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On the stability of set-valued functional equations with the fixed point alternative

Hassan Azadi Kenary[1], Hamid Rezaei[1], Yousof Gheisari[2] and Choonkil Park[3*]

* Correspondence: baak@hanyang.ac.kr
[3]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea
Full list of author information is available at the end of the article

Abstract

Using the fixed point method, we prove the Hyers-Ulam stability of a Cauchy-Jensen type additive set-valued functional equation, a Jensen type additive-quadratic set-valued functional equation, a generalized quadratic set-valued functional equation and a Jensen type cubic set-valued functional equation.

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1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering article by Aumann [1] and Debreu [2] were inspired by problems arising in control theory and mathematical economics. We can refer to the articles by Arrow and Debreu [3], McKenzie [4], the monographs by Hindenbrand [5], Aubin and Frankow [6], Castaing and Valadier [7], Klein and Thompson [8] and the survey by Hess [9]. Let Y be a Banach space. We define the following:

2^Y : the set of all subsets of Y;
$C_b(Y)$: the set of all closed bounded subsets of Y;
$C_c(Y)$: the set of all closed convex subsets of Y;
$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.

On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C,\ x' \in C'\},\ \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda (C + C'),\quad (\lambda + \mu) C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$d(x, C) = \inf\{\|x - y\| : y \in C\},\ x \in Y,$$

$$s\left(x^*, C\right) = \sup\left\{\langle x^*, x\rangle : x \in C\right\}, \; x^* \in Y^*.$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h\left(C, C'\right) = \inf\left\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \; C' \subseteq C + \lambda B_Y\right\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.1. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*

(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;
(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [7]). Debreu [2] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

Lemma 1.2. [2]*Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*

(a) $j(A \oplus B) = j(A) + j(B)$;
(b) $j(\lambda A) = \lambda j(A)$;
(c) $h(A, B) = \|j(A) - j(B)\|_u$;
(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$

for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [10]). In this case, the Debreu integral of f in Ω is the unique element $(D)\int_\Omega f d\nu \in C_{cb}(Y)$ such that $j((D)\int_\Omega f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [11-18]).

The stability problem of functional equations was originated from a question of Ulam [19] concerning the stability of group homomorphisms. Hyers [20] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [21] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The article of Rassias [22] has provided a lot of influence in the development of what we call *Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [23] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [24] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [25] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [26] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [11,21-23,27-61]).

In [52], Najati considered the following functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2\left[f(x)+f(y)+f(z)\right] \quad (1.1)$$

for all $x, y, z \in X$. It is easy to show that the function $f(x) = x$ satisfies the functional Equation (1.1), which is called a *Cauchy-Jensen type additive functional equation* and every solution of the Cauchy-Jensen type additive functional equation is said to be a *Cauchy-Jensen type additive mapping*.

In [57], Park considered the following functional equation

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y) \quad (1.2)$$

for all $x, y \in X$. It is easy to show that the function $f(x) = x+x^2$ satisfies the functional Equation (1.2), which is called a *Jensen type additive-quadratic functional equation* and every solution of the Jensen type additive-quadratic functional equation is said to be a *Jensen type additive-quadratic mapping*.

In [48], Jun and Cho considered the following functional equation

$$f\left(\frac{x+y}{r}+sz\right)+f\left(\frac{x+y}{r}-sz\right)+f\left(\frac{x-y}{r}+sz\right)+f\left(\frac{x-y}{r}-sz\right)=\frac{4}{r^2}f(x)+\frac{4}{r^2}f(y)+4s^2f(z) \quad (1.3)$$

for all $x, y, z \in X$, where r, s are real numbers with $r, s \neq 0$. It is easy to show that the function $f(x) = x^2$ satisfies the functional Equation (1.3), which is called a *generalized quadratic functional equation* and every solution of the generalized quadratic functional equation is said to be a *generalized quadratic mapping*.

In [62], Kim et al. considered the following functional equation

$$f\left(\frac{3x+y}{2}\right)+f\left(\frac{x+3y}{2}\right)=12f\left(\frac{x+y}{2}\right)+f(x)+f(y) \quad (1.4)$$

for all $x, y \in X$. It is easy to show that the function $f(x) = x^3$ satisfies the functional Equation (1.4), which is called a *Jensen type cubic functional equation* and every solution of the Jensen type cubic functional equation is said to be a *Jensen type cubic mapping*.

Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.3. [31,63]*Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \left\{y \in X \mid d\left(J^{n_0} x, y\right) < \infty\right\}$;*
(4) $d\left(y, y^*\right) \leq \frac{1}{1-L} d\left(y, Jy\right)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [47] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [32,33,56,58,64]).

Using the fixed point method, we prove the Cauchy-Jensen type additive set-valued functional equation, the Jensen type additive-quadratic set-valued functional equation, the generalized quadratic set-valued functional equation and the Jensen type cubic set-valued functional equation.

Throughout this article, let X be a real vector space and Y a Banach space.

2. Stability of the Cauchy-Jensen type additive set-valued functional Equation (1.1)

Using the fixed point method, we prove the Hyers-Ulam stability of the Cauchy-Jensen type additive set-valued functional equation.

Definition 2.1. [52] Let $f : X \to C_{cb}(Y)$. The Cauchy-Jensen type additive set-valued functional equation is defined by

$$f\left(\frac{x+y}{2} + z\right) \oplus f\left(\frac{x+z}{2} + y\right) \oplus f\left(\frac{y+z}{2} + x\right) = 2\left[f(x) \oplus f(y) \oplus f(z)\right]$$

for all $x, y, z \in X$. Every solution of the Cauchy-Jensen type additive set-valued functional equation is called an *Cauchy-Jensen type additive set-valued mapping*.

Theorem 2.2. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq \frac{L}{2}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying

$$h\left(f\left(\frac{x+y}{2}+z\right) \oplus f\left(\frac{x+z}{2}+y\right) \oplus f\left(\frac{y+z}{2}+x\right), 2\left[f(x) \oplus f(y) \oplus f(z)\right]\right) \leq \varphi(x, y, z) \qquad (2.1)$$

for all $x, y, z \in X$. Then

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right),$$

exists for each $x \in X$ and defines a unique Cauchy-Jensen type additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), A(x)) \leq \frac{L}{6-6L}\varphi(x, x, x) \qquad (2.2)$$

for all $x \in X$.

Proof. Let $x = y = z$ in (2.1). Since $f(x)$ is convex, we get

$$h(f(2x), 2f(x)) \leq \frac{1}{3}\varphi(x, x, x) \qquad (2.3)$$

and so

$$h\left(f(x), 2f\left(\frac{x}{2}\right)\right) \leq \frac{1}{3}\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{6}\varphi(x, x, x) \qquad (2.4)$$

for all $x \in X$.

Consider

$$S := \{g : g : X \to C_{cb}(Y), g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = \inf\{\mu \in (0, \infty) : h(g(x), f(x)) \leq \mu\varphi(x, x, x), x \in X\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see [41, Theorem 2.4]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g, f) = \varepsilon$. Then

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, x, x)$$

for all $x \in X$. Hence

$$h(Jg(x), Jf(x)) = h\left(2g\left(\frac{x}{2}\right), 2f\left(\frac{x}{2}\right)\right) = 2h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \leq L\varphi(x, x, x)$$

for all $x \in X$. So $d(g, f) = \varepsilon$ implies that $d(Jg, Jf) \leq L\varepsilon$. This means that

$$d\left(Jg, Jf\right) \leq Ld\left(g, f\right)$$

for all $g, f \in S$.

It follows from (2.4) that $d\left(f, Jf\right) \leq \frac{L}{6}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \left\{g \in S : d\left(f, g\right) < \infty\right\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$h\left(f(x), A(x)\right) \leq \mu\varphi(x, x, x)$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d\left(f, A\right) \leq \frac{1}{1-L}d\left(f, Jf\right)$, which implies the inequality

$$d\left(f, A\right) \leq \frac{L}{6 - 6L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$\begin{aligned}
&h\left(2^n f\left(\frac{x+y}{2^{n+1}} + \frac{z}{2^n}\right) \oplus 2^n f\left(\frac{x+z}{2^{n+2}} + \frac{y}{2^n}\right) \oplus 2^n f\left(\frac{y+z}{2^{n+1}} + \frac{x}{2^n}\right),\right.\\
&\left.2^{n+1} f\left(\frac{x}{2^n}\right) \oplus 2^{n+1} f\left(\frac{y}{2^n}\right) \oplus 2^{n+1} f\left(\frac{z}{2^n}\right)\right)\\
&\leq 2^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \leq L^n \phi\left(x, y, z\right),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y, z \in X$. Thus

$$A\left(\frac{x+y}{2} + z\right) \oplus A\left(\frac{x+z}{2} + y\right) \oplus A\left(\frac{y+z}{2} + x\right) = 2\left[A(x) \oplus A\left(y\right) \oplus A(z)\right],$$

as desired. □

Corollary 2.3. *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$\begin{aligned}
&h\left(f\left(\frac{x+y}{2} + z\right) \oplus f\left(\frac{x+z}{2} + y\right) \oplus f\left(\frac{y+z}{2} + x\right), 2\left[f(x) \oplus f\left(y\right) \oplus f(z)\right]\right)\\
&\leq \theta\left(\|x\|^p + \left\|y\right\|^p + \|z\|^p\right)
\end{aligned} \tag{2.6}$$

for all $x, y, z \in X$. Then

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in X$ and defines a unique Cauchy-Jensen type additive set-valued mapping $A : X \to Y$ satisfying

$$h\left(f(x), A(x)\right) \le \frac{\theta\|x\|^p}{2^p - 2}$$

and for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y, z) := \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.4. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (2.1). Then there exists a unique Cauchy-Jensen type additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h\left(f(x), A(x)\right) \le \frac{1}{6 - 6L}\varphi(x, x, x)$$

and

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$h\left(f(x), \frac{1}{2}f(2x)\right) \le \frac{1}{6}\varphi(x, x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $0 < p < 1$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (2.6). Then*

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ and defines a unique Cauchy-Jensen type additive set-valued mapping $A : X \to Y$ satisfying

$$h\left(f(x), A(x)\right) \le \frac{\theta\|x\|^p}{2 - 2^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y, z) := \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

3. Stability of the Jensen type AQ set-valued functional Equation (1.2)

Using the fixed point method, we prove the Hyers-Ulam stability of the Jensen type additive-quadratic set-valued functional equation.

3.1. An odd case

Theorem 3.1. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is an odd mapping satisfying

$$h\left(2f\left(\frac{x+y}{2}\right) \oplus f\left(\frac{x-y}{2}\right) \oplus f\left(\frac{y-x}{2}\right), f(x) \oplus f(y)\right) \le \varphi(x, y) \tag{3.1}$$

for all $x, y \in X$. Then

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in X$ and defines a unique Jensen type additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h\left(f(x), A(x)\right) \le \frac{1}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.1). Since $f(x)$ is convex, we get

$$h\left(f(x), 2f\left(\frac{x}{2}\right)\right) \le \varphi(x, 0) \tag{3.2}$$

for all $x \in X$.

Consider

$$S := \left\{g : g : X \to C_{cb}(Y), g(0) = \{0\}\right\}$$

and introduce the generalized metric on X,

$$d(g, f) = \inf\left\{\mu \in (0, \infty) : h\left(g(x), f(x)\right) \le \mu\varphi(x, 0), x \in X\right\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see [41, Theorem 2.4]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.2) that $d(f, Jf) \le 1$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h\left(2f\left(\frac{x+y}{2}\right) \oplus f\left(\frac{x-y}{2}\right) \oplus f\left(\frac{y-x}{2}\right), f(x) \oplus f(y)\right) \leq \theta\left(\|x\|^p + \|y\|^p\right) \qquad (3.3)$$

for all $x, y \in X$. Then

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in X$ and defines a unique Jensen type additive set-valued mapping $A : X \to Y$ satisfying

$$h\left(f(x), A(x)\right) \leq \frac{2^p\theta\|x\|^p}{2^p - 2}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 3.3. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(2x, 2y) \leq 2L\varphi(x, y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (3.1). *Then there exists a unique Jensen type additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that*

$$h\left(f(x), A(x)\right) \leq \frac{L}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.2) that

$$h\left(f(x), \frac{1}{2}f(2x)\right) \leq \frac{1}{2}\varphi(2x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

Corollary 3.4. *Let $0 < p < 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying* (3.3). *Then there exists a unique Jensen type additive set-valued mapping $A : X \to Y$ satisfying*

$$h\left(f(x), A(x)\right) \leq \frac{2^p\theta\|x\|^p}{2 - 2^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

3.2. An even case

Theorem 3.5. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ satisfying (3.1). *Then there exists a unique Jensen type quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that*

$$h\left(f(x), Q(x)\right) \le \frac{1}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.1). Since $f(x)$ is convex, we get

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \le \varphi(x, 0) \tag{3.4}$$

for all $x \in X$.

Consider

$$S := \left\{g : g : X \to C_{cb}(Y), g(0) = \{0\}\right\}$$

and introduce the generalized metric on X,

$$d\left(g, f\right) = inf\left\{\mu \in (0, \infty) : h\left(g(x), f(x)\right) \le \mu\varphi(x, 0), x \in X\right\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see [41, Theorem 2.4]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.4) that $d(f, Jf) \le 1$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.6. *Let $p > 2$ and $\theta \ge 0$ be real numbers, and let* **X** *be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ satisfying* (3.3). *Then there exists a unique Jensen type quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h\left(f(x), Q(x)\right) \le \frac{2^p\theta\|x\|^p}{2^p - 4}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.5 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.7. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(2x, 2y) \leq 4L\varphi(x, y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ and satisfying (3.1). *Then there exists a unique Jensen type quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that*

$$h(f(x), Q(x)) \leq \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.4) that

$$h\left(f(x), \frac{1}{4}f(2x)\right) \leq \frac{1}{4}\varphi(2x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.5. □

Corollary 3.8. *Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let **X** be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ and satisfying* (3.3). *Then there exists a unique Jensen type quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h(f(x), Q(x)) \leq \frac{2^p\theta\|x\|^p}{4 - 2^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.7 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

4. Stability of the generalized quadratic set-valued functional Equation (1.3)

Using the fixed point method, we define a generalized quadratic set-valued functional equation and prove the Hyers-Ulam stability of the generalized quadratic set-valued functional equation.

Remark 4.1. For convenience, let $f(u \pm v) = f(u + v) \oplus f(u - v)$.

Definition 4.2. Let $f : X \to C_{cb}(Y)$. The generalized quadratic set-valued functional equation is defined by

$$f\left(\frac{x + y}{r} \pm sz\right) \oplus f\left(\frac{x - y}{r} \pm sz\right) = \frac{4}{r^2}f(x) \oplus \frac{4}{r^2}f(y) \oplus 4s^2f(z)$$

for all $x, y, z \in X$. Every solution of the generalized quadratic set-valued functional equation is called a *generalized quadratic set-valued mapping.*

Theorem 4.3. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{rx}{2}, \frac{ry}{2}, \frac{rz}{2}\right) \le \frac{4L}{r^2}\varphi(x, y, z)$$

for all $x, y, z \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ and satisfying

$$h\left(f\left(\frac{x+y}{r} \pm sz\right) \oplus f\left(\frac{x-y}{r} \pm sz\right), \frac{4}{r^2}f(x) \oplus \frac{4}{r^2}f(y) \oplus 4s^2 f(z)\right) \le \varphi(x, y, z) \tag{4.1}$$

for all $x, y, z \in X$, and $r, s \neq 0$. Then there exists a unique generalized quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{2L}{r^2(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. Let $x = y$ and $z = 0$ in (4.1). Since $f(x)$ is convex, we get

$$h\left(2f\left(\frac{2x}{r}\right), \frac{8}{r^2}f(x)\right) \le \varphi(x, x, 0) \tag{4.2}$$

and so

$$h\left(f(x), \frac{4}{r^2}f\left(\frac{rx}{2}\right)\right) \le \frac{1}{2}\varphi\left(\frac{rx}{2}, \frac{rx}{2}, 0\right) \tag{4.3}$$

for all $x \in X$.

Consider

$$S := \{g : g : X \to C_{cb}(Y), g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : h(g(x), f(x)) \le \mu\varphi(x, x, 0), x \in X\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see [41, Theorem 2.4]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{4}{r^2}g\left(\frac{rx}{2}\right)$$

for all $x \in X$.

It follows from (4.3) that $d(f, Jf) \le \frac{2L}{r^2}$.

The rest of the proof is similar to the proof of Theorem 2.2.

Corollary 4.4. *Let $p > 2$, $\theta \ge 0$ be real numbers and $0 < |r| < 2$. Let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ and satisfying*

$$\begin{aligned} &h\left(f\left(\frac{x+y}{r} \pm sz\right) \oplus f\left(\frac{x-y}{r} \pm sz\right), \frac{4}{r^2}f(x) \oplus \frac{4}{r^2}f(y) \oplus 4s^2 f(z)\right) \\ &\le \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \end{aligned} \tag{4.4}$$

for all $x, y, z \in X$, and $r, s \neq 0$. Then there exists a unique generalized quadratic set-valued mapping $Q : X \to Y$ satisfying

$$h\left(f(x), Q(x)\right) \leq \frac{16|r|^{p-2}\theta}{|r|^2\left(2^p - 4|r|^{p-2}\right)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.3 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = \left(\frac{|r|}{2}\right)^{p-2}$ and we get the desired result. □

Theorem 4.5. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{2x}{r}, \frac{2y}{r}, \frac{2z}{r}\right) \leq \frac{r^2L}{4}\varphi(x, y, z)$$

for all $x, y, z \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ and satisfying (4.1). Then there exists a unique generalized quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h\left(f(x), Q(x)\right) \leq \frac{r^2}{8 - 8L}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (4.2) that

$$h\left(f(x), \frac{r^2}{4}f\left(\frac{2x}{r}\right)\right) \leq \frac{r^2}{8}\varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 4.3. □

Corollary 4.6. *Let $0 < p < 2$, $\theta \geq 0$ be real numbers and $|r| > 2$. Let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ and satisfying (4.4). Then there exists a unique generalized quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h\left(f(x), Q(x)\right) \leq \frac{|r|^{4-p}\theta\|x\|^p}{4\left(|r|^{2-p} - 2^{2-p}\right)}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.5 by taking

$$\varphi(x, y, z) := \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = \left(\frac{2}{|r|}\right)^{2-p}$ and we get the desired result. □

5. Stability of the Jensen type cubic set-valued functional Equation (1.4)

Definition 5.1. Let $f : X \to C_{cb}(Y)$. The Jensen type cubic set-valued functional equation is defined by

$$f\left(\frac{3x+y}{2}\right) \oplus f\left(\frac{x+3y}{2}\right) = 12f\left(\frac{x+y}{2}\right) \oplus f(x) \oplus f(y)$$

for all $x, y \in X$. Every solution of the Jensen type cubic set-valued functional equation is called a *Jensen type cubic set-valued mapping.*

Theorem 5.2. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le 8L\varphi(x, y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying

$$h\left(f\left(\frac{3x+y}{2}\right) \oplus f\left(\frac{x+3y}{2}\right), 12f\left(\frac{x+y}{2}\right) \oplus f(x) \oplus f(y)\right) \le \varphi(x, y) \quad (5.1)$$

for all $x, y \in X$. Then there exists a unique Jensen type cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), C(x)) \le \frac{4L}{1-L}\varphi(x, x) \quad (5.2)$$

for all $x \in X$.

Proof. Let $x = y$ in (5.1). Since $f(x)$ is convex, we get

$$h(f(2x), 8f(x)) \le \frac{1}{2}\varphi(x, x) \quad (5.3)$$

and so

$$h\left(f(x), 8f\left(\frac{x}{2}\right)\right) \le \frac{1}{2}\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le 4L\varphi(x, x) \quad (5.4)$$

for all $x \in X$.

Consider

$$S := \{g : g : X \to C_{cb}(Y), g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = \inf\{\mu \in (0, \infty) : h(g(x), f(x)) \le \mu\varphi(x, x), x \in X\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see [41, Theorem 2.4]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (5.4) that $d(f, Jf) \le 4L$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 5.3. *Let $p > 3$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h\left(f\left(\frac{3x+y}{2}\right)\oplus f\left(\frac{x+3y}{2}\right),\ 12f\left(\frac{x+y}{2}\right)\oplus f(x)\oplus f(y)\right)\le\theta\left(\|x\|^p+\|y\|^p\right) \quad (5.5)$$

for all $x, y \in X$ *Then there exists a unique Jensen type cubic set-valued mapping.* $C : X \to Y$ *satisfying*

$$h\left(f(x),\ C(x)\right)\le\frac{64\theta\|x\|^p}{2^p-8}$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.2 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p+\|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 5.4. *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(2x, 2y)\le\frac{L}{8}\varphi(x, y)$$

for all $x, y \in X$. *Suppose that* $f : X \to (C_{cb}(Y), h)$ *is a mapping satisfying (5.1). Then there exists a unique Jensen type cubic set-valued mapping* $C : X \to (C_{cb}(Y), h)$ *such that*

$$h\left(f(x), C(x)\right)\le\frac{1}{16-16L}\varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$h\left(f(x),\ \frac{1}{8}f(2x)\right)\le\frac{1}{16}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 5.5. *Let* $0 < p < 3$ *and* $\theta \ge 0$ *be real numbers, and let* $\boldsymbol{X}$ *be a real normed space. Suppose that* $f : X \to (C_{cb}(Y), h)$ *is a mapping satisfying* (5.5). *Then there exists a unique Jensen type cubic set-valued mapping* $C : X \to Y$ *satisfying*

$$h\left(f(x), C(x)\right)\le\frac{\theta\|x\|^p}{8-2^p}$$

for all $\times \in X$.

Proof. The proof follows from Theorem 5.4 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p+\|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

6. Conclusions

We have defined the Cauchy-Jensen type additive set-valued functional equation, the Jensen type additive-quadratic set-valued functional equation, the generalized quadratic set-valued functional equation and the Jensen type cubic set-valued functional equation, and we have proved the Hyers-Ulam stability of the Cauchy-Jensen type additive set-valued functional equation, the Jensen type additive-quadratic set-valued functional equation, the generalized quadratic set-valued functional equation and the Jensen type cubic set-valued functional equation by using the fixed point method.

Author details
[1]Department of Mathematics, College of Science, Yasouj University, Yasouj 75914-353, Iran [2]Department of Mathematics, Islamic Azad University, Bushehr Branch, Bushehr, Iran [3]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea

Authors' contributions
All authors read and approved the final manuscript.

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