Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Almost partial generalized Jordan derivations: a fixed point approach

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Abstract

Using fixed point method, we investigate the Hyers-Ulam stability and the superstability of partial generalized Jordan derivations on Banach modules related to Jensen type functional equations.

Mathematics Subject Classification 2010: Primary, 39B52; 47H10; 47B47; 13N15; 39B72; 17C50; 39B82; 17C65.

Keywords: Hyers-Ulam stability, superstability, partial Jordan derivation, partial generalized Jordan derivation, Jensen type functional equation

1. Introduction and preliminaries

The following question posed by Ulam [1] in 1940: "When is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$?". Hyers [2] proved the problem for the Cauchy functional equation. In 1978, Rassias [3] proved the following theorem.

Theorem 1.1. *Let f: E → E' be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p) \tag{1.1}$$

for all x, y ∈ E, where ε and p are constants with ε > 0 and p < 1. Then there exists a unique additive mapping T: E → E' such that

$$\|f(x)-T(x)\| \leq \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all x ∈ E. If p < 0 then inequality (1.1) *holds for all x, y ≠ 0, and* (1.2) *for x ≠ 0. Also, if the function t α f(tx) from $\mathbb{R}$ into E' is continuous in real t for each x∈ E, then T is $\mathbb{R}$-linear.*

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. In 1994, a generalization of the Rassias' theorem was obtained by Găvruta as follows [5].

Stability of the Jensen functional equation, $2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$, where f is a mapping between linear spaces, has been investigated by several mathematicians (see [6,7]). During the last decades several stability problems of functional equations have been investigated by a number of mathematicians. See [8-17] and references therein for more detailed information.

Let A, B be two Banach algebras. A $\mathbb{C}$-linear mapping $d: A \to B$ is called a generalized Jordan derivation if there exists a Jordan derivation (in the usual sense) $\delta: A \to X$ such that $d(a^2) = ad(a) + \delta(a)a$ for all $a \in A$.

Generalized derivations and generalized Jordan derivations first appeared in the context of operator algebras [18]. Later, these were introduced in the framework of pure algebra [19,20].

Recently, Badora [21] proved the stability of ring derivations (see also [22,23]). More recently, Eshaghi Gordji and Ghobadipour [24] investigated the stability of generalized Jordan derivations on Banach algebras.

Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be normed algebras over the complex field $\mathbb{C}$ and let $\mathcal{B}$ be a Banach algebra over $\mathbb{C}$. A mapping $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$ is called a k-th partial derivation if

$$d_k(x_1, \ldots, \gamma a_k+\mu b_k, \ldots, x_n) = \gamma d_k(x_1, \ldots, a_k, \ldots, x_n)+\mu d_k(x_1, \ldots, b_k, \ldots, x_n)$$

and there exists a mapping $f_k : \mathcal{A}_k \to \mathcal{B}$ such that

$$d_k(x_1, \ldots, a_k b_k, \ldots, x_n) = f_k(a_k)d_k(x_1, \ldots, b_k, \ldots, x_n)+d_k(x_1, \ldots, a_k, \ldots, x_n)f_k(b_k)$$

for all $a_k, b_k \in \mathcal{A}_k$ and $x_i \in \mathcal{A}_i (i \neq k)$ and all $\gamma, \mu \in \mathbb{C}$.

Chu et al. [25] established the Hyers-Ulam stability of partial derivations.

Definition 1.2. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be normed algebras over the complex field $\mathbb{C}$ and let $\mathcal{X}$ be a Banach module over $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ and $\mathcal{A}_n$. Then

(i) A mapping $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ is called a k-th partial Jordan derivation of Jensen type if

$$2d_k\left(x_1, \ldots, \frac{\gamma a_k + \gamma b_k}{2}, \ldots, x_n\right) = \gamma d_k(x_1, \ldots, a_k, \ldots, x_n)+\gamma d_k(x_1, \ldots, b_k, \ldots, x_n)$$

and

$$d_k(x_1, \ldots, a_k^2, \ldots, x_n) = a_k d_k(x_1, \ldots, a_k, \ldots, x_n) + d_k(x_1, \ldots, a_k, \ldots, x_n)a_k$$

for all $a_k, b_k \in \mathcal{A}_k$ and $x_i \in \mathcal{A}_i (i \neq k)$ and all $\gamma \in \mathbb{C}$.

(ii) A mapping $\delta_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ is called a k-th partial generalized Jordan derivation of Jensen type if

$$2\delta_k\left(x_1, \ldots, \frac{\gamma a_k + \gamma b_k}{2}, \ldots, x_n\right) = \gamma \delta_k(x_1, \ldots, a_k, \ldots, x_n)+\gamma \delta_k(x_1, \ldots, b_k, \ldots, x_n)$$

and there exists a k-th partial Jordan derivation $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ such that

$$\delta_k(x_1, \ldots, a_k^2, \ldots, x_n) = \delta_k(x_1, \ldots, a_k, \ldots, x_n)a_k + a_k d_k(x_1, \ldots, a_k, \ldots, x_n)$$

for all $a_k, b_k \in \mathcal{A}_k$ and $x_i \in \mathcal{A}_i (i \neq k)$ and all $\gamma \in \mathbb{C}$.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [26,27]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [28].

Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies:

(GM_1) $d(x, y) = 0$ if and only if $x = y$;
(GM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(GM_3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Let (X, d) be a generalized metric space. An operator $T: X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \ge 0$ such that

$$d(Tx,\ Ty) \le Ld(x,\ y)$$

for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator.

We recall the following theorem by Diaz and Margolis [26].

Theorem 1.3. *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive function* $T: \Omega \to \Omega$ *with Lipschitz constant L. Then for each given* $x \in \Omega$, *either*

$$d\left(T^m x, T^{m+1} x\right) = \infty \quad \text{for all } m \ge 0,$$

or other exists a natural number m_0 *such that*

$\star d(T^m x,\ T^{m+1} x) < \infty$ *for all* $m \ge m_0$;
$\star$ *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T;
$\star y^*$ *is the unique fixed point of* T *in*

$$\Lambda = \{y \in \Omega : d(T^{m_0} x,\ y) < \infty\};$$

$\star d(y,\ y^*) \le \frac{1}{1-L} d(y,\ Ty) \quad$ *for all* $y \in \Lambda$.

The equation (ξ) is called superstable if every approximate solution of (ξ) is an exact solution.

We use the fixed point method to investigate the Hyers-Ulam stability and the superstability of partial generalized Jordan derivations of Jensen type.

2. Main results

For $n_0 \in \mathbb{N}$, we define

$$\mathbb{T}^1_{\frac{1}{n_O}} := \left\{ e^{i\theta}\ 0 \le \theta \le \frac{2\pi}{n_o} \right\}$$

and we denote $\mathbb{T}^1_{\frac{1}{1}}$ by $\mathbb{T}^1$. Also, we suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are normed algebras over the complex field $\mathbb{C}$ and $\mathcal{X}$ is a Banach module over $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ and $\mathcal{A}_n$. We denote that 0_k, $0_{\mathcal{X}}$ are zero elements of $\mathcal{A}_k$, $\mathcal{X}$, respectively.

Theorem 2.1. *Let* $T_k, F_k : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *be mappings with* $\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1,\ \ldots,\ a_k,\ \ldots,\ x_n) - \lambda S_k(x_1,\ \ldots,\ b_k,\ \ldots,\ x_n) \right\| \le \varphi_k(a_k,\ b_k),$. *Assume that* *there exist functions* $\Psi_k : \mathcal{A}_k \to [0, \infty)$, $\varphi_k : \mathcal{A}_k^2 \to [0, \infty)$ *satisfying*

$$\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n) \right\| \le \varphi_k(a_k, b_k), \tag{2.1}$$

$$\begin{aligned} \max\{ &\| F_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n) a_k \|, \\ &\| T_k(x_1, \ldots, a_k^2, \ldots, x_n) - T_k(x_1, \ldots, a_k, \ldots, x_n) a_k - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) \| \} \\ &\le \Psi_k(a_k) \end{aligned} \tag{2.2}$$

for $S_k \in \{F_k, T_k\}$ *and for all* $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and all* $a_k, b_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. *If there exists a constant* $0 < L < 1$ *such that* $\phi_k(a_k, b_k) \le 2L\phi_k(2^{-1}a_k, 2^{-1}b_k)$, $\Psi_k(a_k) \le 2L\Psi_k(2^{-1}a_k)$ *for all* $a_k, b_k \in \mathcal{A}_k$, *then there exist a unique partial Jordan derivation of Jensen type with respect to k-th variable* $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *and a unique partial generalized Jordan derivation of Jensen type with respect to k-th variable (related to* d_k*)* $D_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *such that*

$$\begin{aligned} \max\{ &\| F_k(x_1, x_2, \ldots, x_n) - d_k(x_1, x_2, \ldots, x_n) \|, \| T_k(x_1, x_2, \ldots, x_n) - D_k(x_1, x_2, \ldots, x_n) \| \} \\ &\le \frac{L}{1-L}\varphi_k(x_k, 0) \end{aligned}$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$.

Proof. It follows from (2.1) that

$$\begin{aligned} &\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n) \right\| \\ &\le \varphi_k(a_k, b_k), \end{aligned} \tag{2.3}$$

for $S_k \in \{F_k, T_k\}$ and for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $a_k, b_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$.

In the inequality (2.3), put $S_k = F_k$, $b_k = 0$, $\lambda = 1$ and replace a_k with $2x_k$. Then we obtain

$$\| F_k(x_1, \ldots, x_k, \ldots, x_n) - 2^{-1}F_k(x_1, \ldots, 2x_k, \ldots, x_n) \| \le 2^{-1}\varphi_k(2x_k, 0) \le L\varphi(x_k, 0) \tag{2.4}$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$. Put $\Omega := \{G_k | G_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}\}$ and define $d: \Omega \times \Omega \to [0, \infty]$ by

$$\begin{aligned} d(H_k, G_k) := \inf\{\alpha \in \mathbb{R}^+ ;\ &\| G_k(x_1, \ldots, x_k, \ldots, x_n) - H_k(x_1, \ldots, x_k, \ldots, x_n) \| \\ &\le \alpha\varphi_k(x_k, 0) \forall x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)\}. \end{aligned}$$

It is easy to show that (Ω, d) is a complete generalized metric space. We define the mapping $J: \Omega \to \Omega$ by

$$J(H_k)(x_1, \ldots, x_k, \ldots, x_n) = 2^{-1}H_k(x_1, \ldots, 2x_k, \ldots, x_n)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$. Let $G_k, H_k \in \Omega$ and let $\alpha \in (0, \infty)$ be arbitrary with $d(G_k, H_k) \le \alpha$. From the definition of d, we have

$$\| G_k(x_1, \ldots, x_k, \ldots, x_n) - H_k(x_1, \ldots, x_k, \ldots, x_n) \| \le \alpha\varphi_k(x_k, 0)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$. Hence we have

$$\begin{aligned}&\|(JG_k)(x_1, \ldots, x_k, \ldots, x_n) - (JH_k)(x_1, \ldots, x_k, \ldots, x_n)\| \\ &= 2^{-1} \|G_k(x_1, \ldots, 2x_k, \ldots, x_n) - H_k(x_1, \ldots, 2x_k, \ldots, x_n)\| \\ &\leq 2^{-1}\alpha\varphi_k(2x_k, 0) \leq \alpha L\varphi_k(x_k, 0)\end{aligned}$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$. So

$$d(J(G_k), J(H_k)) \leq Ld(G_k, H_k)$$

for all G_k, $H_k \in \Omega$. It follows from (2.4) that

$$d(F_k, J(F_k)) \leq L.$$

By Theorem 1.3, J has a unique fixed point in the set $\Omega_1 := \{ H_k \in \Omega; d(F_k, H_k) < \infty\}$. Let d_k be the fixed point of J. d_k is the unique mapping which satisfies

$$d_k(x_1, \ldots, 2x_k, \ldots, x_n) = 2d_k(x_1, \ldots, x_k, \ldots, x_n)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$, and there exists $\alpha \in (0, \infty)$ such that

$$\|d_k(x_1, \ldots, x_k, \ldots, x_n) - F_k(x_1, \ldots, x_k, \ldots, x_n)\| \leq \alpha\varphi_k(x_k, 0)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$.

On the other hand, we have $\lim_{m\to\infty} d(J^m(F_k), d_k) = 0$. It follows that

$$\lim_{m\to\infty} 2^{-m}F_k(x_1, \ldots, 2^m x_k, \ldots, x_n) = d_k(x_1, \ldots, x_k, \ldots, x_n)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$. It follows from that $d(F_k, d_k) \leq \frac{1}{1-L} d(F_k, J(F_k))$ that

$$d(F_k, d_k) \leq \frac{L}{1-L}.$$

This means that

$$\|F_k(x_1, x_2, \ldots, x_n) - d_k(x_1, x_2, \ldots, x_n)\| \leq \frac{L}{1-L}\varphi_k(x_k, 0)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$. By the inequality $\phi_k(a_k, b_k) \leq 2L\phi_k(2^{-1}a_k, 2^{-1}b_k)$, we conclude that

$$\lim_{m\to\infty} 2^{-m}\varphi(2^m a_k, 2^m b_k) = 0$$

for all $a_k, b_k \in \mathcal{A}_k$. In the inequality (2.3), replacing a_k, b_k by $2^m a_k$, $2^m b_k$, respectively, we obtain that

$$\begin{aligned}2^{-m} &\left\| 2F_k\left(x_1, \ldots, \frac{\lambda 2^m a_k + \lambda 2^m b_k}{2}, \ldots, x_n\right) - F_k(x_1, \ldots, 2^m a_k, \ldots, x_n)\right. \\ &\left. - F_k(x_1, \ldots, 2^m b_k, \ldots, x_n)\right\| \leq 2^{-m}\varphi_k(2^m a_k, 2^m b_k).\end{aligned}$$

Passing the limit $m \to \infty$, we obtain

$$2d_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) = \lambda d_k(x_1, \ldots, a_k, \ldots, x_n) + \lambda d_k(x_1, \ldots, b_k, \ldots, x_n)$$

for all $a_k, b_k \in \mathcal{A}_k$ and all $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}}$. Now, we show that d_k is $\mathbb{C}$-linear with respect to k-th variable. First suppose that λ belongs to T^1. Then $\lambda = e^{i\theta}$ for some $0 \le \theta \le 2\pi$. We set $\lambda_1 = e^{\frac{i\theta}{n_o}}$. Then λ_1 belongs to $T^1_{\frac{1}{n_o}}$ and

$$2d_k\left(x_1, \ldots, \frac{\lambda_1 a_k + \lambda_1 b_k}{2}, \ldots, x_n\right) = \lambda_1 d_k(x_1, \ldots, a_k, \ldots, x_n) + \lambda_1 d_k(x_1, \ldots, b_k, \ldots, x_n)$$

for all $a_k, b_k \in \mathcal{A}_k$. It is easy to show that d_k is additive with respect to k-th variable. Moreover, if λ belongs to $nT^1 = \{nz\ z \in T^1\}$ then by additivity of d_k on k-th variable, we have

$$d_k(x_1, \ldots, \lambda a_k, \ldots, x_n) = \lambda d_k(x_1, \ldots, a_k, \ldots, x_n)$$

for all $a_k \in \mathcal{A}_k$. If $t \in (0, \infty)$, then by Archimedean property of $\mathbb{C}$, there exists an $n \in \mathbb{N}$ such that the point $(t, 0)$ lies in the interior of circle with center at origin and radius n. Let $t_1 = t + \sqrt{n^2 - t^2}i \in nT^1$ and $t_2 = t - \sqrt{n^2 - t^2}i \in nT^1$. We have $t = \frac{t_1 + t_2}{2}$. Then

$$d_k(x_1, \ldots, ta_k, \ldots, x_n) = d_k\left(x_1, \ldots, \frac{t_1 + t_2}{2}a_k, \ldots, x_n\right) = \frac{t_1 + t_2}{2}d_k(x_1, \ldots, a_k, \ldots, x_n)$$

for all $a_k \in \mathcal{A}_k$. Let $\lambda \in \mathbb{C}$. Then $\lambda = |\lambda|e^{i\lambda_1}$ and so

$$d_k(x_1, \ldots, \lambda a_k, \ldots, x_n) = |\lambda|e^{i\lambda_1}d_k(x_1, \ldots, a_k, \ldots, x_n) = \lambda d_k(x_1, \ldots, a_k, \ldots, x_n)$$

for all $a_k \in \mathcal{A}_k$. It follows that d_k is $\mathbb{C}$-linear with respect to k-th variable.

By the same reasoning as above, we can show that the limit

$$D_k(x_1, \ldots, x_k, \ldots, x_n) := \lim_{m\to\infty} 2^{-m}T_k(x_1, \ldots, 2^m x_k, \ldots, x_n)$$

exists for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$ and that D_k is $\mathbb{C}$-linear with respect to k-th variable.

By te inequality $\Psi_k(a_k) \le 2L\Psi_k(2^{-1}a_k)$, we conclude that

$$\lim_{m\to\infty} 2^{-m}\Psi_k(2^m a_k) = 0$$

for all $a_k \in \mathcal{A}_k$.

Now, by (2.2), we have

$$\begin{aligned}&\|F_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n)a_k\|\\ &\le \Psi_k(a_k)\end{aligned}$$

for all $a_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. Replacing a_k by $2^m a_k$ in the above inequality, we obtain that

$$\begin{aligned}&\|F_k(x_1, \ldots, 2^{2m}a_k^2, \ldots, x_n) - 2^m a_k F_K(x_1, \ldots, 2^m a_k, \ldots, x_n)\\ &-F_k(x_1, \ldots, 2^m a_k, \ldots, x_n)2^m a_k\| \le \Psi_k(2^m a_k).\end{aligned}$$

Then we have

$$\begin{aligned}&\|2^{-2m}F_k(x_1, \ldots, 2^{2m}a_k^2, \ldots, x_n) - 2^{-m}a_k F_K(x_1, \ldots, 2^m a_k, \ldots, x_n)\\ &-F_k(x_1, \ldots, 2^m a_k, \ldots, x_n)2^{-m}a_k\| \le 2^{-2m}\Psi_k(2^m a_k)\end{aligned}$$

for all $a_k \in \mathcal{A}_k$. Passing $m \to \infty$, we obtain

$$d_k(x_1, \ldots, a_k^2, \ldots, x_n) = a_k d_k(x_1, \ldots, a_k, \ldots, x_n) + d_k(x_1, \ldots, a_k, \ldots, x_n)a_k$$

for all $a_k \in \mathcal{A}_k$ and all $x_i \in \mathcal{A}_i (i \neq k)$. This shows that d_k is a partial Jordan derivation. We have to show that D_k is a partial generalized Jordan derivation related to d_k. By (2.2), we have

$$\|T_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k T_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n)a_k\| \\ \leq \Psi_k(a_k)$$

for all $a_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. Replacing a_k by $2^m a_k$ in the last inequality, we get

$$\|T_k(x_1, \ldots, 2^{2m} a_k^2, \ldots, x_n) - 2^m a_k T_K(x_1, \ldots, 2^m a_k, \ldots, x_n) \\ - F_k(x_1, \ldots, 2^m a_k, \ldots, x_n) 2^m a_k\| \leq \Psi_k(2^m a_k)$$

for all $a_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. Then we have

$$\|2^{-2m} T_k(x_1, \ldots, 2^{2m} a_k^2, \ldots, x_n) - 2^{-m} a_k T_K(x_1, \ldots, 2^m a_k, \ldots, x_n) \\ - F_k(x_1, \ldots, 2^m a_k, \ldots, x_n) 2^{-m} a_k\| \leq 2^{-2m} \Psi_k(2^m a_k)$$

for all $a_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. Passing $m \to \infty$, we obtain that

$$D_k(x_1, \ldots, a_k^2, \ldots, x_n) = a_k D_k(x_1, \ldots, a_k, \ldots, x_n) + d_k(x_1, \ldots, a_k, \ldots, x_n)a_k$$

for all $a_k \in \mathcal{A}_k$ and all $x_i \in \mathcal{A}_i (i \neq k)$. Hence D_k is a partial generalized Jordan derivation related to d_k.

Corollary 2.2. *Let $p \in (0, 1)$ and $\theta \in [0, \infty)$ be real numbers. Let $T_k, F_k : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ be mappings such that $T_k(x_1, \ldots, 0_k, \ldots, x_n) = F_k(x_1, \ldots, 0_k, \ldots, x_n) = 0_{\mathcal{X}}$ and that*

$$\left\|2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n)\right\| \\ \leq \varphi_k(a_k, b_k),$$

$$\max\{\|F_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n)a_k\|, \\ \|T_k(x_1, \ldots, a_k^2, \ldots, x_n) - T_k(x_1, \ldots, a_k, \ldots, x_n)a_k - a_k F_k(x_1, \ldots, a_k, \ldots, x_n)\|\} \\ \leq \theta(\|a_k\|^p)$$

for $S_k \in \{F_k, T_k\}$ and for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}}$ and all $a_k, b_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. Then there exist a unique partial Jordan derivation of Jensen type with respect to k-th variable $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ and a unique partial generalized Jordan derivation of Jensen type with respect to k-th variable (related to d_k) $D_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ such that

$$\max\{\|F_k(x_1, x_2, \ldots, x_n) - d_k(x_1, x_2, \ldots, x_n)\|, \|T_k(x_1, x_2, \ldots, x_n) - D_k(x_1, x_2, \ldots, x_n)\|\} \\ \leq \frac{2^p}{2 - 2^p} \theta \|x_k\|^p$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$.

Proof. It follows from Theorem 2.1 by putting $\Psi_k(a_k) = \theta(||a_k||^p)$, $\phi_k(a_k, b_k) = \theta(|| a_k ||^p + ||b_k||^p)$ and $L = 2^{p-1}$. □

Theorem 2.3. *Let* $T_k, F_k : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *be mappings with* $T_k(x_1, \ldots, 0_k, \ldots, x_n) = F_k(x_1, \ldots, 0_k, \ldots, x_n) = 0_{\mathcal{X}}$. *Assume that there exist functions* $\Psi_k : \mathcal{A}_k \to [0, \infty)$, $\varphi_k : \mathcal{A}_k^2 \to [0, \infty)$ *satisfying*

$$\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n) \right\|, \\ \le \varphi_k(a_k, b_k),$$

$$\max\{\|F_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n)a_k\|, \\ \|T_k(x_1, \ldots, a_k^2, \ldots, x_n) - T_k(x_1, \ldots, a_k, \ldots, x_n)a_k - a_k F_k(x_1, \ldots, a_k, \ldots, x_n)\|\} \\ \le \Psi_k(a_k)$$

for $S_k \in \{F_k, T_k\}$ *and for all* $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and all* $a_k, b_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. *If there exists a constant* $0 < L < 1$ *such that* $\phi_k(a_k, b_k) \le 2^{-1}L\phi_k(2a_k, 2b_k)$, $\Psi_k(a_k) \le 2^{-1}L\Psi_k(2a_k)$ *for all* $a_k, b_k \in \mathcal{A}_k$, *then there exist a unique partial Jordan derivation of Jensen type with respect to k-th variable* $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *and a unique partial generalized Jordan derivation of Jensen type with respect to k-th variable (related to* d_k*)* $D_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *such that*

$$\max\{\|F_k(x_1, x_2, \ldots, x_n) - d_k(x_1, x_2, \ldots, x_n)\|, \|T_k(x_1, x_2, \ldots, x_n) - D_k(x_1, x_2, \ldots, x_n)\|\} \\ \le \frac{L}{2-2L}\varphi_k(2x_k, 0)$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let* $p \in (1, \infty)$ *and* $\theta \in [0, \infty)$ *be real numbers. Let* $T_k, F_k : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *be mappings such that* $\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n) \right\| \le \theta(\|a_k\|^p + \|b_k\|^p)$, *and*

$$\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n) \right\| \\ \le \theta(\|a_k\|^p + \|b_k\|^p),$$

$$\max\{\|F_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n)a_k\|, \\ \|T_k(x_1, \ldots, a_k^2, \ldots, x_n) - T_k(x_1, \ldots, a_k, \ldots, x_n)a_k - a_k F_k(x_1, \ldots, a_k, \ldots, x_n)\|\} \\ \le \theta(\|a_k\|^p)$$

for $S_k \in \{F_k, T_k\}$ *and for all* $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and all* $a_k, b_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. *Then there exist a unique partial Jordan derivation of Jensen type with respect to k-th variable* $d_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *and a unique partial generalized Jordan derivation of Jensen type with respect to k-th variable (related to* d_k*)* $D_k : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ *such that*

$$\max\{\|F_k(x_1, x_2, \ldots, x_n) - d_k(x_1, x_2, \ldots, x_n)\|, \|T_k(x_1, x_2, \ldots, x_n) - D_k(x_1, x_2, \ldots, x_n)\|\} \\ \le \frac{\theta}{1-2^{p-1}}\|x_k\|^p$$

for all $x_i \in \mathcal{A}_i (i = 1, 2, \ldots, n)$.

Proof. It follows from Theorem 2.3 by putting $\Psi_k(a_k)=\theta(||a_k||^p)$, $\phi_k(a_k, b_k) = \theta(||a_k||^p + ||b_k||^p)$ and $L = 2^{1-p}$ for each $a_k, b_k \in \mathcal{A}_k$. □

Moreover, we have the following result for the superstability of partial generalized Jordan derivations of Jensen type.

Corollary 2.5. *Let* $p \in (0, \frac{1}{2})$ *and* $\theta \in [0, \infty)$ *be real numbers. Let* $T_k, F_k : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{X}$ be mappings such that $T_k(x_1, \ldots, 0_k, \ldots, x_n) = F_k(x_1, \ldots, 0_k, \ldots, x_n) = 0_{\mathcal{X}}$ and

$$\left\| 2S_k\left(x_1, \ldots, \frac{\lambda a_k + \lambda b_k}{2}, \ldots, x_n\right) - \lambda S_k(x_1, \ldots, a_k, \ldots, x_n) - \lambda S_k(x_1, \ldots, b_k, \ldots, x_n) \right\| \leq \theta(\|a_k\|^p \|b_k\|^p),$$

$$\max\{\|F_k(x_1, \ldots, a_k^2, \ldots, x_n) - a_k F_k(x_1, \ldots, a_k, \ldots, x_n) - F_k(x_1, \ldots, a_k, \ldots, x_n) a_k\|, \|T_k(x_1, \ldots, a_k^2, \ldots, x_n) - T_k(x_1, \ldots, a_k, \ldots, x_n) a_k - a_k F_k(x_1, \ldots, a_k, \ldots, x_n)\|\} \leq \theta(\|a_k\|^p)$$

for $S_k \in \{F_k, T_k\}$ *and for all* $\lambda \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and all* $a_k, b_k \in \mathcal{A}_k$, $x_i \in \mathcal{A}_i (i \neq k)$. *Then* F_k *is a partial Jordan derivation of Jensen type with respect to k-th variable and* T_k *is a partial generalized Jordan derivation of Jensen type with respect to k-th variable (related to* F_k*).*

Proof. It follows from Theorem 2.1 by putting $\Psi_k(a_k)=\theta(||a_k||^p)$, $\phi_k(a_k, b_k) = \theta(||a_k||^p + ||b_k||^p)$, and $L = 2^{2p-1}$. □

Acknowledgements
This work was supported by the Daejin University Research Grants in 2012.

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 27 October 2011 Accepted: 29 May 2012 Published: 29 May 2012

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doi:10.1186/1029-242X-2012-119
Cite this article as: Gordji *et al.*: Almost partial generalized Jordan derivations: a fixed point approach. *Journal of Inequalities and Applications* 2012 **2012**:119.