Journal of Combinatorial Theory, Series A 119 (2012) 397–419

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Young tableaux and crystal $\mathcal{B}(\infty)$ for the exceptional Lie algebra types

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ARTICLE INFO

Article history:
Received 12 March 2011
Available online 12 October 2011

Keywords:
Crystal base
Quantum group
Young tableau
Exceptional type Lie algebra

ABSTRACT

We study the crystal base $\mathcal{B}(\infty)$ associated with the negative part of the quantum group for finite simple Lie algebras of types E_6, E_7, E_8, and F_4. After describing the crystal $\mathcal{B}(\infty)$ as a union of highest weight crystals, we give an explicit description of $\mathcal{B}(\infty)$ in terms of Young tableaux.

1. Introduction

The quantum group $U_q(\mathfrak{g})$ is a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ and the crystal base $\mathcal{B}(\infty)$ presents the bare skeleton structure of its negative part $U_q^-(\mathfrak{g})$. The crystal $\mathcal{B}(\infty)$ has received attention since the very birth of crystal base theory [11,22] as an integral part of the grand loop argument proving the existence of crystal bases and substantial efforts have been made to give explicit descriptions of the crystal $\mathcal{B}(\infty)$. A variety of tools, such as Kashiwara embedding [1,12,24], Littelmann's path [19–21], and quiver varieties [16,25], were used for this purpose and there were many other approaches, with [3,7,9,18] being a partial list.

Most of these are specific to a certain class of Lie algebra types and explicit descriptions of $\mathcal{B}(\infty)$ based on the Young tableaux for the classical simple Lie algebras and the G_2 type Lie algebra were presented in [3] by the authors of this paper. The current paper provides analogous realizations for the remaining finite simple Lie algebras of types E_6, E_7, E_8, and F_4. Even though this work extends

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[1] This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0004561).
[2] This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (KRF-2008-313-C00018).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2011.09.009

the result of [3] to other Lie algebra types, as we shall soon explain, the approaches used in the two works are different.

The previous work [3] started from the concrete Young tableau realizations $\mathcal{T}(\lambda)$ of the highest weight crystals $\mathcal{B}(\lambda)$ that were announced in [15,10]. First, tableaux in $\mathcal{T}(\lambda)$ of certain shapes were named as *large* tableaux with $\mathcal{T}(\lambda)^L$ denoting their collection and these were gathered into a single set $\bigcup_\lambda \mathcal{T}(\lambda)^L$. Then, many of the large tableaux were identified with each other to form a set of equivalence classes $\bigcup_\lambda \mathcal{T}(\lambda)^L/\sim$. Finally, a crystal structure was given to $\bigcup_\lambda \mathcal{T}(\lambda)^L/\sim$ and this was shown to be crystal isomorphic to $\mathcal{B}(\infty)$.

We start the current paper with a result that captures the structure that is common to the five separate realizations of [3]. The claim is that, if the full sets $\mathcal{B}(\lambda)$ are gather into a single set $\bigcup_\lambda \mathcal{B}(\lambda)$ and certain identifications are made within this infinite union, then the resulting set of equivalence classes $\bigcup_\lambda \mathcal{B}(\lambda)/\sim$ has a natural crystal structure induced from those on each $\mathcal{B}(\lambda)$ and is crystal isomorphic to $\mathcal{B}(\infty)$. The previous realization that had gathered only the smaller subsets $\mathcal{T}(\lambda)^L \subset \mathcal{B}(\lambda)$ consisting of large tableaux had worked because each final equivalence class in $\bigcup_\lambda \mathcal{B}(\lambda)/\sim$ contained at least one large tableau. The description of $\mathcal{B}(\infty)$ as $\bigcup_\lambda \mathcal{B}(\lambda)/\sim$ is valid for all symmetrizable Kac–Moody algebras and a very similar result had appeared in [13].

Based on this preliminary result, one can attempt to construct a realization of $\mathcal{B}(\infty)$ as follows. One first develops a way to express elements of the crystals $\mathcal{B}(\lambda)$ for whatever Lie algebra type that is under consideration. Then, the identifications made within $\bigcup_\lambda \mathcal{B}(\lambda)$ are translated to those on the set of developed expressions. Finally, a suitable set of representatives for each equivalence class is collected as an explicit realization.

Contrary to the situation in [3], Young tableau realizations of the highest weight crystals $\mathcal{B}(\lambda)$, of the style given by [15], are currently not available for the E_6, E_7, E_8, and F_4 types. This difference makes the first of the above mentioned three steps difficult to achieve, but our previous experience with the classical types allows us to expect that it may be enough to express only certain subsets of each $\mathcal{B}(\lambda)$ as tableaux to achieve our goal, and this is the approach we take. Using the Nakajima monomial theory [14,23], we explicitly draw the crystal graph for one of the simplest crystals available for each Lie algebra type and refer to these as the basic crystals. Then, the crystal $\mathcal{B}(\lambda)$ is located as a sub-crystal in the tensor product of a suitable number of copies of the basic crystal. Thus elements of $\mathcal{B}(\lambda)$ can be viewed as tableaux with entries filled with elements of the basic crystal. Instead of trying to characterize all elements of $\mathcal{B}(\lambda)$ as tableaux, we show that certain *large* tableaux must belong to $\mathcal{B}(\lambda)$. Finally, the equivalence among crystal elements in $\bigcup_\lambda \mathcal{B}(\lambda)$ is related to the equivalence among large tableaux and we fix an explicit representative tableau from each equivalence class to arrive at an explicit realization of $\mathcal{B}(\infty)$.

Let us briefly discuss two possible applications of this work. Nakajima monomial descriptions of $\mathcal{B}(\infty)$ for the classical and G_2 types were found in [17] by locating the Nakajima monomials that are in natural correspondence with elements of the Young tableau realization given by [3]. We expect a similar approach based on results of this paper to work for the remaining exceptional Lie algebra types. Another direction of future study is to use the large tableaux defined in this paper to describe $\mathcal{B}(\infty)$ as the image of the Kashiwara embedding in a manner analogous to what was done in [1].

The rest of this paper is organized as follows. In the next section, we show that the crystal $\mathcal{B}(\infty)$ can be expressed as $\bigcup_\lambda \mathcal{B}(\lambda)/\sim$. In Section 3 we define the large tableaux and prepare some lemmas whose proofs are specific to each Lie algebra type under consideration. In the final section, we discuss the identification among tableaux and present an explicit realization of $\mathcal{B}(\infty)$ in terms of tableaux. The basic crystals that form the entries of all tableaux are explicitly given in Appendix A.

2. Crystal $\mathcal{B}(\infty)$ as a union of $\mathcal{B}(\lambda)$

We assume knowledge of the basic theory of crystal bases. Standard notation, as may be found in the textbooks [2,5], will be used. In particular, we assume familiarity with the following notions and notation: index set I, simple root α_i, coroot h_i, fundamental weight Λ_i, positive root lattice Q_+, set of dominant integral weights P^+, quantum group $U_q(\mathfrak{g})$, irreducible highest weight module $V(\lambda)$, crystal lattice $L(\lambda)$, abstract crystal with associated Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ and maps wt, ε_i, φ_i, irreducible highest weight crystal $\mathcal{B}(\lambda)$, tensor product rule, i-signature, negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$, crystal lattice $L(\infty)$ of $U_q^-(\mathfrak{g})$, and crystal base $\mathcal{B}(\infty)$ of $U_q^-(\mathfrak{g})$.

In this section, we show that the crystal $\mathcal{B}(\infty)$ can be expressed as a union of highest weight crystals $\mathcal{B}(\lambda)$. The arguments of this section are for all symmetrizable Kac–Moody algebras. The following statement is a slight modification of Theorem 5 appearing in [11].

Theorem 2.1. *For any weight $\lambda \in P^+$, let $\pi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ be the $U_q^-(\mathfrak{g})$-linear homomorphism that sends* 1 *to the highest weight vector v_λ.*

1. *We have $\pi_\lambda(L(\infty)) = L(\lambda)$ and this implies that π_λ induces a surjective homomorphism $\bar{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda)$.*
2. *The induced mapping $\bar{\pi}_\lambda$ is a bijection between $\{b \in \mathcal{B}(\infty) \mid \bar{\pi}_\lambda(b) \neq 0\}$ and $\mathcal{B}(\lambda)$.*
3. *We have $\tilde{f}_i \circ \bar{\pi}_\lambda = \bar{\pi}_\lambda \circ \tilde{f}_i$, so that the mapping $\bar{\pi}_\lambda$ sends $\tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} b_\infty$ to $\tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} b_\lambda$.*
4. *If $b \in \mathcal{B}(\infty)$ satisfies $\bar{\pi}_\lambda(b) \neq 0$, then $\tilde{e}_i \bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_i b)$.*

The notation $\bar{\pi}_\lambda$ appearing in this theorem is used throughout this paper. The theorem implies that every element of $\mathcal{B}(\lambda)$ is an image of exactly one element from $\mathcal{B}(\infty)$ under $\bar{\pi}_\lambda$.

Definition 2.2. Two elements from the disjoint union $\bigcup_{\lambda \in P^+} \mathcal{B}(\lambda)$ are defined to be equivalent if and only if they correspond to the same element of $\mathcal{B}(\infty)$. This is clearly an equivalence relation. We fix the notation

$$\mathcal{B}(\cup) = \bigcup_{\lambda \in P^+} \mathcal{B}(\lambda)/\sim$$

for the set of such equivalence classes.

Any equivalence class $\overline{\bar{\pi}_\lambda(b)} \in \mathcal{B}(\cup)$ designates a unique element $b \in \mathcal{B}(\infty)$ and hence there is a natural mapping from $\mathcal{B}(\cup)$ to $\mathcal{B}(\infty)$. The following restatement of Corollary 4.4.5 from [11] allows us to show that this mapping is bijective.

Lemma 2.3. *Given any $\xi \in Q_+$, we have $|\mathcal{B}(\infty)_{-\xi}| = |\mathcal{B}(\lambda)_{\lambda-\xi}|$, for all $\lambda \in P^+$ such that $\lambda(h_i) \gg 0$ for every $i \in I$.*

This lemma and the fact that $\bar{\pi}_\lambda$ is surjective implies that the restricted map $\bar{\pi}_\lambda : \mathcal{B}(\infty)_{-\xi} \to \mathcal{B}(\lambda)_{\lambda-\xi}$ is bijective for all sufficiently large λ. Hence, given any element $b \in \mathcal{B}(\infty)$, we can always find a sufficiently large $\lambda \in P^+$ such that $\bar{\pi}_\lambda(b)$ is nonzero. Since every nonzero $\bar{\pi}_\lambda(b)$ belongs to the same equivalence class in $\mathcal{B}(\cup)$, the mapping $b \mapsto \overline{\bar{\pi}_\lambda(b)}$ is well defined. Let us use the notation

$$\bar{\pi}_\cup : \mathcal{B}(\infty) \to \mathcal{B}(\cup)$$

for this map. This is the inverse map to the natural map from $\mathcal{B}(\cup)$ to $\mathcal{B}(\infty)$ and hence is bijective.

The bijective map $\bar{\pi}_\cup$ allows us to copy the crystal structure of $\mathcal{B}(\infty)$ onto that of $\mathcal{B}(\cup)$. Let us explain the Kashiwara operator $\tilde{e}_i$ on $\mathcal{B}(\cup)$ as an example. Given any $\overline{\bar{\pi}_\lambda(b)} \in \mathcal{B}(\cup)$, if $\tilde{e}_i b$ is zero, we define the $\tilde{e}_i(\overline{\bar{\pi}_\lambda(b)})$ also to be zero. If otherwise, we choose a $\mu \in P^+$ that is large enough to make $\bar{\pi}_\mu(\tilde{e}_i b)$ nonzero and define $\tilde{e}_i(\overline{\bar{\pi}_\lambda(b)}) = \overline{\bar{\pi}_\mu(\tilde{e}_i b)}$. This process is well defined since an equivalence class $\overline{\bar{\pi}_\lambda(b)} \in \mathcal{B}(\cup)$ uniquely identifies an element $b \in \mathcal{B}(\infty)$ and every nonzero $\bar{\pi}_\mu(\tilde{e}_i b)$ gives the same equivalence class in $\mathcal{B}(\cup)$.

We would like to related the crystal structure we have just given to $\mathcal{B}(\cup)$ with that found on $\mathcal{B}(\lambda)$.

Lemma 2.4. *Fix an $i \in I$ and let $u \in \mathcal{B}(\lambda)$.*

1. *If $\tilde{f}_i u$ is nonzero, then $\tilde{f}_i \bar{u}$ is equal to $\overline{\tilde{f}_i u}$, which is nonzero.*
2. *If $\tilde{e}_i u$ is nonzero, then $\tilde{e}_i \bar{u}$ is equal to $\overline{\tilde{e}_i u}$, which is nonzero.*
3. *If $\tilde{e}_i u$ is zero, then $\tilde{e}_i \bar{u}$ is zero.*

Proof. According to Theorem 2.1, we may express an arbitrary nonzero element $u \in \mathcal{B}(\lambda)$ as $\bar{\pi}_\lambda(b)$ for some $b \in \mathcal{B}(\infty)$. For the remainder of this proof, we take $u = \bar{\pi}_\lambda(b)$.

To show the first claim, we start from

$$\tilde{f}_i u = \tilde{f}_i\big(\bar{\pi}_\lambda(b)\big) = \bar{\pi}_\lambda(\tilde{f}_i b),$$

where the second equality follows from Theorem 2.1. Since $\tilde{f}_i u \in \mathcal{B}(\lambda)$ is nonzero, the first of the following sequence of equalities is true.

$$\overline{\tilde{f}_i u} = \bar{\pi}_\cup(\tilde{f}_i b) = \tilde{f}_i\big(\bar{\pi}_\cup(b)\big) = \tilde{f}_i \bar{u}.$$

The second equality is a consequence of the crystal structure on $\mathcal{B}(\cup)$ being a copy of that on $\mathcal{B}(\infty)$ and the final equality depends on $u = \bar{\pi}_\lambda(b)$ being nonzero.

Similarly, we can start from

$$\tilde{e}_i u = \tilde{e}_i \bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_i b),$$

and, when $\tilde{e}_i u$ is nonzero, write

$$\overline{\tilde{e}_i u} = \bar{\pi}_\cup(\tilde{e}_i b) = \tilde{e}_i \bar{\pi}_\cup(b) = \tilde{e}_i \bar{u}$$

to arrive at the second claim.

As for the final claim, when $\tilde{e}_i u$ is zero for a nonzero $u \in \mathcal{B}(\lambda)$, we note that

$$0 = \tilde{f}_i \tilde{e}_i u = \tilde{f}_i \tilde{e}_i \bar{\pi}_\lambda(b) = \tilde{f}_i \bar{\pi}_\lambda(\tilde{e}_i b) = \bar{\pi}_\lambda(\tilde{f}_i \tilde{e}_i b).$$

Now, unless $\tilde{e}_i b$ is zero, the final term of this sequence of equalities is equal to the nonzero value $\bar{\pi}_\lambda(b) = u$. Hence $\tilde{e}_i b$ can only be zero and $\tilde{e}_i \bar{u} = \tilde{e}_i \bar{\pi}_\cup(b)$ must be zero by definition of the crystal structure on $\mathcal{B}(\cup)$. □

This lemma shows that the Kashiwara operator $\tilde{e}_i$ on $\mathcal{B}(\lambda)$ is identical to that on $\mathcal{B}(\cup)$. As for the $\tilde{f}_i$ operator, the following lemma fills in the missing part.

Lemma 2.5. *Fix an $i \in I$ and choose any $\bar{\pi}_\cup(b) \in \mathcal{B}(\cup)$. Then $\tilde{f}_i(\bar{\pi}_\lambda(b)) = \bar{\pi}_\lambda(\tilde{f}_i b)$ is nonzero for all sufficiently large $\lambda \in P^+$.*

Proof. Referring to Eq. (3.5.6) of [11], we know that $\tilde{f}_i$ is never zero on $\mathcal{B}(\infty)$, so that $\tilde{f}_i b$ is nonzero. Then, as was discussed while defining the map $\bar{\pi}_\cup$, Lemma 2.3 can be used to show that $\bar{\pi}_\lambda(\tilde{f}_i b)$ is nonzero for all sufficiently large λ. It now suffices to utilize $\bar{\pi}_\lambda \circ \tilde{f}_i = \tilde{f}_i \circ \bar{\pi}_\lambda$ from Theorem 2.1 to arrive at our claim. □

Let us now define a second crystal structure on $\mathcal{B}(\cup)$. We start with the Kashiwara operator $\tilde{f}_i$. Given any $\bar{\pi}_\cup(b) \in \mathcal{B}(\cup)$, Lemma 2.5 allows us to choose a λ such that $\tilde{f}_i(\bar{\pi}_\lambda(b))$ is nonzero. We define $\tilde{f}_i \bar{\pi}_\cup(b) = \overline{\tilde{f}_i(\bar{\pi}_\lambda(b))}$, using any such λ. Since $\tilde{f}_i(\bar{\pi}_\lambda(b)) = \bar{\pi}_\lambda(\tilde{f}_i b)$, the equivalence class does not depend on the choice of λ.

The definition of $\tilde{e}_i$ on $\bar{\pi}_\cup(b) \in \mathcal{B}(\cup)$ is even more simple. We fix any representative $\bar{\pi}_\lambda(b) \in \mathcal{B}(\lambda)$. If $\tilde{e}_i\bar{\pi}_\lambda(b)$ is zero, we define $\tilde{e}_i(\bar{\pi}_\cup(b))$ to be zero, and, if otherwise, we define $\tilde{e}_i(\bar{\pi}_\cup(b)) = \overline{\tilde{e}_i(\bar{\pi}_\lambda(b))}$. To see that this is well defined, we first observe from Lemma 2.4 that $\tilde{e}_i\bar{\pi}_\lambda(b)$ is zero for some $\lambda \in P^+$ if and only if $\tilde{e}_i\bar{\pi}_\mu(b)$ is zero for every $\mu \in P^+$. Furthermore, since we know that the previously defined $\tilde{e}_i$ on $\mathcal{B}(\cup)$ is a well defined operator, the lemma implies that the equivalence class $\overline{\tilde{e}_i u}$ does not depend on the choice of the representative $u \in \mathcal{B}(\lambda)$. Hence, the equivalence class $\overline{\tilde{e}_i(\bar{\pi}_\lambda(b))}$ does not depend on λ.

The weight function on $\mathcal{B}(\cup)$ can be given by setting $\mathrm{wt}(\bar{\pi}_\cup(b)) = \mathrm{wt}(b)$. This is equivalent to setting $\mathrm{wt}(\bar{\pi}_\cup(b)) = \mathrm{wt}(\bar{\pi}_\lambda(b)) - \lambda$, for any choice of $\lambda \in P^+$ such that $\bar{\pi}_\lambda(b) \neq 0$. The fact that $\tilde{e}_i$ on $\bar{\pi}_\cup(b) \in \mathcal{B}(\cup)$ can be computed from any nonzero representative $\bar{\pi}_\lambda(b) \in \mathcal{B}(\lambda)$ allows us to define $\varepsilon_i(\bar{\pi}_\cup(b)) = \varepsilon_i(\bar{\pi}_\lambda(b))$ using any such choice of $\lambda \in P^+$. The final component can be defined through $\varphi_i(\bar{\pi}_\cup(b)) = \varepsilon_i(\bar{\pi}_\cup(b)) + \mathrm{wt}(b)(h_i)$.

Let us refer to the crystal structure on $\mathcal{B}(\cup)$ that we have just described as the crystal structure *induced* from $\mathcal{B}(\lambda)$. The previous two lemmas imply that the two crystal structures given to the set of equivalence classes $\mathcal{B}(\cup)$ are identical. We write what we have discussed as a theorem.

Theorem 2.6. *The set of equivalence classes $\mathcal{B}(\cup) = \bigcup_{\lambda \in P^+} \mathcal{B}(\lambda)/\sim$ can be given a crystal structure induced from those on each $\mathcal{B}(\lambda)$. When $\mathcal{B}(\cup)$ is given this crystal structure, the bijection $\bar{\pi}_\cup : \mathcal{B}(\infty) \to \mathcal{B}(\cup)$ is an isomorphism of crystals.*

We remark that this result is essentially identical to the expression

$$\mathcal{B}(\infty) = \varinjlim_{\lambda \in P^+} \mathcal{B}(\lambda) \otimes T_{-\lambda},$$

which appears in [13, Chapitre 7]. We refer the interested readers to [13] for the notation $T_{-\lambda}$ and the various mappings associated with this direct limit expression.

Let us briefly return to the fact that the mapping $\bar{\pi}_\lambda$ gives a bijection between $\{b \in \mathcal{B}(\infty) \mid \bar{\pi}_\lambda(b) \neq 0\}$ and $\mathcal{B}(\lambda)$. This allows us to view each $\mathcal{B}(\lambda)$ as a subset of crystal $\mathcal{B}(\infty)$ and, under this identification, Lemma 2.3 implies that

$$\mathcal{B}(\infty) = \bigcup_{\lambda \in P^+} \mathcal{B}(\lambda).$$

The above theorem amounts to showing that this identity of sets can be viewed as that between crystals if slight care is taken, for example, with the $\tilde{f}_i$ action.

For later use, we provide a slight generalization of this result. Let us say that a subset $\tilde{P}$ of P^+ *contains sufficiently large weights*, if it satisfies the property that, given any set $\{n_i\}_{i \in I}$ of positive integers, there exists $\lambda \in \tilde{P}$ satisfying $\lambda(h_i) > n_i$ for all $i \in I$.

Proposition 2.7. *If $\tilde{P} \subset P^+$ contains sufficiently large weights, then the set of equivalence classes $\bigcup_{\lambda \in \tilde{P}} \mathcal{B}(\lambda)/\sim$ is identical to $\mathcal{B}(\cup)$. The various functions providing $\mathcal{B}(\cup)$ with a crystal structure can be computed using representatives belonging to $\mathcal{B}(\lambda)$ with λ restricted to $\tilde{P}$.*

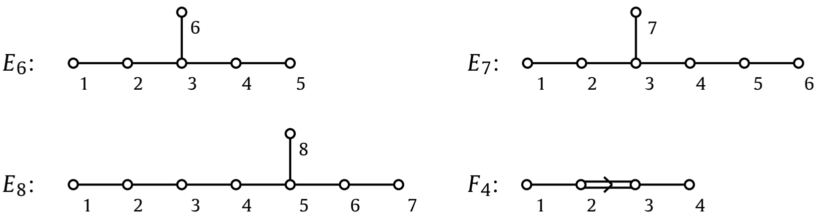
Proof. The validity of the first statement follows from Lemma 2.3. As for the rest, it suffices to review the definitions giving the induced crystal structure, taking note of the fact that Lemma 2.5 will provide at least one λ belonging to the smaller set $\tilde{P}$ for which $\tilde{f}_i(\bar{\pi}_\lambda(b))$ is nonzero. □

The task of finding an explicit description of $\mathcal{B}(\infty)$ is now transformed to choosing a good set of representatives for $\mathcal{B}(\cup)$ and describing the Kashiwara operators on the representative set.

3. Large tableaux

For the remainder of this paper, we restrict ourselves to the finite simple Lie algebras of types E_6, E_7, E_8, and F_4. Much of the arguments in this section needs to be given separately for each Lie algebra type. We will always work mainly with the E_6 type and provide additional remarks concerning other types if proofs or arguments concerning them contain any marked differences.

We first fix the indices for each of the four Lie algebra types under consideration as follows.



The direction of the arrow appearing in the Dynkin diagram for the F_4 type follows the setting of [8].

For the purpose of this paper, it will suffice to consider highest weight crystals for which the highest weights belong to a certain subset $\hat{P}^+$ of P^+. We define these subsets $\hat{P}^+$ for each Lie algebra type as follows

$$E_6:\quad \hat{P}^+ = \left\{ \sum_{i=1}^{6} c_i\Lambda_i \in P^+ \;\middle|\; c_6 = c_4 + 2c_5 \right\},$$

$$E_7:\quad \hat{P}^+ = \left\{ \sum_{i=1}^{7} c_i\Lambda_i \in P^+ \;\middle|\; c_7 = 2c_1 + c_2 \right\},$$

$$E_8:\quad \hat{P}^+ = \left\{ \sum_{i=1}^{8} c_i\Lambda_i \in P^+ \;\middle|\; c_8 = c_6 + 2c_7 \right\},$$

$$F_4:\quad \hat{P}^+ = \left\{ \sum_{i=1}^{4} c_i\Lambda_i \in P^+ \;\middle|\; c_3 \geqslant c_1 \right\}.$$

Each $\hat{P}^+$ may equivalently be given in the following form

$$E_6:\quad \mathbf{Z}_{\geqslant 0}\Lambda_1 + \mathbf{Z}_{\geqslant 0}\Lambda_2 + \mathbf{Z}_{\geqslant 0}\Lambda_3 + \mathbf{Z}_{\geqslant 0}(\Lambda_4 + \Lambda_6) + \mathbf{Z}_{\geqslant 0}(\Lambda_5 + 2\Lambda_6),$$

$$E_7:\quad \mathbf{Z}_{\geqslant 0}\Lambda_6 + \mathbf{Z}_{\geqslant 0}\Lambda_5 + \mathbf{Z}_{\geqslant 0}\Lambda_4 + \mathbf{Z}_{\geqslant 0}\Lambda_3 + \mathbf{Z}_{\geqslant 0}(\Lambda_2 + \Lambda_7) + \mathbf{Z}_{\geqslant 0}(\Lambda_1 + 2\Lambda_7),$$

$$E_8:\quad \mathbf{Z}_{\geqslant 0}\Lambda_1 + \mathbf{Z}_{\geqslant 0}\Lambda_2 + \cdots + \mathbf{Z}_{\geqslant 0}\Lambda_5 + \mathbf{Z}_{\geqslant 0}(\Lambda_6 + \Lambda_8) + \mathbf{Z}_{\geqslant 0}(\Lambda_7 + 2\Lambda_8),$$

$$F_4:\quad \mathbf{Z}_{\geqslant 0}\Lambda_4 + \mathbf{Z}_{\geqslant 0}\Lambda_3 + \mathbf{Z}_{\geqslant 0}\Lambda_2 + \mathbf{Z}_{\geqslant 0}(\Lambda_1 + \Lambda_3).$$

Notice that each $\hat{P}^+$ contains sufficiently large weights, in the sense described above Proposition 2.7.

The goal of this paper is to express $\mathcal{B}(\infty)$ in terms of tableaux. Access to realizations of the crystals $\mathcal{B}(\lambda)$, given in terms of tableaux, would allow us to achieve this goal through the statement $\mathcal{B}(\cup) \cong \mathcal{B}(\infty)$, given by Theorem 2.6. Such tableau realizations are available for the classical finite types [15] and the G_2 type [10], and these provided the basis of the work [3], but analogous information for the Lie algebra types that are being considered in this paper is not yet available. Below, we consider $\mathcal{B}(\lambda)$ for $\lambda \in \hat{P}^+$ and express only some of its elements as tableaux. Such partial information will be enough for the purpose of this paper.

The first step to obtaining this partial information is to show that every $\mathcal{B}(\lambda)$ with $\lambda \in \hat{P}^+$ can be found within a tensor product of suitably many of the following crystal, given separately for each type.

$$E_6\colon\ \mathcal{B}(\Lambda_1), \qquad E_7\colon\ \mathcal{B}(\Lambda_6), \qquad E_8\colon\ \mathcal{B}(\Lambda_1), \qquad F_4\colon\ \mathcal{B}(\Lambda_4).$$

We will refer to these building blocks of the tableaux as the *basic crystals*. The crystal graphs for these basic crystals are given in Appendix A. Some of our later proofs will require direct access to these explicit crystal graphs. Elements of the basic crystals will be written as given in Appendix A. In particular, the highest weight element b_{Λ_1} of the basic crystal $\mathcal{B}(\Lambda_1)$ for type E_6 will be denoted by 1.

All the crystal graphs of Appendix A were obtained through applications of the Nakajima monomial [14] theory. Note that the crystal graphs we give for types E_6 and E_7 are identical to those found in [6] except for the labeling of arrows and crystal elements. In fact, the naming of crystal elements in our graphs follows the idea of [6], and the differences mainly originate from whether the Dynkin diagram labeling followed that of [4] or [8]. However, the graphs of [6] and those of this paper were constructed through different techniques.

A tableau with its boxes filled with elements from a single basic crystal will be viewed as a tensor product of crystal elements through the *far eastern reading*. For example, we are using the identification

$$\begin{array}{|c|c|c|c|}\hline x_5 & x_3 & x_2 & x_1 \\ \hline x_6 & x_4 & \multicolumn{2}{c}{} \\ \cline{1-2} x_7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \otimes x_6 \otimes x_7.$$

Applications of the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$ on a tableau will follow the tensor product rule, with the tableau seen as an element of the tensor product of suitably many basic crystals. The Kashiwara operators will act on one of the box entries constituting the tableau without changing the shape or entry arrangements of the tableau.

We first identify the highest weight tableaux for each weight belonging to the reduced set of dominant integral weights $\hat{P}^+$.

Proposition 3.1. *The following is a list of highest weight elements, which may be found in a tensor product of appropriately many basic crystals.*

(E6) *For Lie algebra type E_6, the tableau*

$$b_{\lambda_{E6}} := \begin{array}{lllll}
\overbrace{\qquad\qquad}^{a_5} & \overbrace{\qquad\qquad}^{a_4} & \overbrace{\qquad\qquad}^{a_3} & \overbrace{\qquad\qquad}^{a_2} & \overbrace{\qquad\qquad}^{a_1} \\
1 \cdots & & & & \cdots\ 1 \\
\bar{1}2 \cdots & & & \cdots\ \bar{1}2 & \\
\bar{2}3 \cdots & & \cdots\ \bar{2}3 & & \\
\bar{3}46 \cdots & \cdots\ \bar{3}46 & & & \\
\bar{4}56 \cdots \bar{4}56 & & & &
\end{array}$$

viewed as an element of the crystal $\mathcal{B}(\Lambda_1)^{\otimes(a_1+2a_2+\cdots+5a_5)}$, is a highest weight element of weight

$$\lambda_{E6} := a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3 + a_4(\Lambda_4 + \Lambda_6) + a_5(\Lambda_5 + 2\Lambda_6) \in \hat{P}^+.$$

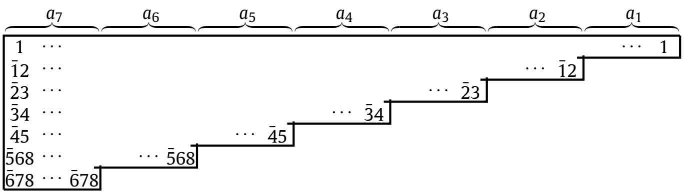
(E7) *For Lie algebra type E_7, the tableau*

$$b_{\lambda_{E7}} := \begin{array}{llllll}
\overbrace{\qquad\qquad}^{a_6} & \overbrace{\qquad\qquad}^{a_5} & \overbrace{\qquad\qquad}^{a_4} & \overbrace{\qquad\qquad}^{a_3} & \overbrace{\qquad\qquad}^{a_2} & \overbrace{\qquad\qquad}^{a_1} \\
6 \cdots & & & & & \cdots\ 6 \\
\bar{6}5 \cdots & & & & \cdots\ \bar{6}5 & \\
\bar{5}4 \cdots & & & \cdots\ \bar{5}4 & & \\
\bar{4}3 \cdots & & \cdots\ \bar{4}3 & & & \\
\bar{3}27 \cdots & \cdots\ \bar{3}27 & & & & \\
\bar{2}17 \cdots \bar{2}17 & & & & &
\end{array}$$

viewed as an element of the crystal $\mathcal{B}(\Lambda_6)^{\otimes(a_1+2a_2+3a_3+4a_4+5a_5+6a_6)}$*, is a highest weight element of weight*

$$\lambda_{E7} := a_1\Lambda_6 + a_2\Lambda_5 + a_3\Lambda_4 + a_4\Lambda_3 + a_5(\Lambda_2 + \Lambda_7) + a_6(\Lambda_1 + 2\Lambda_7) \in \hat{P}^+.$$

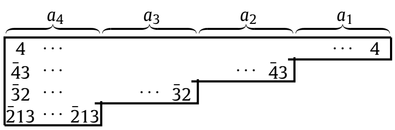
(E8) *For Lie algebra type* E_8*, the tableau*

$b_{\lambda_{E8}} :=$ 

viewed as an element of the crystal $\mathcal{B}(\Lambda_1)^{\otimes(a_1+2a_2+\cdots+7a_7)}$*, is a highest weight element of weight*

$$\lambda_{E8} := a_1\Lambda_1 + a_2\Lambda_2 + \cdots + a_5\Lambda_5 + a_6(\Lambda_6 + \Lambda_8) + a_7(\Lambda_7 + 2\Lambda_8) \in \hat{P}^+.$$

(F4) *For Lie algebra type* F_4*, the tableau*

$b_{\lambda_{F4}} :=$ 

viewed as an element of the crystal $\mathcal{B}(\Lambda_4)^{\otimes(a_1+2a_2+3a_3+4a_4)}$*, is a highest weight element of weight*

$$\lambda_{F4} := a_1\Lambda_4 + a_2\Lambda_3 + a_3\Lambda_2 + a_4(\Lambda_1 + \Lambda_3) \in \hat{P}^+.$$

Proof. Let us explain only the E_6 case, as the other cases may be approached similarly. Recall the notation for elements of $\mathcal{B}(\Lambda_1)$, as given in Appendix A. First, the tableau or crystal element $\boxed{1} = 1 \in \mathcal{B}(\Lambda_1)$ is a highest weight element of weight Λ_1. Next, one can directly check through an application of the tensor product rule that the tableau $\begin{array}{|c|}\hline 1 \\ \hline \bar{1}2 \\ \hline\end{array} = 1 \otimes \bar{1}2$ is a highest weight element of weight Λ_2. Similarly, one can check that each of the single column tableaux $1 \otimes \bar{1}2 \otimes \bar{2}3$, $1 \otimes \bar{1}2 \otimes \bar{2}3 \otimes \bar{3}46$, and $1 \otimes \bar{1}2 \otimes \bar{2}3 \otimes \bar{3}46 \otimes \bar{4}56$ are highest weight elements of weights Λ_3, $\Lambda_4 + \Lambda_6$, and $\Lambda_5 + 2\Lambda_6$, respectively. The tableau $b_{\lambda_{E6}}$ presented by this proposition is a tensor product of the highest weight elements we have discussed, hence must also be a highest weight element. It now suffices to add the weights to verify the final claim concerning the weight. □

In the above proof we first constructed certain single column highest weight elements and combined them into highest weight elements of more general weights. We shall refer to these single column highest weight tableaux as the *basic columns*. Specifically, they are given as follows for each Lie algebra type

$$\begin{aligned}
E_6&: \quad b_{\Lambda_1}, b_{\Lambda_2}, b_{\Lambda_3}, b_{\Lambda_4+\Lambda_6}, b_{\Lambda_5+2\Lambda_6}, \\
E_7&: \quad b_{\Lambda_6}, b_{\Lambda_5}, b_{\Lambda_4}, b_{\Lambda_3}, b_{\Lambda_2+\Lambda_7}, b_{\Lambda_1+2\Lambda_7}, \\
E_8&: \quad b_{\Lambda_1}, b_{\Lambda_2}, b_{\Lambda_3}, b_{\Lambda_4}, b_{\Lambda_5}, b_{\Lambda_6+\Lambda_8}, b_{\Lambda_7+2\Lambda_8}, \\
F_4&: \quad b_{\Lambda_4}, b_{\Lambda_3}, b_{\Lambda_2}, b_{\Lambda_1+\Lambda_3}.
\end{aligned} \tag{3.2}$$

If a tableau contains a column that looks identical to any one of the basic columns listed for its Lie algebra type, the column within the tableau is also referred to as a basic column. A column within a tableau can be a basic column only if it is a whole column and not a part of a column.

The notation introduced in Proposition 3.1, namely, the weights λ_{E6}, λ_{E7}, λ_{E8}, λ_{F4} and the highest weight elements $b_{\lambda_{E6}}$, $b_{\lambda_{E7}}$, $b_{\lambda_{E8}}$, $b_{\lambda_{F4}}$, will be used throughout this paper.

Let us present our next argument in terms of the E_6 type. The expression

$$\lambda_{E6} = a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3 + a_4(\Lambda_4 + \Lambda_6) + a_5(\Lambda_5 + 2\Lambda_6)$$

presents the most general element of $\hat{P}^+$. Note that the highest weight tableaux $b_{\lambda_{E6}}$ corresponding to different weights $\lambda_{E6} \in \hat{P}^+$ will have different outer silhouettes. If the outer silhouette of a tableau is identical to that of a $b_{\lambda_{E6}}$ and the tableau has its boxes filled with elements from the basic crystal $\mathcal{B}(\Lambda_1)$, we shall say that it is a tableau of *shape* λ_{E6}.

For each weight $\lambda_{E6} \in \hat{P}^+$, we define $\mathcal{T}(\lambda_{E6})$ to be the connected component of the crystal $\mathcal{B}(\Lambda_1)^{\otimes(a_1+2a_2+\cdots+5a_5)}$, containing the highest weight element $b_{\lambda_{E6}}$. Since $b_{\lambda_{E6}}$ is of shape λ_{E6} and we take the Kashiwara operators as not changing the shape of its operand, the elements of $\mathcal{T}(\lambda_{E6})$ will be viewed as tableaux of shape λ_{E6}, rather than simply as tensor products of elements from the basic crystal.

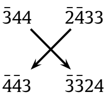
The notion of the *shape* of a tableau and the definition of the crystal $\mathcal{T}(\lambda)$, that we have just described for type E_6, may easily be extended to those for Lie algebra types E_7, E_8, and F_4. In this paper, whenever we refer to a tableau, it will implicitly be assumed to be of a shape that belongs to $\hat{P}^+$.

It is clear from the definition of $\mathcal{T}(\lambda)$ that it is isomorphic to $\mathcal{B}(\lambda)$ as a crystal. Hence, if one is able to find an explicit description of the set $\mathcal{T}(\lambda)$, it could be used as a realization of the crystal $\mathcal{B}(\lambda)$. Such a full description of $\mathcal{T}(\lambda)$ is not required for the purpose of this work and we will only identify certain tableaux as elements of $\mathcal{T}(\lambda)$. Our description of these elements will involve a partial ordering on each of the basic crystals.

Let us describe this partial ordering, starting with the E_6 type. We first refer to the crystal graph given in Appendix A and view the basic crystal $\mathcal{B}(\Lambda_1)$ for the E_6 type as a directed graph by forgetting all the colors on the arrows describing the $\tilde{f}_i$ operator actions. Then, two elements $b_1, b_2 \in \mathcal{B}(\Lambda_1)$ are defined to be related as $b_1 \leqslant b_2$ if and only if either $b_1 = b_2$ or there is a sequence of arrows which can be traversed from b_1 to b_2 in the pointed directions. Since the set of simple roots is linearly independent and each $\tilde{f}_i$ action reduces the weight of the input element by a simple root, the directed graph $\mathcal{B}(\Lambda_1)$ cannot contain any closed loops that may be circled in the pointed direction, and this ensures that the described approach defines a partial ordering on $\mathcal{B}(\Lambda_1)$.

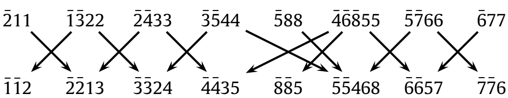
The same approach is taken with the E_7 type. Two elements $b_1, b_2 \in \mathcal{B}(\Lambda_6)$ are said to satisfy $b_1 \leqslant b_2$ if and only if $b_1 = b_2$ or there is a sequence of arrows in the directed graph $\mathcal{B}(\Lambda_6)$ starting from b_1 and ending at b_2.

Before discussing the E_8 type, we describe the partial ordering for the F_4 type. The directed graph $\mathcal{B}(\Lambda_4)$ for the F_4 type is first supplemented with the following two extra arrows.



These two arrows are not related to any Kashiwara operator actions, but one can visually check that their introduction does not create any directed loops. The resulting directed graph is used to define a partial ordering on the set $\mathcal{B}(\Lambda_4)$ in the same manner as was described for the previous two cases.

Finally, in the E_8 case, the many arrows



are added to $\mathcal{B}(\Lambda_1)$, before the resulting directed graph is used to define a partial ordering.

Given any tableau, we shall use $x_{i,j}$ to denote the basic crystal entry occupying its j-th box from the right in the i-th row from the top. For example, in the E_6 type case, the basic crystal entries in a general tableau of shape λ_{E6} would be labeled as follows.

$$\begin{array}{|lllllllllllll|}
\hline
x_{1,a_1+\cdots+a_5} & & & & & & & & & \cdots & x_{1,a_1} & \cdots & x_{1,1} \\
 & & & & & & & \cdots & x_{2,a_2} & \cdots & x_{2,1} & & \\
 & & & & & \cdots & x_{3,a_3} & \cdots & x_{3,1} & & & & \\
 & & & \cdots & x_{4,a_4} & \cdots & x_{4,1} & & & & & & \\
x_{5,a_5} & \cdots & x_{5,1} & & & & & & & & & & \\
\hline
\end{array} \tag{3.3}$$

The notion of large tableaux was first introduced in [1] for semi-standard tableaux of the classical finite types. Using the partial ordering on each basic crystal, we now define the set of large tableaux of shape $\lambda \in \hat{P}^+$, for each of the Lie algebra types under consideration.

Definition 3.4. An E_6 type tableau $T = (x_{i,j})$ of shape $\lambda_{E6} \in \hat{P}^+$ is *large*, if it satisfies the following three sets of conditions.

1. The entries written in the tableau boxes are weakly increasing to the right on each row of T.
2. $\bar{1}2 \leqslant x_{2,j}$, $\bar{2}3 \leqslant x_{3,j}$, $\bar{3}46 \leqslant x_{4,j}$, $\bar{4}56 \leqslant x_{5,j}$, $x_{2,j} \not\geqslant \bar{1}5$, $x_{3,j} \not\geqslant \bar{2}15$, $x_{4,j} \not\geqslant \bar{3}25$, and $x_{5,j} \not\geqslant \bar{4}\bar{6}35$, for every possible j.
3. All five kinds of basic columns listed in (3.2) for type E_6 appear among the columns of the tableau T.

Let us provide some explanations concerning this definition. Recall that a tableau T being of shape λ_{E6} requires all its entries to belong to the basic crystal $\mathcal{B}(\Lambda_1)$. Every order relation mentioned in the above definition refers to the partial ordering we have given to $\mathcal{B}(\Lambda_1)$. For every $x_{i,j+1}, x_{i,j} \in \mathcal{B}(\Lambda_1)$ pair that appear horizontally next to each other in T, the first condition explicitly requires them to be related as $x_{i,j+1} \leqslant x_{i,j}$. Since the set $\mathcal{B}(\Lambda_1)$ is not linearly ordered, this is a stronger condition than the expression $x_{i,j+1} \not> x_{i,j}$. On the other hand, requirements of the form $x_{i,j} \not\geqslant b$, that are listed in the second set of conditions, do not prevent elements that cannot be compared with b from being used as $x_{i,j}$. For example, the element $\bar{6}1$, which cannot be compared with $\bar{1}5$, may appear in the second row of T.

The three sets of conditions jointly imply that T must take the following form.

$$\begin{array}{|lllllllllll|}
\hline
1 & \cdots & & & & & & \cdots & 1 & \cdots & x_{1,1} \\
\bar{1}2 & \cdots & & & & \cdots & \bar{1}2 & \cdots & x_{2,1} & & \\
\bar{2}3 & \cdots & & & \cdots & \bar{2}3 & \cdots & x_{3,1} & & & \\
\bar{3}46 & \cdots & \cdots & \bar{3}46 & \cdots & x_{4,1} & & & & & \\
\bar{4}56 & \cdots & x_{5,1} & & & & & & & & \\
\hline
\end{array} \tag{3.5}$$

That is, most of the entries positioned inside a large tableau of shape λ_{E6} must be identical to those of the highest weight tableau $b_{\lambda_{E6}}$ and differences may only appear within short horizontal strips at the right end of each row. The degenerate example is the highest weight tableau $b_{\lambda_{E6}}$ itself, which is large if and only if every $a_i > 0$.

If all 1-arrows are removed from the crystal graph of $\mathcal{B}(\Lambda_1)$ for the E_6 type, given in Appendix A, the graph breaks into three connected components. Imposing the two conditions $\bar{1}2 \leqslant x_{2,j} \not\geqslant \bar{1}5$ concerning entries of the second row is equivalent to requiring that every $x_{2,j}$ belong to the connected component containing $\bar{1}2$. Similarly, one can visualize the two conditions $\bar{2}3 \leqslant x_{3,j} \not\geqslant \bar{2}15$ by first removing every 1-arrow and 2-arrow from $\mathcal{B}(\Lambda_1)$ and then taking the connected component containing $\bar{2}3$. Conditions concerning the fourth row corresponds to a connected component after removal of every 1-, 2-, and 3-arrows. Finally, the conditions concerning the bottom row corresponds to a connected component that remains after removal of all 1-, 2-, 3-, 4-, and 6-arrows, i.e., all arrows other than 5-arrows. In fact, the connected component consists of just $\bar{4}56$ and $\bar{5}6$, and only these two elements may appear in the bottom row.

The weakly increasing condition imposed on the second row allows us to replace the multiple conditions $\bar{1}2 \leqslant x_{2,j}$, which was stated for every j, with the single condition $\bar{1}2 \leqslant x_{2,a_2+\cdots+a_5}$ concerning the leftmost entry. Similarly, since each of the conditions $x_{2,j} \not\geqslant \bar{1}5$ can be deduced from the conditions $x_{2,j} \leqslant x_{2,1}$ and $x_{2,1} \ngeqslant \bar{1}5$, we may replace all $x_{2,j} \ngeqslant \bar{1}5$ with the single $x_{2,1} \ngeqslant \bar{1}5$ condition. Analogous comments for each of the lower rows may also be made.

It should be noted that the form (3.5) of the tableau together with the weakly increasing condition on the rows imply that the crystal elements placed in each box are strictly increasing as we follow down each column. Hence, every large tableau is a semi-standard tableau in the most classical sense.

Let us continue to define largeness for the other Lie algebra types.

Definition 3.6. An E_7 type tableau $T = (x_{i,j})$ of shape $\lambda_{E7} \in \hat{P}^+$ is *large*, if it satisfies the following three sets of conditions.

1. Entries in each row are weakly increasing to the right.
2. $\bar{6}5 \leqslant x_{2,j}$, $\bar{5}4 \leqslant x_{3,j}$, $\bar{4}3 \leqslant x_{4,j}$, $\bar{3}27 \leqslant x_{5,j}$, $\bar{2}17 \leqslant x_{6,j}$, $x_{2,j} \ngeqslant \bar{6}1$, $x_{3,j} \ngeqslant \bar{5}16$, $x_{4,j} \ngeqslant \bar{4}15$, $x_{5,j} \ngeqslant \bar{3}14$, and $x_{6,j} \ngeqslant \bar{2}\bar{7}13$.
3. The tableau contains every kind of basic column.

Remarks similar to those made for the E_6 type are applicable here. In particular, the second set of conditions can be expressed in terms of certain connected components, and the conditions $\bar{2}17 \leqslant x_{6,j} \ngeqslant \bar{2}\bar{7}13$ imply that only $\bar{2}17$ and $\bar{1}7$ may appear in the bottom row of a large E_7 type tableau.

We discuss the largeness of F_4 type tableaux before describing the E_8 case.

Definition 3.7. An F_4 type tableau $T = (x_{i,j})$ of shape $\lambda_{F4} \in \hat{P}^+$ is *large*, if it satisfies the following three sets of conditions.

1. Entries in each row are weakly increasing to the right, except that each of the two elements $\bar{3}3$ and $\bar{4}4$ may not appear more than once.
2. $\bar{4}3 \leqslant x_{2,j}$, $\bar{3}2 \leqslant x_{3,j}$, $\bar{2}13 \leqslant x_{4,j}$, $x_{2,j} \ngeqslant \bar{4}1$, $x_{2,j} \ngeqslant \bar{4}4$, $x_{3,j} \ngeqslant \bar{3}14$, and $x_{4,j} \ngeqslant \bar{3}14$.
3. The tableau contains every kind of basic column.

Remarks analogous to those made for the E_6 type are all applicable here, and we have some additional comments. The second set of conditions prevents the elements $\bar{3}3$ and $\bar{4}4$ from appearing in any row other than the top row, so that the exception announced in the first set of conditions concerns only the top row. We also caution the reader that there are *three* sub-conditions concerning the second row listed in the second set of conditions. A careful study of the partial ordering definition given to $\mathcal{B}(\Lambda_4)$ shows that the condition $x_{2,1} \ngeqslant \bar{4}1$ allows the condition $x_{2,1} \ngeqslant \bar{4}4$ to be replaced with the simpler relation $x_{2,1} \neq \bar{4}4$. As before, each of these range conditions can be visualized as connected components after removal of certain arrows. For example, the set $\{b \in \mathcal{B}(\Lambda_4) \mid \bar{4}3 \leqslant b, b \ngeqslant \bar{4}1,\ b \ngeqslant \bar{4}4\}$, to which every $x_{2,j}$ must belong, is the connected component containing $\bar{4}3$ after removal of all 4-arrows from $\mathcal{B}(\Lambda_4)$.

Definition 3.8. An E_8 type tableau $T = (x_{i,j})$ of shape $\lambda_{E8} \in \hat{P}^+$ is *large*, if it satisfies the following three sets of conditions.

1. Entries in each row are weakly increasing to the right, except that each of the elements $\bar{1}1$, $\bar{2}2$, $\ldots$, $\bar{8}8$ may not appear more than once.
2. $\bar{1}2 \leqslant x_{2,j}$, $\bar{2}3 \leqslant x_{3,j}$, $\bar{3}4 \leqslant x_{4,j}$, $\bar{4}5 \leqslant x_{5,j}$, $\bar{5}68 \leqslant x_{6,j}$, $\bar{6}78 \leqslant x_{7,j}$, $x_{2,j} \ngeqslant \bar{1}7$, $x_{2,j} \ngeqslant \bar{1}1$, $x_{3,j} \ngeqslant \bar{2}17$, $x_{4,j} \ngeqslant \bar{3}27$, $x_{5,j} \ngeqslant \bar{4}37$, $x_{6,j} \ngeqslant \bar{5}47$, and $x_{7,j} \ngeqslant \bar{6}\bar{8}57$.
3. The tableau contains every kind of basic column.

As with the F_4 type case, the second set of conditions prevents the elements $\bar{1}1, \bar{2}2, \ldots, \bar{8}8$ from appearing in any row other than the top row. Also, the condition $x_{2,1} \ngeqslant \bar{1}1$ may be replaced by $x_{2,1} \neq \bar{1}1$.

We want to collect some information concerning the action of $\tilde{f}_i$ on large tableaux. Adhering to the far eastern reading, the general E_6 type large tableau of (3.5) can be rewritten in the tensor form

$$\begin{aligned}
&x_{1,1} \otimes \cdots \otimes x_{1,k_1} \otimes \{b_{\Lambda_1} \otimes \cdots \otimes b_{\Lambda_1}\} \\
&\quad \otimes (b_{\Lambda_1} \otimes x_{2,1}) \otimes \cdots \otimes (b_{\Lambda_1} \otimes x_{2,k_2}) \otimes \{b_{\Lambda_2} \otimes \cdots \otimes b_{\Lambda_2}\} \\
&\quad \otimes (b_{\Lambda_2} \otimes x_{3,1}) \otimes \cdots \otimes (b_{\Lambda_2} \otimes x_{3,k_3}) \otimes \{b_{\Lambda_3} \otimes \cdots \otimes b_{\Lambda_3}\} \\
&\quad \otimes (b_{\Lambda_3} \otimes x_{4,1}) \otimes \cdots \otimes (b_{\Lambda_3} \otimes x_{4,k_4}) \otimes \{b_{\Lambda_4+\Lambda_6} \otimes \cdots \otimes b_{\Lambda_4+\Lambda_6}\} \\
&\quad \otimes (b_{\Lambda_4+\Lambda_6} \otimes x_{5,1}) \otimes \cdots \otimes (b_{\Lambda_4+\Lambda_6} \otimes x_{5,k_5}) \otimes \{b_{\Lambda_5+2\Lambda_6} \otimes \cdots \otimes b_{\Lambda_5+2\Lambda_6}\},
\end{aligned} \tag{3.9}$$

where $x_{1,k_1} \neq 1$, $x_{2,k_2} \neq \bar{1}2$, $x_{3,k_3} \neq \bar{2}3$, $x_{4,k_4} \neq \bar{3}46$, and $x_{5,k_5} \neq \bar{4}56$, so that the terms placed within the braces are all the basic columns that appear in the tableau T. The largeness condition ensures that each pair of matching braces are nonempty, but terms outside the braces may be nonexistent in some or even all rows of this equation.

Lemma 3.10. *Let $T = (x_{j,k})$ be an E_6 type large tableau. Depending on the index i, the $\tilde{f}_i$ action on T will take place on one of the entries listed below, where the $x_{j,k}$ notation follows* (3.3) *and the indices k_j are defined through the expression* (3.9) *for T.*

$$\begin{aligned}
&\tilde{f}_1: \quad x_{1,1}, \ldots, x_{1,k_1}, x_{1,k_1+1}, \\
&\tilde{f}_2: \quad x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, x_{2,k_2+1}, \\
&\tilde{f}_3: \quad x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, x_{3,1}, \ldots, x_{3,k_3}, x_{3,k_3+1}, \\
&\tilde{f}_4: \quad x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, x_{3,1}, \ldots, x_{3,k_3}, x_{4,1}, \ldots, x_{4,k_4}, x_{4,k_4+1}, \\
&\tilde{f}_5: \quad x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, x_{3,1}, \ldots, x_{3,k_3}, x_{4,1}, \ldots, x_{4,k_4}, x_{5,1}, \ldots, x_{5,k_5}, x_{5,k_5+1}, \\
&\tilde{f}_6: \quad x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, x_{3,1}, \ldots, x_{3,k_3}, x_{4,1}, \ldots, x_{4,k_4}, x_{4,k_4+1}.
\end{aligned}$$

The resulting $\tilde{f}_i T$ is always nonzero.

Proof. Let us first consider the action of $\tilde{f}_1$ on the most general large tableau given by (3.9). While computing the 1-signature for this element, we will originally place a + sign under $x_{1,k_1+1} = b_{\Lambda_1}$. To prove our claim for the $i = 1$ case, it suffices to show that this + sign will not be removed during the erasure of all $(+,-)$ pairs. One can quickly verify from the crystal graph $\mathcal{B}(\Lambda_1)$, explicitly given in Appendix A, that none of its elements can be assigned more than one + or − sign. Since each b_{Λ_1} would be assigned a + sign, under each $b_{\Lambda_1} \otimes x_{2,k}$, we would originally place either a single + sign, two + signs, or a $(+,-)$ pair. The last of these cases reduces to the empty signature and none of these cases can leave a − sign that might later cancel out the + sign placed under x_{1,k_1+1}. As for entries appearing in the remaining three rows, one can check from the crystal graph of Appendix A that the largeness condition restricts $x_{3,k}$, $x_{4,k}$, and $x_{5,k}$ to elements of $\mathcal{B}(\Lambda_1)$ that are unrelated to 1-arrows, so that no + or − signs will be placed under any of these entries. We have thus shown that no − sign will remain at the right of x_{1,k_1+1} to possibly cancel out the + sign originally placed under it. The action of $\tilde{f}_1$ will be nonzero and take place on the $x_{1,1} \otimes \cdots \otimes x_{1,k_1+1}$ part.

We next consider the $\tilde{f}_2$ case. First, recall that the leftmost b_{Λ_2} appearing between the braces in the second row of (3.9) is a condensed expression for

$$b_{\Lambda_2} = b_{\Lambda_1} \otimes \bar{1}2 = b_{\Lambda_1} \otimes x_{2,k_2+1}.$$

When computing the 2-signature for T, a + sign will originally be placed under $x_{2,k_2+1} = \bar{1}2$. Since no signs will originally be placed under any of the b_{Λ_1}'s appearing in the top two rows of (3.9), to prove our claim for $i = 2$, it suffices to show that the + sign under x_{2,k_2+1} will not be canceled out by a − sign from a later row. As before, any − sign from the third row is canceled out by the + sign of its matching b_{Λ_2} and the largeness condition restricts $x_{4,k}$ and $x_{5,k}$ to ranges where the appearance of − signs is impossible.

We may deal with each of the remaining indices i through similar approaches. □

Analogous results for types E_7, E_8, and F_4 are provided below, together with a restatement of the above E_6 type case. As with the E_6 type, proofs for the E_7, E_8, and F_4 types require case by case checking of small details that can be obtained from the explicit crystal graphs of the basic crystals, which is provided in Appendix A.

Lemma 3.11. *If a tableau T is large, then $\tilde{f}_i T$ is nonzero. For each Lie algebra type and index i, the $\tilde{f}_i$ action on a large tableau $T = (x_{j,k})$ will take place on an entry $x_{j,k}$, where the row index j is restricted as follows for each situation.*

(E_6) $\tilde{f}_1 : 1$; $\tilde{f}_2 : 1 \sim 2$; $\tilde{f}_3 : 1 \sim 3$; $\tilde{f}_4 : 1 \sim 4$; $\tilde{f}_5 : 1 \sim 5$; $\tilde{f}_6 : 1 \sim 4$;
(E_7) $\tilde{f}_6 : 1$; $\tilde{f}_5 : 1 \sim 2$; $\tilde{f}_4 : 1 \sim 3$; $\tilde{f}_3 : 1 \sim 4$; $\tilde{f}_2 : 1 \sim 5$; $\tilde{f}_1 : 1 \sim 6$; $\tilde{f}_7 : 1 \sim 5$;
(E_8) $\tilde{f}_1 : 1$; $\tilde{f}_2 : 1 \sim 2$; $\tilde{f}_3 : 1 \sim 3$; $\tilde{f}_4 : 1 \sim 4$; $\tilde{f}_5 : 1 \sim 5$; $\tilde{f}_6 : 1 \sim 6$; $\tilde{f}_7 : 1 \sim 7$; $\tilde{f}_8 : 1 \sim 6$;
(F_4) $\tilde{f}_4 : 1$; $\tilde{f}_3 : 1 \sim 2$; $\tilde{f}_2 : 1 \sim 3$; $\tilde{f}_1 : 1 \sim 4$.

For each fixed $\tilde{f}_i$ and a row index j that is admissible for that $\tilde{f}_i$, the range of possible indices k is $1 \leqslant k \leqslant k_j + 1$, when the j value is the largest among all admissible j-indices for that $\tilde{f}_i$, and $1 \leqslant k \leqslant k_j$, when otherwise.

The definition of largeness given in the work [3], that dealt with the classical and G_2 types, only listed what is essentially the last of our three sets of conditions, at the expense of requiring a large tableau to be an element of $\mathcal{B}(\lambda)$. The next two lemmas show that the last of our three sets of conditions is the most significant requirement for largeness.

Lemma 3.12. *If a tableau T is large, then $\tilde{f}_i T$ satisfies the first set of conditions for largeness.*

Proof. Let us focus on the E_6 case. We first consider the $\tilde{f}_1$ action. Based on Lemma 3.11, we know that it acts on the first row of T. Consider any $x_{1,j-1}$ and $x_{1,j}$. We know that they satisfy $x_{1,j} \leqslant x_{1,j-1}$. It suffices to show that, whenever the tensor product rule states that the $\tilde{f}_1$ action on $x_{1,j-1} \otimes x_{1,j}$ operates on the entry $x_{1,j}$, we have $\tilde{f}_1 x_{1,j} \leqslant x_{1,j-1}$.

Note that the 1-signature of no element from $\mathcal{B}(\Lambda_1)$ contains both a $+$ and a $-$ sign simultaneously. Hence, for the action of $\tilde{f}_1$ to be on $x_{1,j}$, the 1-signature of $x_{1,j}$ must be a $+$ sign and that of $x_{1,j-1}$ must be either a $-$ sign or empty. Now, one can check from the crystal graph of Appendix A that whenever $b_1, b_2 \in \mathcal{B}(\Lambda_1)$ are such that $b_1 \geqslant b_2$ and a 1-arrow leaves from b_2 but not from b_1, then we have $b_1 \geqslant \tilde{f}_1 b_2$. This proves our claim for $\tilde{f}_1$.

Let us next consider the action of $\tilde{f}_2$. We know from Lemma 3.11 that it must act on either the first or the second row of T. If it acts on the first row, we may deal with it as we have already done with $\tilde{f}_1$. If the action is on the second row, we must consider the tensor product $x_{2,j-1} \otimes b_{\Lambda_1} \otimes x_{2,j}$. Since the 2-signature of b_{Λ_1} is empty, in particular, since it does not contain any $-$ signs, for the $\tilde{f}_2$ action on this tensor product to be on $x_{2,j}$, the 2-signature of $x_{2,j-1}$ cannot be a $+$ sign. Hence, it suffices to show that if $b_1, b_2 \in \mathcal{B}(\Lambda_1)$ are such that $b_1 \geqslant b_2$ and a 2-arrow leaves from b_2 but not from b_1, then we have $b_1 \geqslant \tilde{f}_2 b_2$. This can be done by explicitly checking for all such situations using Appendix A. The remaining $\tilde{f}_i$ cases are analogous.

The other Lie algebra types may also be approached similarly, but in the cases of E_8 and F_4 types, unlike the E_6 type that we have explained, one will encounter $-+$ signs as signatures of certain basic crystal elements, so let us briefly comment on this. For any fixed i, there is at most one element of the basic crystal whose i-signature is $-+$. Largeness implies that this element can appear at most once and only in the top row of the tableau. Hence, situations of $x_{1,j-1} \otimes x_{1,j}$, where the signature of $x_{1,j}$ is $-+$ and that of $x_{1,j-1}$ is one of $+$, $-$, or empty, need to be considered. In other words, one checks from Appendix A that whenever b_2 is the unique element of $-+$ signature for the $\tilde{f}_i$ under consideration and $b_1 \gneqq b_2$, we have $b_1 \geqslant \tilde{f}_i b_2$. One must remember to consider the extra arrows defining the partial orderings when checking this. □

Lemma 3.13. *If a tableau T is large, then $\tilde{f}_i T$ satisfies the second set of conditions for largeness.*

Proof. Let us focus on the E_6 type case as the proofs for the remaining three Lie algebra types are almost identical. We first deal with the two conditions $\bar{1}2 \leqslant x_{2,j} \not\geqslant \bar{1}5$ concerning the second row entries of the tableau. Below Definition 3.4, we explained that the set $\{b \in \mathcal{B}(\Lambda_1) \mid \bar{1}2 \leqslant b \not\geqslant \bar{1}5\}$ is a connected component of $\mathcal{B}(\Lambda_1)$ that remains after removal of every 1-arrow. This implies that an element of this set can only be sent to an element outside this set by a 1-arrow. On the other hand, Lemma 3.11 shows that an $\tilde{f}_1$ operator cannot act on any of the second row entries $x_{2,j}$. Hence, the conditions $\bar{1}2 \leqslant x_{2,j} \not\geqslant \bar{1}5$ are preserved under any $\tilde{f}_i$ action on T.

Similarly, the boundary defined by the conditions $\bar{2}3 \leqslant x_{3,j} \not\geqslant \bar{2}15$ can only be crossed over by $\tilde{f}_1$ or $\tilde{f}_2$ actions and we know that these cannot act on the third row of T. The conditions concerning the remaining rows can be dealt with in the same way and the second set of conditions for largeness of an E_6 type tableau are preserved under any $\tilde{f}_i$ action on a large tableau. □

The $\tilde{f}_i$ operator does not necessarily preserve the third condition for largeness, but the $\tilde{e}_i$ operator satisfies the following stronger statement.

Lemma 3.14. *If the tableau T is large, then $\tilde{e}_i T$ is either zero or large.*

One can prove this claim by following along the lines of argument taken in the proofs of the previous three lemmas. In particular, when the claim of Kashiwara operator actions being nonzero is removed from Lemma 3.11, it holds true with every $\tilde{f}_i$ changed to $\tilde{e}_i$ and the range of k set to $1 \leqslant k \leqslant k_j$ for every j.

The next lemma shows that b_λ are the only large tableaux with the highest weight property.

Lemma 3.15. *If a tableau T of shape λ is large and different from the highest weight tableau b_λ, then there exists an i for which $\tilde{e}_i T \neq 0$.*

Proof. Let tableau T be of shape λ, large, and different from the highest weight tableau b_λ. Suppose that the topmost row of the tableau T that contains any differences with b_λ is the j-th row. The first set of conditions for largeness implies that the rightmost entry $x_{j,1}$ of this row will be different from the corresponding entry of b_λ. Below, we shall show that there exists an i for which the i-signature of $x_{j,1}$ is a minus sign and for which this sign is not removed during the erasure of $(+,-)$ pairs done while computing the i-signature for T. This does not imply that the $\tilde{e}_i$ action on T will be applied to $x_{j,1}$, but it does guarantee that the action will be nonzero.

Let us explain just the E_6 type case, as the other cases can be dealt with similarly. Suppose that the topmost difference happens to be in the top row of T. The rightmost entry $x_{1,1}$ of this row will be different from the corresponding entry of b_λ, which is $b_{\Lambda_1} = 1 \in \mathcal{B}(\Lambda_1)$. The entry $x_{1,1}$ becomes the leftmost or first entry when the tableau entries are rearranged in the tensor product form. Now, since $x_{1,1} \neq b_{\Lambda_1}$, there exists an i for which $\tilde{e}_i x_{1,1} \neq 0$. Let us fix such an i. When working out the tensor product rule for this i, a minus sign will be placed under the first tensor entry $x_{1,1}$ and this minus sign cannot be removed during the removal of $(+,-)$ sign pairs, since it is the leftmost sign. Hence, $\tilde{e}_i T$ will be nonzero for the index i we have chosen.

Next, suppose that the topmost difference between T and b_λ appears in the second row, so that $x_{2,1} \neq \bar{1}2$. One can directly verify[3] through Appendix A that every element $b \in \mathcal{B}(\Lambda_1)$ satisfying $\bar{1}2 \lesssim b \not\geqslant \bar{1}5$ will have an $i \neq 1$, for which $\tilde{e}_i b \neq 0$. Let us fix such an i for $x_{2,1}$. Now, the tensor product form of T will start with

[3] Recall that the set $\mathcal{B}_{\mathrm{tmp}} = \{b \in \mathcal{B}(\Lambda_1) \mid \bar{1}2 \leqslant b \not\geqslant \bar{1}5\}$ is one of three connected components of the basic crystal $\mathcal{B}(\Lambda_1)$ that appear with the removal of all 1-arrows. By ignoring all 1-arrows, we are treating the E_6 type crystal $\mathcal{B}(\Lambda_1)$ as a D_5 type crystal. Based on the basic theory concerning the irreducible decomposition of finite crystals, we know that each of the connected components must be a D_5 type highest weight crystal. Hence, the only fact that needs to be verified here is whether $\bar{1}2$ is the unique highest weight element of the D_5 type highest weight crystal $\mathcal{B}_{\mathrm{tmp}}$.

$$b_{\Lambda_1} \otimes \cdots \otimes b_{\Lambda_1} \otimes b_{\Lambda_1} \otimes x_{2,1} \otimes b_{\Lambda_1} \otimes x_{2,2} \otimes \cdots,$$

and the i-signature to be placed under the entries that sit to the left of $x_{2,1}$ will be empty for any $i \neq 1$. Thus, the minus sign to be placed under $x_{2,1}$ cannot be canceled out while working out the tensor product rule. This shows that $\tilde{e}_i T$ will be nonzero for at least one $i \neq 1$ in the current situation.

Similarly, if the top row of distinction is the third row, the corresponding tensor product form will be

$$b_{\Lambda_1} \otimes \cdots \otimes b_{\Lambda_1} \otimes b_{\Lambda_2} \otimes \cdots \otimes b_{\Lambda_2} \otimes b_{\Lambda_2} \otimes x_{3,1} \otimes b_{\Lambda_2} \otimes x_{3,2} \otimes \cdots.$$

For $i \neq 1, 2$, the i-signature under the entries that sit to the left of $x_{3,1}$ will be empty, and one can check[4] that every $b \in \mathcal{B}(\Lambda_1)$ satisfying $\bar{2}3 \lneq b \ngeq \bar{2}15$ has at least one $i \neq 1, 2$ such that $\tilde{e}_i b \neq 0$. This shows that the $\tilde{e}_i$ action on T for this case will be nonzero for at least one $i \neq 1, 2$.

We may deal with the remaining lower rows successively in similar manners and finally conclude that the $\tilde{e}_i$ action on $T \neq b_\lambda$ will be nonzero for at least one i in all cases. $\square$

Since we view Kashiwara operators as not changing the shape of a tableau and since a highest weight element can always be reached within a finite number of applications of the $\tilde{e}_i$ operator, iterative applications of Lemma 3.14 and Lemma 3.15 results in the following statement.

Lemma 3.16. *Every large tableau of shape $\lambda \in \hat{P}^+$ belongs to the connected component $\mathcal{T}(\lambda)$ containing b_λ.*

Justified by this observation, we use $\mathcal{T}(\lambda)^L$ to denote the set of large tableaux of shape λ. In this notation, Lemma 3.16 claims that $\mathcal{T}(\lambda)^L \subset \mathcal{T}(\lambda)$ for every $\lambda \in \hat{P}^+$. The collection of all large tableaux for each Lie algebra type will be denoted by

$$\mathcal{T}^L := \bigcup_{\lambda \in \hat{P}^+} \mathcal{T}(\lambda)^L. \tag{3.17}$$

Some of the $\mathcal{T}(\lambda)^L$ appearing on the right-hand side will be empty.

4. Equivalence of tableaux and crystal $\mathcal{T}(\cup)$

In this section, we define an equivalence relation between large tableaux. We will arrive at a new realization $\mathcal{T}(\cup)$ of the crystal $\mathcal{B}(\infty)$ by showing that the newly defined equivalence relation corresponds directly to the equivalence relation used in defining $\mathcal{B}(\cup) \cong \mathcal{B}(\infty)$.

Arguments of the previous section were mostly provided separately for each Lie algebra type and many of the proofs depended on meticulous checking of details made available by explicit crystal graphs for each Lie algebra type. In contrast, arguments of this section will be rather independent of the Lie algebra type, relying mostly on features that are common to all four Lie algebra types. Our discussions will still be given mainly in terms of the E_6 type, but this is due to lack of notation that can commonly be used with all four Lie algebra types, rather than caused by the need to check for details separately for each type.

At the end of the previous section, we verified that large tableaux of shape λ belong to the connected component $\mathcal{T}(\lambda) \cong \mathcal{B}(\lambda)$. We now provide an equivalence relation among large tableaux and will eventually show that they correspond directly to the equivalence relation on $\bigcup_{\lambda \in P^+} \mathcal{B}(\lambda)$ given by Definition 2.2.

[4] By ignoring all 1-arrows and 2-arrows, we are treating $\mathcal{B}(\Lambda_1)$ as an A_4 type crystal. It suffices to check whether $\bar{2}3$ is the unique highest weight element in $\{b \in \mathcal{B}(\Lambda_1) \mid \bar{2}3 \leqslant b \ngeq \bar{2}15\}$, which we know to be an irreducible A_4 type crystal.

Recall that we only deal with tableaux of shapes that belong to $\hat{P}^+$.

Definition 4.1. Two tableaux T_1 and T_2, for the same Lie algebra type, are *equivalent*, if what remains of the two tableaux after removal of all their basic columns are identical. This is clearly an equivalence relation. Equivalence of the tableaux are expressed as $T_1 \sim T_2$ and the notation

$$\mathcal{T}(\cup) := \mathcal{T}^L/{\sim} = \bigcup_{\lambda\in\hat{P}^+} \mathcal{T}(\lambda)^L/{\sim}$$

will be used for the set of equivalence classes among all large tableaux.

Note that the definition of tableau equivalence is not restricted to large tableaux. For example, any two highest weight tableaux b_λ and b_μ are trivially equivalent to each other regardless of whether they are large.

Let us now work to connect the equivalence relation of this section to that of Section 2.

Lemma 4.2. *If two large tableaux T_1 and T_2 are equivalent, then $\tilde{f}_i T_1$ and $\tilde{f}_i T_2$ are equivalent.*

Proof. We write the proof in terms of the E_6 type for lack of notation that can be used commonly over all types, but proofs for all Lie algebra types are almost identical.

The equivalence of T_1 and T_2 imply that expressions of the form (3.9) for the two tableaux will be identical except possibly in the number of entries within the braces. Hence the i-signatures for T_1 and T_2 will be in natural correspondence except possibly in the number of $+$ signs provided by the basic columns. Since the action of $\tilde{f}_i$ will take place on the leftmost $+$ sign, such signature differences will be immaterial to the $\tilde{f}_i$ action, and what is obtained after the $\tilde{f}_i$ actions on T_1 and T_2 will also be in good correspondence, implying that $\tilde{f}_i T_1$ and $\tilde{f}_i T_2$ are equivalent. □

An analogous statement for the $\tilde{e}_i$ action follows below. Its validity is immediate from an understanding of the proof to Lemma 4.2. Since the $\tilde{e}_i$ action corresponds to the rightmost $-$ sign, the difference in the number of $+$ signs appearing within matching braces are even less important for its proof than in the $\tilde{f}_i$ case.

Lemma 4.3. *If two large tableaux T_1 and T_2 are equivalent, then either $\tilde{e}_i T_1$ and $\tilde{e}_i T_2$ are both zero, or $\tilde{e}_i T_1$ and $\tilde{e}_i T_2$ are equivalent.*

Readers should recall the equivalence relation on $\bigcup_{\lambda\in\hat{P}^+} \mathcal{B}(\lambda)$ we had defined in Section 2 using correspondences between $\mathcal{B}(\infty)$ and $\mathcal{B}(\lambda)$. When $\mathcal{T}(\lambda)$ is identified with $\mathcal{B}(\lambda)$, the equivalence relation on $\bigcup_{\lambda\in\hat{P}^+} \mathcal{B}(\lambda)$ can be carried over to that on the subset $\mathcal{T}^L \subset \bigcup_{\lambda\in\hat{P}^+} \mathcal{T}(\lambda)$. We are now ready to connect the two equivalence relations given to $\mathcal{T}^L$.

Proposition 4.4. *Two large tableaux T_1 and T_2 are equivalent in the sense that removal of all basic columns make them identical if and only if they are equivalent in the sense that they correspond to the same element of $\mathcal{B}(\infty)$.*

Proof. If two tableaux are equivalent in the sense that removal of all basic columns make them identical, let us temporarily use the phrase that the two are *equivalent as tableaux*. If two tableaux are equivalent in the sense that they correspond to the same element of $\mathcal{B}(\infty)$, let us temporarily say that the two are *equivalent as crystal elements*.

Suppose that the large tableaux T_1 and T_2 of respective shapes λ_1 and λ_2 are equivalent as tableaux. By Lemma 3.16, we know that there is a sequence of Kashiwara operators $\tilde{e}_i$'s such that $\tilde{e}_{i_t}\cdots\tilde{e}_{i_1} T_1 = b_{\lambda_1}$. Now, iterative applications of Lemma 3.14 and Lemma 4.3 imply that b_{λ_1} is equivalent to $\tilde{e}_{i_t}\cdots\tilde{e}_{i_1} T_2$ as tableaux and that $\tilde{e}_{i_t}\cdots\tilde{e}_{i_1} T_2$ is large. Applying Lemma 4.3 once more, since $\tilde{e}_i b_{\lambda_1}$ is zero for every i, the same must be true of $\tilde{e}_{i_t}\cdots\tilde{e}_{i_1} T_2$, and we may use Lemma 3.15 to state that $\tilde{e}_{i_t}\cdots\tilde{e}_{i_1}T_2 = b_{\lambda_2}$. In summary, we have shown that $T_1 = \tilde{f}_{i_1}\cdots\tilde{f}_{i_t}b_{\lambda_1}$ and $T_2 = \tilde{f}_{i_1}\cdots\tilde{f}_{i_t}b_{\lambda_2}$, i.e., that they correspond to the same element $\tilde{f}_{i_1}\cdots\tilde{f}_{i_t}u_\infty \in \mathcal{B}(\infty)$.

Let us now discuss the converse statement. We may assume that two large tableaux, equivalent as crystal elements, are given as $T_1 = \tilde{f}_{i_t}\cdots\tilde{f}_{i_1}b_{\lambda_1}$ and $T_2 = \tilde{f}_{i_t}\cdots\tilde{f}_{i_1}b_{\lambda_2}$. Iterative applications of Lemma 3.14 show both $\tilde{f}_{i_k}\cdots\tilde{f}_{i_1}b_{\lambda_1}$ and $\tilde{f}_{i_k}\cdots\tilde{f}_{i_1}b_{\lambda_2}$ to be large for every $0 \leqslant k \leqslant t$. Since the highest weight elements b_{λ_1} and b_{λ_2} are trivially equivalent as tableaux, iterative applications of Lemma 4.2 imply that T_1 and T_2 are equivalent as tableaux. □

Through the discussions of Section 2, we saw that the crystal $\mathcal{B}(\infty)$ is isomorphic to

$$\mathcal{B}(\cup) = \bigcup_{\lambda\in P^+}\mathcal{B}(\lambda)/\sim = \bigcup_{\lambda\in\hat{P}^+}\mathcal{B}(\lambda)/\sim$$

where the crystal structure on $\mathcal{B}(\cup)$ is that induced from those on each $\mathcal{B}(\lambda)$. We have also seen that the equivalence relation defined on $\bigcup_{\lambda\in\hat{P}^+}\mathcal{B}(\lambda)$ is identical to that defined on

$$\mathcal{T}^L = \bigcup_{\lambda\in\hat{P}^+}\mathcal{T}(\lambda)^L \subset \bigcup_{\lambda\in\hat{P}^+}\mathcal{B}(\lambda),$$

under the identification $\mathcal{B}(\lambda) = \mathcal{T}(\lambda)$. Hence, if we can show that the largeness restriction does not make the set of equivalence classes on $\mathcal{T}^L$ strictly smaller than that on $\bigcup_{\lambda\in\hat{P}^+}\mathcal{B}(\lambda)$, then $\mathcal{T}(\cup)$ could be used as another realization of the set $\mathcal{B}(\infty) \cong \mathcal{B}(\cup)$. The next lemma resolves the remaining small gap.

Lemma 4.5. *Given any $b \in \mathcal{B}(\infty)$, there exists a $\lambda \in \hat{P}^+$, for which the tableau $\bar{\pi}_\lambda(b)$ is large.*

Proof. Let us provide arguments for just the E_6 type. The other cases may be approached almost identically. Suppose

$$b = \tilde{f}_{i_t}\cdots\tilde{f}_{i_2}\tilde{f}_{i_1}u_\infty \in \mathcal{B}(\infty)_{-\xi}$$

is given, with $\xi = n_1\alpha_1 + \cdots + n_6\alpha_6 \in Q_+$. It is always possible to choose a weight $\lambda_{E6} \in \hat{P}^+$ such that

$$a_1 > n_1, \qquad a_2 > n_2, \qquad a_3 > n_3, \qquad a_4 > n_4 + n_6, \quad \text{and} \quad a_5 > n_5.$$

Let us fix any such λ_{E6} and consider the sequence of tableaux

$$T_0 = b_{\lambda_{E6}}, \qquad T_1 = \tilde{f}_{i_1}b_{\lambda_{E6}}, \qquad T_2 = \tilde{f}_{i_2}\tilde{f}_{i_1}b_{\lambda_{E6}}, \quad \ldots, \quad T_t = \tilde{f}_{i_t}\cdots\tilde{f}_{i_1}b_{\lambda_{E6}}.$$

Suppose that the tableau T_k is large. Then, Lemma 3.12 and Lemma 3.13 imply that the next tableau T_{k+1} will also be large, as long as it contains every basic column. On the other hand, Lemma 3.11 shows that the set of basic columns contained in T_{k+1} will be either identical to or one less than that of T_k. The first tableau T_0 is certainly large and contains many basic columns. Hence, it suffices to check whether enough basic columns remain in each tableaux T_k throughout the above sequence of tableaux.

The condition $a_1 > n_1$ and Lemma 3.11 implies that $T_0 = b_{\lambda_{E6}}$ may receive n_1-many $\tilde{f}_1$ actions and still retain at least one basic column that looks identical to b_{Λ_1}. This is true even when the n_1-many $\tilde{f}_1$ actions are intermixed with other $\tilde{f}_i$ actions in any order, as long as we do not run out of basic columns of other kinds. Similar statements may be made concerning conditions on a_2 and a_3. The condition $a_4 > n_4 + n_6$ implies that $b_{\lambda_{E6}}$ may receive n_4-many $\tilde{f}_4$ actions together with n_6-many $\tilde{f}_6$ actions and still contain at least one basic column that is identical to $b_{\Lambda_4+\Lambda_6}$. Finally, the condition $a_5 > n_5$ insures existence of at least one basic column identical to $b_{\Lambda_5+2\Lambda_6}$ after n_5-many $\tilde{f}_5$ actions.

Since n_i-many indices among the indices $i_1, \dots, i_t$ are equal to i, the discussion so far shows that the tableau $\bar{\pi}_\lambda(b) = \tilde{f}_{i_t} \cdots \tilde{f}_{i_1} b_{\lambda_{E6}}$ retains at least one copy of every kind of basic column and that it is large. □

We can now say that the set of equivalence classes $\mathcal{T}(\cup)$ is equal to $\mathcal{B}(\cup)$, under the identification $\mathcal{T}(\lambda) \cong \mathcal{B}(\lambda)$. However, we must keep in mind that this is currently an equality of sets and not an isomorphism of crystals. We must provide $\mathcal{T}(\cup)$ with a crystal structure and compare it with that on $\mathcal{B}(\cup)$. A natural definition for the Kashiwara operator $\tilde{f}_i$ on $\mathcal{T}(\cup)$ is supplied by Lemma 4.2 and the following lemma, whose truth is evident from the proof of Lemma 4.5.

Lemma 4.6. *Given any element of $\mathcal{T}(\cup)$, it is always possible to choose its representative $T \in \mathcal{T}^L$ in such a way that $\tilde{f}_i T$ is large.*

The corresponding support for the $\tilde{e}_i$ operator is provided by Lemma 3.14 and Lemma 4.3. Since there are no difficulties in defining the remaining maps wt, ε_i, and φ_i, there is a crystal structure on $\mathcal{T}(\cup)$ induced from those on each $\mathcal{T}(\lambda)$.

Since the crystal structures on $\mathcal{T}(\cup)$ and $\mathcal{B}(\cup)$ were both derived from those on each $\mathcal{T}(\lambda)$ and $\mathcal{B}(\lambda)$, the two crystals $\mathcal{T}(\cup)$ and $\mathcal{B}(\cup)$ will be isomorphic under the identification $\mathcal{T}(\lambda) = \mathcal{B}(\lambda)$ of crystals. We have arrived at our main result.

Theorem 4.7. *The set of equivalence classes $\mathcal{T}(\cup) = \bigcup_{\lambda \in \hat{P}^+} \mathcal{T}(\lambda)^L / \sim$ can be given a crystal structure that is induced from those on each $\mathcal{T}(\lambda)$. When $\mathcal{T}(\cup)$ is given this crystal structure, it is isomorphic to $\mathcal{B}(\infty)$ as a crystal.*

To achieve our final goal of giving an explicit description of the crystal $\mathcal{B}(\infty)$ in terms of tableaux, it suffices to provide an explicit set of representatives for $\mathcal{T}(\cup) = \bigcup_{\lambda \in \hat{P}^+} \mathcal{T}(\lambda)^L / \sim$ and translate the various maps on $\mathcal{T}(\cup)$ to those on the representative set. To choose a set of representatives for $\mathcal{T}(\cup)$ from the set of all large tableaux, it suffices to make the following definition.

Definition 4.8. A large tableau is *marginally large*, if removing any one of its basic columns destroys the largeness of the tableau.

For example, a highest weight tableau of weight λ_{E6} is marginally large if and only if every $a_i = 1$.

Theorem 4.9. *The set of marginally large tableaux forms a set of representatives for $\mathcal{T}(\cup) \cong \mathcal{B}(\infty)$.*

Proof. Given any large tableaux, we can arrive at a marginally large tableau by successively removing any of its excess basic column copies. Removal of any set of basic columns always results in a tableau that is equivalent to the original tableau. This shows that every large tableau can be represented by a marginally large tableau. On the other hand, the definition of equivalence between tableaux implies that any two equivalent marginally large tableaux must be identical. □

The definition of marginally large tableaux we gave requires the tableaux to be as close as possible to becoming not large, but working with a different definition is also possible. One can consider fixing the numbers of basic columns to be found within a large tableau to any positive integer, for each basic column type, and any set of such choices would give a set of representatives for $\mathcal{T}(\cup)$.

We now discuss how the crystal structure on $\mathcal{T}(\cup)$ carries over to the set of marginally large tableaux. To apply $\tilde{f}_i$ to a marginally large tableau T, we could rely on Lemma 4.6 to choose an equivalent large tableaux T' such that $\tilde{f}_i T'$ is large and then remove basic columns from $\tilde{f}_i T'$ until we arrive at a marginally large tableau. This process is the correct $\tilde{f}_i$ action on the set of marginally large tableaux, but let us provide a slightly more efficient way to compute $\tilde{f}_i T$.

Given a specific basic column and a large tableau there is only a small number of places within the tableau where one may insert the basic column and still maintain the largeness. For example, in the

E_6 case, when the general large tableau is written as (3.9), a basic column can only be inserted within the pair of matching braces to which it corresponds. Inserting the basic column anywhere away from these braces will always break the first condition for largeness. Recalling Lemma 3.11, one sees that, when T is large, the tableau $\tilde{f}_i T$ will still be of the form (3.9), or its analogue for other Lie algebra types, except that one of the pairs of matching braces may be empty. Combining this observation with Lemma 3.12 and Lemma 3.13, one can claim that, when T is large, the tableau $\tilde{f}_i T$ is either large, or can be made large by inserting a single basic column at an appropriate place. The proof to Lemma 3.11 contains enough information about the workings of the $\tilde{f}_i$ action for us to realize that this act of inserting a basic column commutes with the application of the $\tilde{f}_i$ operator.

The $\tilde{f}_i$ action on the set of marginally large tableaux may now be computed alternatively as follows.

1. Given a marginally large tableau T, compute $\tilde{f}_i T$ as usual.
2. If the result is large, it is marginally large, and we are done.
3. If otherwise, insert one appropriate basic column into $\tilde{f}_i T$ to arrive at a marginally large tableau.

Analogous process for the $\tilde{e}_i$ operator is as follows.

1. Given a marginally large tableau T, compute $\tilde{e}_i T$ as usual.
2. If the result is zero or a marginally large tableau, we are done.
3. If otherwise, remove one appropriate basic column from $\tilde{e}_i T$ to arrive at a marginally large tableau.

Note that, since the $\tilde{e}_i$ action will change only one tableau entry, at most one column needs to be removed.

Examples of the Kashiwara operator action on the set of marginally large tableaux are given in Appendix B.

Acknowledgments

We thank the two anonymous reviewers for helpful comments which have improved this paper and also for bringing the reference [13] to our attention.

Appendix A. Crystal graphs of the basic crystals

The explicit crystal graphs for the basic crystals used in this work are presented in this section. Let us first very briefly explain how these were obtained.

The set of *Nakajima monomials* in the variables $Y_i(m)$, with $i \in I$, $m \in \mathbf{Z}$, is defined to be

$$\mathcal{M} = \left\{ \prod_{(i,m)\in I\times\mathbf{Z}} Y_i(m)^{y_i(m)} \,\middle|\, y_i(m) \in \mathbf{Z} \text{ vanishes except at finitely many } (i,m) \right\},$$

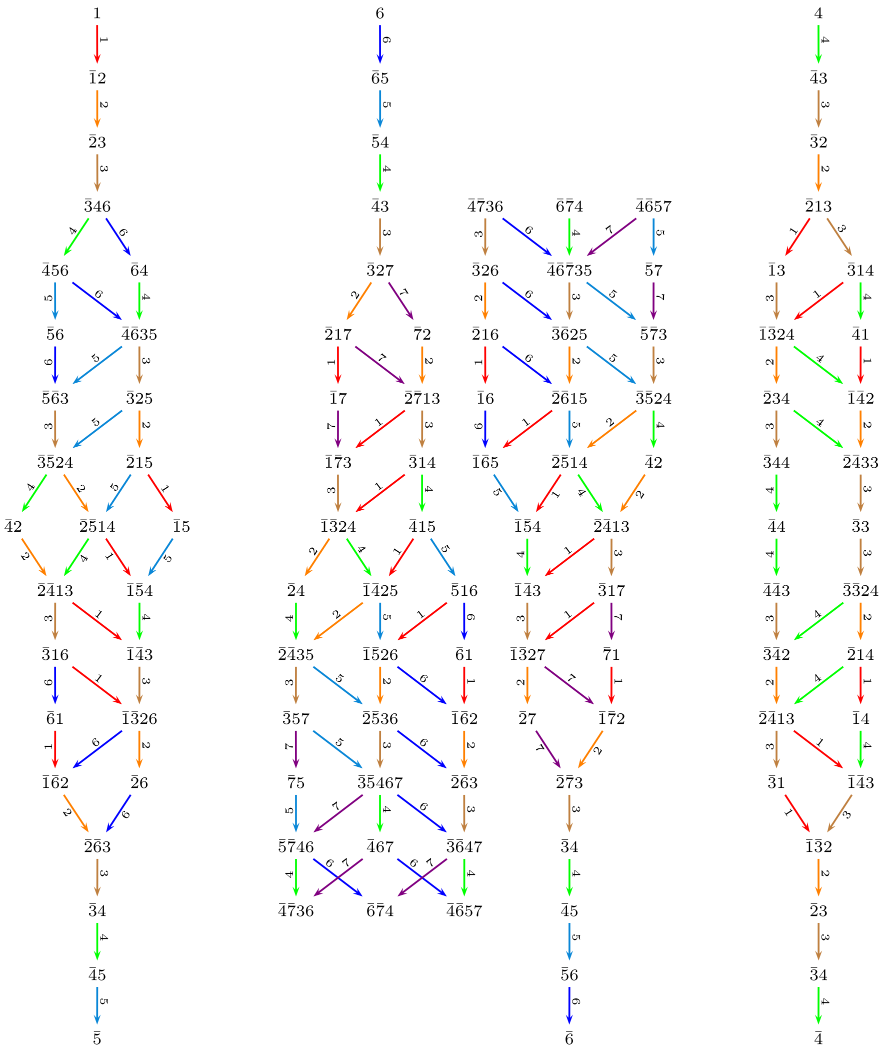
and this set can be given a crystal structure, whenever a set of integers $(c_{ij})_{i\neq j\in I}$ such that $c_{ij}+c_{ji}=1$ is fixed. In our computations, we always used the explicit numbers

$$c_{ij} = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i < j. \end{cases}$$

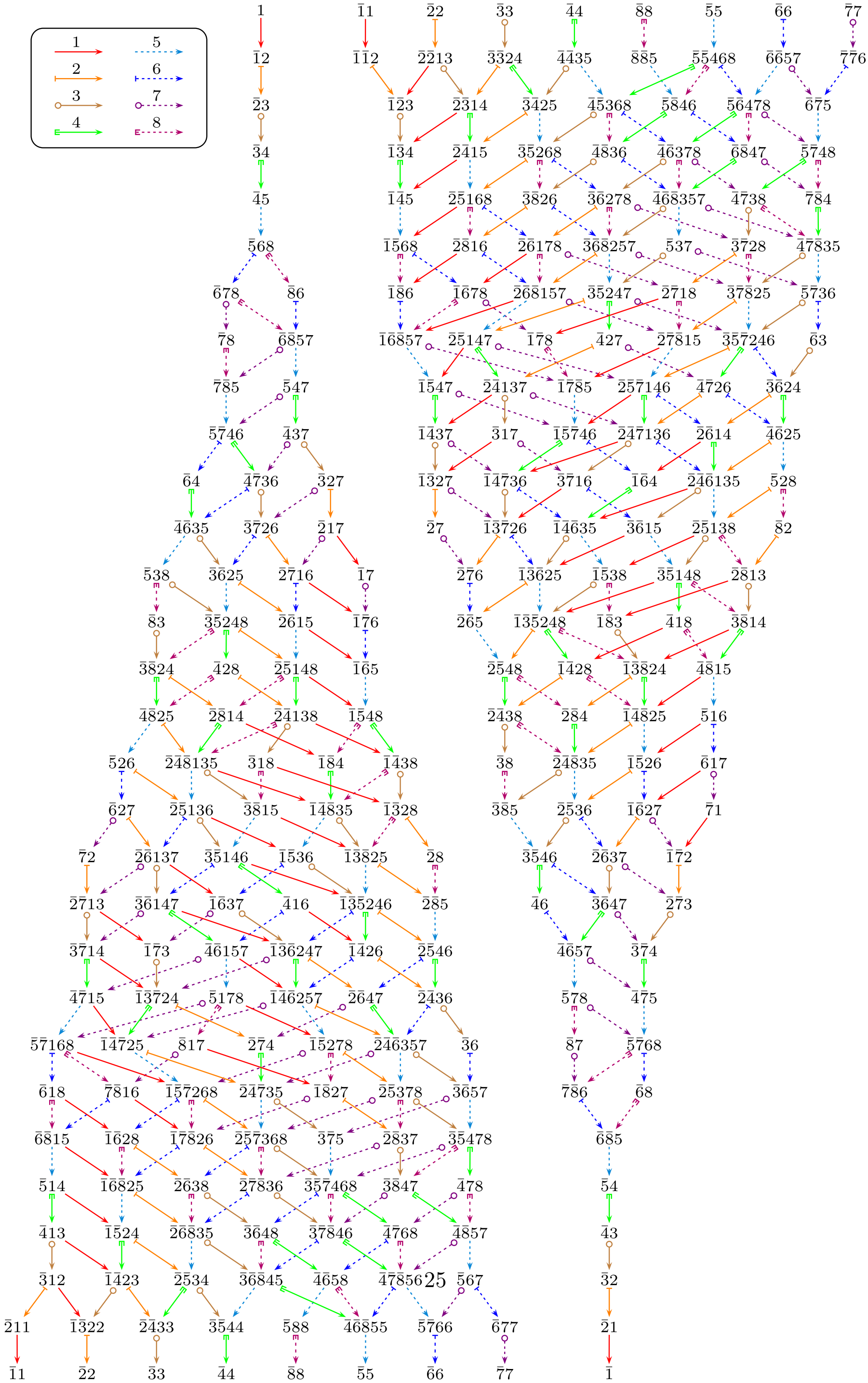
We rely on the following theorem from [14] to draw the basic crystals.

Theorem A.1. *For a maximal vector $M \in \mathcal{M}$, the connected component of $\mathcal{M}$ containing M is isomorphic to $\mathcal{B}(\mathrm{wt}(M))$.*

For each Lie algebra type and basic crystal under consideration, we chose an explicit highest weight vector of appropriate weight, and computed its connected component through direct computation, following the explicit Kashiwara operator definitions given to $\mathcal{M}$. For example, in the case of E_6, the connected component that starts from the highest weight vector $b_{\Lambda_1} = Y_1(0)$ was computed. After each explicit crystal graph was completely obtained, we adopted the idea of [6] in renaming the crystal elements. That is, each Nakajima monomial was replaced with a simpler notation that reflected its weight. For example, the monomial $Y_4(1)^{-1}Y_5(0)Y_6(0)$, which is of weight $-\Lambda_4 + \Lambda_5 + \Lambda_6$, that appeared in the E_6 type basic crystal $\mathcal{B}(\Lambda_1)$, was replaced by $\bar{4}56$. The resulting crystal graphs are given below.



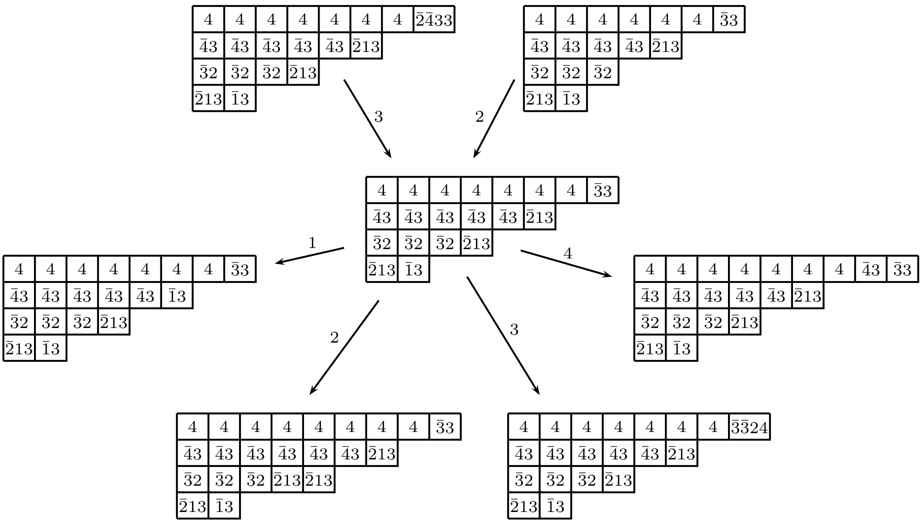
$\mathcal{B}(\Lambda_1)$ for E_6 $\mathcal{B}(\Lambda_6)$ for E_7 $\mathcal{B}(\Lambda_4)$ for F_4



$\mathcal{B}(\Lambda_1)$ for E_8

Appendix B. Example of Kashiwara operator action

The diagram below gives examples of the Kashiwara operator action on the set of marginally large tableaux for Lie algebra type F_4. All $\tilde{e}_i$ and $\tilde{f}_i$ actions going into and leading out of the tableau placed at the center are presented.



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