z)}{s}\right) - \gamma f(2^n x) - \gamma f(2^n y) - \gamma f(2^n z)\right\|_Y \\
&\le \lim_{n\to\infty} \frac{1}{|2|^n} \varphi(2^n x, 2^n y, 2^n z) \\
&\le \lim_{n\to\infty} \frac{1}{|2|^n} \cdot |2|^n \alpha^n \varphi(x,y,z) \\
&= 0
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
&rA\left(\frac{x+y+z}{s}\right) + rA\left(\frac{x-y+z}{s}\right) + rA\left(\frac{x+y-z}{s}\right) \\
&\quad + rA\left(\frac{-x+y+z}{s}\right) - \gamma A(x) - \gamma A(y) - \gamma A(z) = 0
\end{aligned}$$

for all $x, y, z \in X$. Hence, $A : X \to Y$ satisfying (1.1). This completes the proof. □

Corollary 2.2 *Let θ be a positive real number and q is a real number with $q > 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}
&\left\| rf\left(\frac{x+y+z}{s}\right) + rf\left(\frac{x-y+z}{s}\right) + rf\left(\frac{x+y-z}{s}\right)\right. \\
&\quad \left. + rf\left(\frac{-x+y+z}{s}\right) - \gamma f(x) - \gamma f(y) - \gamma f(z)\right\|_Y \le \theta\left(\|x\|^q + \|y\|^q + \|z\|^q\right) \qquad (2.8)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f(x) - A(x)\right\|_Y \le \frac{2|2|\theta \|x\|^q}{|2\gamma|(|2| - |2|^q)}$$

for all $x \in X$.

Proof The proof follows from Theorem 2.1 by taking $\varphi(x,y,z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{q-1}$ and we get the desired result. □

Theorem 2.3 *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{\alpha}{|2|}\varphi(x,y,z) \qquad (2.9)$$

for all $x,y,z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{\alpha \max\{\varphi(2x,0,0), \varphi(x,x,0)\}}{|2\gamma|(1-\alpha)}$$

for all $x \in X$.

Proof Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.
Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.
Replacing x by $\frac{x}{2}$ in (2.6) and using (2.9), we have

$$\begin{aligned}\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y &\le \frac{1}{|\gamma|}\max\left\{\varphi(x,0,0), \varphi\left(\frac{x}{2},\frac{x}{2},0\right)\right\} \\ &\le \frac{\alpha}{|2\gamma|}\max\{\varphi(2x,0,0), \varphi(x,x,0)\}.\end{aligned} \tag{2.10}$$

So $d(f,Jf) \le \frac{\alpha}{|2\gamma|}$.
The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4 *Let θ be a positive real number and q is a real number with $0 < q < 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.8). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_Y \le \frac{2|2|\theta\|x\|^q}{|2\gamma|(|2|^q - |2|)}$$

for all $x \in X$.

Proof The proof follows from Theorem 2.3 by taking $\varphi(x,y,z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x,y,z \in X$. Then we can choose $\alpha = |2|^{1-q}$ and we get the desired result. □

Theorem 2.5 *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(2x,2y,2z) \le |4|\alpha\varphi(x,y,z) \tag{2.11}$$

for all $x,y,z \in X$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ and satisfying (2.2). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{\max\{\varphi(2x,0,0), |2|\varphi(x,x,0)\}}{|4\gamma|(1-\alpha)} \tag{2.12}$$

for all $x \in X$.

Proof Consider the set $S^* = \{g : X \to Y; g(0) = 0\}$ and the generalized metric d^* in S^* defined by

$$d^*(g,h) = \inf\{\mu \in (0,+\infty) : \|g(x) - h(x)\|_Y \le \mu \max\{\varphi(2x,0,0), |2|\varphi(x,x,0)\}, \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S^*, d^*) is complete (see [30]). Now we consider the linear mapping $J : (S^*, d^*) \to (S^*, d^*)$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

Putting $y = x$ and $z = 0$ in (2.2), we have

$$\left\| 2rf\left(\frac{2x}{s}\right) - 2\gamma f(x) \right\|_Y \le \varphi(x,x,0) \tag{2.13}$$

for all $x \in X$.

Substituting $y = z = 0$ and then replacing x by $2x$ in (2.2), we obtain

$$\left\| 4rf\left(\frac{2x}{s}\right) - \gamma f(2x) \right\|_Y \le \varphi(2x,0,0). \tag{2.14}$$

By (2.13) and (2.14), we get

$$\begin{aligned} \left\| \frac{f(2x)}{4} - f(x) \right\|_Y &= \frac{1}{|4\gamma|} \left\| 2\left(2rf\left(\frac{2x}{s}\right) - 2\gamma f(x)\right) - \left(4rf\left(\frac{2x}{s}\right) - \gamma f(2x)\right) \right\|_Y \\ &\le \frac{1}{|4\gamma|} \max\left\{ |2| \left\| 2rf\left(\frac{2x}{s}\right) - 2\gamma f(x) \right\|_Y, \left\| 4rf\left(\frac{2x}{s}\right) - \gamma f(2x) \right\|_Y \right\} \\ &\le \frac{1}{|4\gamma|} \max\{\varphi(2x,0,0), |2|\varphi(x,x,0)\}. \end{aligned} \tag{2.15}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6 *Let θ be a positive real number and q is a real number with $q > 2$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ and satisfying (2.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \le \frac{|4|2|2|\theta\|x\|^q}{|4\gamma|(|4| - |2|^q)}$$

for all $x \in X$.

Proof The proof follows from Theorem 2.5 by taking $\varphi(x,y,z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{q-2}$ and we get the desired result. □

Theorem 2.7 *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{\alpha}{|4|} \varphi(x,y,z) \tag{2.16}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ and satisfying (2.2). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha \max\{\varphi(2x,0,0), |2|\varphi(x,x,0)\}}{|4\gamma|(1-\alpha)} \tag{2.17}$$

for all $x \in X$.

Proof It follows from (2.15) that

$$\begin{aligned}\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\|_Y &\leq \frac{1}{|\gamma|}\max\left\{\varphi(x,0,0), |2|\varphi\left(\frac{x}{2},\frac{x}{2},0\right)\right\} \\ &\leq \frac{\alpha}{|4\gamma|}\max\left\{\varphi(2x,0,0), |2|\varphi(x,x,0)\right\}.\end{aligned}$$

The rest of the proof is similar to the proof of Theorems 2.1 and 2.5. □

Corollary 2.8 *Let θ be a positive real number and q is a real number with $0 < q < 2$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ and satisfying (2.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|4|2|2|\theta\|x\|^q}{|4\gamma|(|2|^q - |4|)}$$

for all $x \in X$.

Proof The proof follows from Theorem 2.7 by taking $\varphi(x,y,z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{2-q}$ and we get the desired result. □

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (1.1). Let $f_e(x) := \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_e is an even mapping satisfying (1.1) and f_o is an odd mapping satisfying (1.1) such that $f(x) = f_e(x) + f_o(x)$.

On the other hand

$$\|D_{f_o}(x,y,z)\| \leq \frac{\max\{D_f(x,y,z), D_f(-x,-y,-z)\}}{|2|} \leq \frac{\max\{\varphi(x,y,z), \varphi(-x,-y,-z)\}}{|2|}$$

and

$$\|D_{f_e}(x,y,z)\| \leq \frac{\max\{D_f(x,y,z), D_f(-x,-y,-z)\}}{|2|} \leq \frac{\max\{\varphi(x,y,z), \varphi(-x,-y,-z)\}}{|2|}$$

for all $x, y, z \in X$, where $D_f(x,y,z)$ is the difference operator of the functional equation (1.1). So we obtain the following theorem.

Theorem 2.9 *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(2x,2y,2z) \leq |4|\alpha\varphi(x,y,z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (2.2). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
&\|f(x) - A(x) - Q(x)\|_Y \\
&\quad \le \max\left\{ \left\|\frac{f(x) - f(-x)}{2} - A(x)\right\|_Y, \left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\|_Y \right\} \\
&\quad \le \max\left\{ \frac{\max\{\max\{\varphi(2x, 0, 0), \varphi(-2x, 0, 0)\}, \max\{\varphi(x, x, 0), \varphi(-x, -x, 0)\}\}}{|4\gamma|(1 - \alpha)}, \right. \\
&\qquad \left. \frac{\max\{\max\{\varphi(2x, 0, 0), \varphi(-2x, 0, 0)\}, |2| \max\{\varphi(x, x, 0), \varphi(-x, -x, 0)\}\}}{|8\gamma|(1 - \alpha)} \right\}
\end{aligned}$$

for all $x \in X$.

Theorem 2.10 *Let X is a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{\alpha \varphi(x, y, z)}{|2|}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (2.2). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
&\|f(x) - A(x) - Q(x)\|_Y \\
&\quad \le \alpha \max\left\{ \frac{\max\{\max\{\varphi(2x, 0, 0), \varphi(-2x, 0, 0)\}, \max\{\varphi(x, x, 0), \varphi(-x, -x, 0)\}\}}{|4\gamma|(1 - \alpha)}, \right. \\
&\qquad \left. \frac{\max\{\max\{\varphi(2x, 0, 0), \varphi(-2x, 0, 0)\}, |2| \max\{\varphi(x, x, 0), \varphi(-x, -x, 0)\}\}}{|8\gamma|(1 - \alpha)} \right\}
\end{aligned}$$

for all $x \in X$.

3 Stability of functional equation (1.1): a direct method

In this section, using direct method, we prove the generalized Hyers-Ulam stability of the additive-quadratic functional equation (1.1) in non-Archimedean space.

Theorem 3.1 *Let G be a vector space and that X is a non-Archimedean Banach space. Assume that $\varphi : G^3 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.1}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\Omega(x) = \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right) \right\}; 0 \le k < n \right\} \tag{3.2}$$

exists and $f: G \to X$ be an odd mapping satisfying

$$\begin{aligned} &\left\| rf\left(\frac{x+y+z}{s}\right) + rf\left(\frac{x-y+z}{s}\right) + rf\left(\frac{x+y-z}{s}\right) \right. \\ &\quad \left. + rf\left(\frac{-x+y+z}{s}\right) - \gamma f(x) - \gamma f(y) - \gamma f(z) \right\|_X \le \varphi(x,y,z). \end{aligned} \tag{3.3}$$

Then the limit

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in G$ and defines an additive mapping $A: G \to X$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|\gamma|}\Omega(x). \tag{3.4}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right), \varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\}; j \le k < n+j\right\} = 0$$

then A is the unique additive mapping satisfying (3.4).

Proof By (2.10), we know

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\|_X \le \frac{1}{|\gamma|}\max\left\{\varphi(x,0,0), \varphi\left(\frac{x}{2},\frac{x}{2},0\right)\right\} \tag{3.5}$$

for all $x \in G$. Replacing x by $\frac{x}{2^n}$ in (3.5), we obtain

$$\begin{aligned} &\left\| 2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\|_X \\ &\quad \le \frac{|2|^n}{|\gamma|}\max\left\{\varphi\left(\frac{x}{2^n},0,0\right), \varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},0\right)\right\}. \end{aligned} \tag{3.6}$$

Thus, it follows from (3.1) and (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}_{n\ge 1}$ is a Cauchy sequence. Since X is complete, it follows that $\{2^n f(\frac{x}{2^n})\}_{n\ge 1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

By induction on n, one can show that

$$\begin{aligned} &\left\| 2^n f\left(\frac{x}{2^n}\right) - f(x)\right\|_X \\ &\quad \le \frac{1}{|\gamma|}\max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right), \varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\}; 0 \le k < n\right\} \end{aligned} \tag{3.7}$$

for all $n \geq 1$ and $x \in G$. By taking $n \to \infty$ in (3.7) and using (3.2), one obtains (3.4). By (3.1) and (3.3), we get

$$\begin{aligned}
&\left\| rA\left(\frac{x+y+z}{s}\right) + rA\left(\frac{x-y+z}{s}\right) + rA\left(\frac{x+y-z}{s}\right)\right. \\
&\qquad \left. + rA\left(\frac{-x+y+z}{s}\right) - \gamma A(x) - \gamma A(y) - \gamma A(z)\right\|_X \\
&\quad = \lim_{n\to\infty} |2|^n \left\| rf\left(\frac{x+y+z}{2^n s}\right) + rf\left(\frac{x-y+z}{2^n s}\right) + rf\left(\frac{x+y-z}{2^n s}\right)\right. \\
&\qquad \left. + rf\left(\frac{-x+y+z}{2^n s}\right) - \gamma f\left(\frac{x}{2^n}\right) - \gamma f\left(\frac{y}{2^n}\right) - \gamma f\left(\frac{z}{2^n}\right)\right\|_X \\
&\quad \leq \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\quad = 0
\end{aligned}$$

for all $x, y, z \in X$. Therefore, the mapping $A : G \to X$ satisfies (1.1).

To prove the uniqueness property of A, let L be another mapping satisfying (3.4). Then we have

$$\begin{aligned}
&\|A(x) - L(x)\|_X \\
&\quad = \lim_{n\to\infty} |2|^n \left\| A\left(\frac{x}{2^n}\right) - L\left(\frac{x}{2^n}\right)\right\|_X \\
&\quad \leq \lim_{k\to\infty} |2|^n \max\left\{ \left\| A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_X, \left\| f\left(\frac{x}{2^n}\right) - L\left(\frac{x}{2^n}\right)\right\|_X \right\} \\
&\quad \leq \lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right)\right\}; j \leq k < n+j \right\} \\
&\quad = 0
\end{aligned}$$

for all $x \in G$. Therefore, $A = L$. This completes the proof. □

Corollary 3.2 *Let* $\xi : [0, \infty) \to [0, \infty)$ *be a function satisfying*

$$\xi\left(|2|^{-1}t\right) \leq \xi\left(|2|^{-1}\right)\xi(t), \qquad \xi\left(|2|^{-1}\right) < |2|^{-1}$$

for all $t \geq 0$. *Assume that* $\kappa > 0$ *and* $f : G \to X$ *be a mapping with* $f(0) = 0$ *such that*

$$\begin{aligned}
&\left\| rf\left(\frac{x+y+z}{s}\right) + rf\left(\frac{x-y+z}{s}\right) + rf\left(\frac{x+y-z}{s}\right)\right. \\
&\qquad \left. + rf\left(\frac{-x+y+z}{s}\right) - \gamma f(x) - \gamma f(y) - \gamma f(z)\right\|_X \leq \kappa\left(\xi\left(\|x\|\right) + \xi\left(\|y\|\right) + \xi\left(\|z\|\right)\right)
\end{aligned} \tag{3.8}$$

for all $x, y, z \in G$. *Then there exists a unique additive mapping* $A : G \to X$ *such that*

$$\|f(x) - A(x)\|_X \leq \frac{1}{|\gamma|} \max\left\{ \kappa\zeta\left(\|x\|\right), \frac{2}{|2|}\kappa\zeta\left(\|x\|\right)\right\}.$$

Proof Defining $\varphi : G^3 \to [0,\infty)$ by $\varphi(x,y,z) := \kappa(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|))$, then we have

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \le \lim_{n\to\infty} \left(|2|\xi\left(|2|^{-1}\right)\right)^n \varphi(x,y,z) = 0$$

for all $x,y,z \in G$. The last equality comes form the fact that $|2|\xi(|2|^{-1}) < 1$. On the other hand, it follows that

$$\begin{aligned}\Omega(x) &= \lim_{n\to\infty} \max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right)\right\}; 0 \le k < n\right\} \\ &\le \max\left\{\varphi(x,0,0), \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)\right\} \\ &= \max\left\{\kappa\zeta\left(\|x\|\right), \frac{2}{|2|}\kappa\zeta\left(\|x\|\right)\right\}\end{aligned}$$

exists for all $x \in G$. Also, we have

$$\begin{aligned}&\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right)\right\}; j \le k < n+j\right\} \\ &= \lim_{j\to\infty} |2|^j \max\left\{\varphi\left(\frac{x}{2^j},0,0\right), \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right)\right\} \\ &= 0.\end{aligned}$$

Thus, applying Theorem 3.1, we have the conclusion. This completes the proof. □

Theorem 3.3 *Let G be a vector space and that X is a non-Archimedean Banach space. Assume that $\varphi : G^3 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n} = 0 \tag{3.9}$$

for all $x,y,z \in G$. Suppose that, for any $x \in G$, the limit

$$\Omega(x) = \lim_{n\to\infty} \max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0), \varphi(2^k x, 2^k x, 0)\}}{|2|^k}; 0 \le k < n\right\} \tag{3.10}$$

exists and $f : G \to X$ be an odd mapping satisfying (3.3). Then the limit $A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G$ and

$$\left\|f(x) - A(x)\right\|_X \le \frac{1}{|2\gamma|}\Omega(x) \tag{3.11}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0), \varphi(2^k x, 2^k x, 0)\}}{|2|^k}; j \le k < n+j\right\} = 0,$$

then A is the unique mapping satisfying (3.11).

Proof By (2.6), we get

$$\left\|\frac{f(2x)}{2} - f(x)\right\|_X \le \frac{\max\{\varphi(2x,0,0), \varphi(x,x,0)\}}{|2\gamma|} \tag{3.12}$$

for all $x \in G$. Replacing x by $2^n x$ in (3.12), we obtain

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\|_X \le \frac{\max\{\varphi(2^{n+1}x,0,0), \varphi(2^n x, 2^n x, 0)\}}{|2\gamma||2|^n}. \tag{3.13}$$

Thus, it follows from (3.9) and (3.13) that the sequence $\{\frac{f(2^n x)}{2^n}\}_{n\ge1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

On the other hand, it follows from (3.13) that

$$\begin{aligned}
&\left\|\frac{f(2^p x)}{2^q} - \frac{f(2^q x)}{2^q}\right\|_X \\
&\quad = \left\|\sum_{k=p}^{q-1} \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right\|_X \\
&\quad \le \max\left\{\left\|\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right\|_X ; p \le k < q-1\right\} \\
&\quad \le \frac{1}{|2\gamma|} \max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0), \varphi(2^k x, 2^k x, 0)\}}{|2|^k} ; p \le k < q\right\}
\end{aligned}$$

for all $x \in G$ and $p, q \ge 0$ with $q > p \ge 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (3.10), we obtain (3.11).

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

Theorem 3.4 *Let G be a vector space and that X is a non-Archimedean Banach space. Assume that $\varphi : G^3 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |4|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.14}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\Theta(x) = \lim_{n\to\infty} \max\left\{|4|^k \max\left\{\varphi\left(\frac{x}{2^k}, 0, 0\right), |2|\varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right)\right\} ; 0 \le k < n\right\} \tag{3.15}$$

exists and $f : G \to X$ be an even mapping with $f(0) = 0$ and satisfying (3.3). Then the limit $Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for all $x \in G$ and defines a quadratic mapping $Q : G \to X$ such that

$$\|f(x) - Q(x)\|_X \le \frac{1}{|\gamma|} \Theta(x). \tag{3.16}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{|4|^k\max\left\{\varphi\left(\frac{x}{2^k},0,0\right),|2|\varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\};j\le k<n+j\right\}=0$$

then Q is the unique additive mapping satisfying (3.16).

Proof It follows from (2.15) that

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\|_X\le\frac{1}{|\gamma|}\max\left\{\varphi(x,0,0),|2|\varphi\left(\frac{x}{2},\frac{x}{2},0\right)\right\}. \tag{3.17}$$

Replacing x by $\frac{x}{2^n}$ in (3.18), we have

$$\left\|4^nf\left(\frac{x}{2^n}\right)-4^{n+1}f\left(\frac{x}{2^{n+1}}\right)\right\|_X\le\frac{|4|^n}{|\gamma|}\max\left\{\varphi\left(\frac{x}{2^n},0,0\right),|2|\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},0\right)\right\}. \tag{3.18}$$

It follows from (3.14) and (3.18) that the sequence $\{4^nf(\frac{x}{2^n})\}_{n\ge1}$ is Cauchy sequence. The rest of the proof is similar to the proof of Theorem 3.1. □

Similarly, we can obtain the followings. We will omit the proof.

Theorem 3.5 *Let G be a vector space and that X is a non-Archimedean Banach space. Assume that* $\varphi:G^3\to[0,+\infty)$ *be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny,2^nz)}{|4|^n}=0 \tag{3.19}$$

for all $x,y,z\in G$. *Suppose that, for any* $x\in G$, *the limit*

$$\Theta(x)=\lim_{n\to\infty}\max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^kx,2^kx,0)\}}{|4|^k};0\le k<n\right\} \tag{3.20}$$

exists and $f:G\to X$ *be an even mapping with* $f(0)=0$ *and satisfying (3.3). Then the limit* $Q(x):=\lim_{n\to\infty}\frac{f(2^nx)}{4^n}$ *exists for all* $x\in G$ *and*

$$\|f(x)-Q(x)\|_X\le\frac{1}{|4\gamma|}\Theta(x) \tag{3.21}$$

for all $x\in G$. *Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^kx,2^kx,0)\}}{|4|^k};j\le k<n+j\right\}=0,$$

then Q is the unique mapping satisfying (3.21).

Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (1.1). Let $f_e(x):=\frac{f(x)+f(-x)}{2}$ and $f_o(x)=\frac{f(x)-f(-x)}{2}$. Then f_e is an even mapping satisfying (1.1) and f_o is an odd mapping satisfying (1.1) such that $f(x)=f_e(x)+f_o(x)$. On the other hand,

$$\|D_{f_o}(x,y,z)\|\le\frac{\max\{\varphi(x,y,z),\varphi(-x,-y,-z)\}}{|2|}$$

and

$$\|D_{f_e}(x,y,z)\| \le \frac{\max\{\varphi(x,y,z),\varphi(-x,-y,-z)\}}{|2|}$$

for all $x,y,z \in X$, where $D_f(x,y,z)$ is the difference operator of the functional equation (1.1). So we obtain the following theorem.

Theorem 3.6 *Let G be a vector space and that X is a non-Archimedean Banach space. Assume that $\varphi : G^3 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{|4|^n} = 0$$

for all $x,y,z \in G$. Suppose that the limits

$$\begin{aligned}\Omega^*(x) = \lim_{n\to\infty} \max_{0\le k<n} \bigl\{&\max\bigl\{\max\bigl\{\varphi\bigl(2^{k+1}x,0,0\bigr),\varphi\bigl(-2^{k+1}x,0,0\bigr)\bigr\},\\ &\max\bigl\{\varphi\bigl(2^k x,2^k x,0\bigr),\varphi\bigl(-2^k x,-2^k x,0\bigr)\bigr\}\bigr\}/|2|^{k+1}\bigr\}\end{aligned}$$

and

$$\begin{aligned}\Theta^*(x) = \lim_{n\to\infty} \max_{0\le k<n} \bigl\{&\max\bigl\{\max\bigl\{\varphi\bigl(2^{k+1}x,0,0\bigr),\varphi\bigl(-2^{k+1}x,0,0\bigr)\bigr\},\\ &\max\bigl\{\varphi\bigl(2^k x,2^k x,0\bigr),\varphi\bigl(-2^k x,-2^k x,0\bigr)\bigr\}\bigr\}/\bigl(|2||4|^k\bigr)\bigr\}\end{aligned}$$

exist for all $x \in G$ and $f : G \to X$ be a mapping with $f(0) = 0$ and satisfying (3.3). Then there exist an additive mapping $A : G \to X$ and a quadratic mapping $Q : G \to X$ such that

$$\begin{aligned}&\|f(x) - A(x) - Q(x)\|_X\\ &\quad\le \max\left\{\left\|\frac{f(x)+f(-x)}{2} - Q(x)\right\|_X, \left\|\frac{f(x)-f(-x)}{2} - A(x)\right\|_X\right\}\\ &\quad\le \max\left\{\frac{1}{|2\gamma|}\Omega^*(x), \frac{1}{|4\gamma|}\Theta^*(x)\right\}\end{aligned} \tag{3.22}$$

for all $x \in G$. Moreover, if

$$\begin{aligned}\lim_{j\to\infty}\lim_{n\to\infty} \max_{j\le k<n+j} \bigl\{&\max\bigl\{\max\bigl\{\varphi\bigl(2^{k+1}x,0,0\bigr),\varphi\bigl(-2^{k+1}x,0,0\bigr)\bigr\},\\ &\max\bigl\{\varphi\bigl(2^k x,2^k x,0\bigr),\varphi\bigl(-2^k x,-2^k x,0\bigr)\bigr\}\bigr\}/|2|^{k+1}\bigr\} = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{j\to\infty}\lim_{n\to\infty} \max_{j\le k<n+j} \bigl\{&\max\bigl\{\max\bigl\{\varphi\bigl(2^{k+1}x,0,0\bigr),\varphi\bigl(-2^{k+1}x,0,0\bigr)\bigr\},\\ &\max\bigl\{\varphi\bigl(2^k x,2^k x,0\bigr),\varphi\bigl(-2^k x,-2^k x,0\bigr)\bigr\}\bigr\}/\bigl(|2||4|^k\bigr)\bigr\} = 0\end{aligned}$$

then A, Q are the unique mappings satisfying (3.22).

Theorem 3.7 *Let G be a vector space and that X is a non-Archimedean Banach space. Assume that $\varphi : G^3 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

for all $x, y, z \in G$. Suppose that the limits

$$\Omega^{**}(x) = \frac{1}{|2|} \lim_{n\to\infty} \max_{0\le k<n} \left\{ |2|^k \max\left\{ \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{-x}{2^k}, 0, 0\right)\right\}, \max\left\{ \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right), \varphi\left(\frac{-x}{2^{k+1}}, \frac{-x}{2^{k+1}}, 0\right)\right\}\right\}\right\}$$

and

$$\Theta^{**}(x) = \frac{1}{|2|} \lim_{n\to\infty} \max_{0\le k<n} \left\{ |4|^k \max\left\{ \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{-x}{2^k}, 0, 0\right)\right\}, |2| \max\left\{ \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right), \varphi\left(\frac{-x}{2^{k+1}}, \frac{-x}{2^{k+1}}, 0\right)\right\}\right\}\right\}$$

exist for all $x \in G$ and $f : G \to X$ be a mapping with $f(0) = 0$ and satisfying (3.3). Then there exist an additive mapping $A : G \to X$ and a quadratic mapping $Q : G \to X$ such that

$$\left\| f(x) - A(x) - Q(x) \right\|_X \le \frac{\max\{\Omega^{**}(x), \Theta^{**}(x)\}}{|\gamma|} \tag{3.23}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\le k<n+j} \left\{ |2|^k \max\left\{ \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{-x}{2^k}, 0, 0\right)\right\}, \max\left\{ \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right), \varphi\left(\frac{-x}{2^{k+1}}, \frac{-x}{2^{k+1}}, 0\right)\right\}\right\}\right\} = 0$$

and

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\le k<n+j} \left\{ |4|^k \max\left\{ \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{-x}{2^k}, 0, 0\right)\right\}, |2| \max\left\{ \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right), \varphi\left(\frac{-x}{2^{k+1}}, \frac{-x}{2^{k+1}}, 0\right)\right\}\right\}\right\} = 0$$

then A, Q are the unique mappings satisfying (3.23).

4 Conclusion

We linked here two different disciplines, namely, the non-Archimedean normed spaces and functional equations. We established the generalized Hyers-Ulam stability of the functional equation (1.1) in non-Archimedean normed spaces.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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