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Fuzzy Hyers-Ulam stability of an additive functional equation

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Abstract

In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam stability of the following additive functional equation

$$2f\left(\frac{x+y+z}{2}\right)=f(x)+f(y)+f(z) \tag{0.1}$$

in fuzzy normed spaces.

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1. Introduction

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?* If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In the next year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [3] proved a generalization of the Hyers' theorem for additive mappings.

Theorem 1.1. (Th.M. Rassias) *Let $f : X \to Y$ be a mapping from a normed vector space X into a Banach space Y subject to the inequality*

$$\| f(x+y)-f(x)-f(y) \| \le \varepsilon(\| x\|^p + \| y\|^p)$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $0 \le p < 1$. Then the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ and $L : X \to Y$ is the unique additive mapping which satisfies

$$\| f(x)+L(x) \| \le \frac{2\varepsilon}{2-2^p}\| x\|^p$$

for all $x \in X$. Also, if for each $x \in X$, the function f(tx) is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta [4] by replacing the bound $\varepsilon(||x||^p + ||y||^p)$ by a general control function $\phi(x, y)$.

In 1983, a Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation. The reader is referred to ([8-20]) and references therein for detailed information on stability of functional equations.

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [22,23]). In particular, Bag and Samanta [24], following Cheng and Mordeson [25], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [26]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [27].

Definition 1.2. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

($N1$) $N(x, t) = 0$ *for* $t \le 0$;

($N2$) $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$;

($N3$) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

($N4$) $N(x + y, c + t) \ge min\{N(x, s), N(y, t)\}$;

($N5$) $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$;

($N6$) *for* $x \neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space.

Example 1.3. *Let $(X, ||.||)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|} & t > 0, x \in X \\ 0 & t \le 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.4. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.*

Definition 1.5. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X.

Definition 1.6. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:*

(a) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$;
(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Theorem 1.7. ([28,29]) *Let* (X, d) *be a complete generalized metric space and* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then, for all* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers n or there exists a positive integer* n_0 *such that*

(a) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n_0 \geq n_0$;
(b) the sequence $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(c) y^* *is the unique fixed point of J in the set* $Y = \{y \in X : d(J^{n_0}x,\ y) < \infty\}$;
(d) $d(y,\ y^*) \leq \frac{d(y,Jy)}{1-L}$ *for all* $y \in Y$.

2. Fuzzy stability of the functional Eq. (0.1)

Throughout this section, using the fixed point and direct methods, we prove the Hyers-Ulam stability of functional Eq. (0.1) in fuzzy normed spaces.

2.1. Fixed point alternative approach

Throughout this subsection, using the fixed point alternative approach, we prove the Hyers-Ulam stability of functional Eq. (0.1) in fuzzy Banach spaces.

In this subsection, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 2.1. *Let* $\phi : X^3 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,\ y,\ z) \ \leq \frac{L\varphi(2x, 2y, 2z)}{2}$$

for all $x, y, z \in X$. *Let* $f : X \to Y$ *be a mapping satisfying*

$$N\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z),\ t\right) \geq \frac{t}{t+\varphi(x,y,z)} \tag{2.1}$$

for all $x, y, z \in X$ *and all* $t > 0$. *Then the limit*

$$A(x) \ := N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for each $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x),\ t) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,2x,x)}. \tag{2.2}$$

Proof. Putting $y = 2x$ and $z = x$ in (2.1) and replacing x by $\frac{x}{2}$, we have

$$N\left(2f\left(\frac{x}{2}\right) - f(x),\ t\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2}, x, \frac{x}{2}\right)} \tag{2.3}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S \ := \{g : X \to Y\}$$

and the generalized metric d in S defined by

$$d(f,\ g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x) - h(x),\ \mu t) \ge \frac{t}{t + \varphi(x, 2x, x)},\ \forall x \in X,\ t > 0\right\},$$

where $\inf \varnothing = +\infty$. It is easy to show that (S, d) is complete (see [30, Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x),\ \varepsilon t) \ge \frac{t}{t + \varphi(x, 2x, x)}$$

for all $x \in X$ and $t > 0$. Hence,

$$\begin{aligned} N(Jg(x) - Jh(x),\ L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right),\ L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right),\ \frac{L\varepsilon t}{2}\right) \\ &\ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, x, \frac{x}{2}\right)} \\ &\ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L\varphi(x,2x,x)}{2}} \\ &= \frac{t}{t + \varphi(x, 2x, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg,\ Jh) \le Ld(g,\ h)$$

for all $g, h \in S$. It follows from (2.3) that

$$\begin{aligned} N\left(f(x) - 2f\left(\frac{x}{2}\right),\ t\right) \ge \frac{t}{t + \varphi\left(\frac{x}{2}, x, \frac{x}{2}\right)} &\ge \frac{t}{t + \frac{L\varphi(x,2x,x)}{2}} \\ &= \frac{\frac{2t}{L}}{\frac{2t}{L} + \varphi(x2x, x)}. \end{aligned} \tag{2.4}$$

Therefore,

$$N\left(f(x) - 2f\left(\frac{x}{2}\right),\ \frac{Lt}{2}\right) \ge \frac{t}{t + \varphi(x, 2x, x)}. \tag{2.5}$$

This means that

$$d(f,\ Jf) \le \frac{L}{2}.$$

By Theorem 1.7, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{2}\right) = \frac{A(x)}{2} \tag{2.6}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, 2x, x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.2) holds. Furthermore, since

$$\begin{aligned} &N\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z), t\right) \\ &\geq N - \lim_{n\to\infty}\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), t\right) \\ &\geq \lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n \varphi(x,y,z)}{2^n}} \to 1 \end{aligned}$$

for all $x, y, z \in X$, $t > 0$. So $N\left(A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z), t\right) = 1$ for all $x, y, z \in X$ and all $t > 0$. Thus the mapping $A : X \to Y$ is additive, as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $||.||$. Let $f : X \to Y$ be a mapping satisfying*

$$N\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}$$

for all $x, y, z \in X$ and all $t > 0$. Then the limit

$$A(x) := N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x),\ t) \geq \frac{(2^p - 1)t}{(2^p - 1)t + (2^{r-1} + 1)\theta \parallel x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\phi(x, y, z)\colon = \theta(||x||^p + ||y||^p + ||z||^p)$ for all $x, y, z \in X$. Then we can choose $L = 2^{-p}$ and we get the desired result. □

Theorem 2.3. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(2x,\ 2y,\ 2z)\ \leq 2L\varphi(x,\ y,\ z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (2.1). Then

$$A(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x),\ t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, 2x, x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined as in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{g(2x)}{2}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x),\ \varepsilon t)\ \geq \frac{t}{t + \varphi(x, 2x, x)}$$

for all $x \in X$ and $t > 0$. Hence,

$$\begin{aligned} N(Jg(x) - Jh(x),\ L\varepsilon t) &= N\left(\frac{g(2x)}{2} - \frac{h(2x)}{2},\ L\varepsilon t\right) \\ &= N\left(g(2x) - h(2x),\ 2L\varepsilon t\right) \\ &\geq \frac{2Lt}{2Lt + \varphi(2x,\ ,\ 4x,\ 2x)} \\ &\geq \frac{2Lt}{2Lt + 2L\varphi(x,\ 2x,\ x)} \\ &= \frac{t}{t + \varphi(x,\ 2x,\ x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg,\ Jh) \leq Ld(g,\ h)$$

for all $g, h \in S$. It follows from (2.3) that

$$N\left(\frac{f(2x)}{2} - f(x),\ \frac{t}{2}\right) \geq \frac{t}{t + \varphi(x, 2x, x)}.$$

Therefore,

$$d(f,\ Jf) \leq \frac{1}{2}.$$

By Theorem 1.7, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$2A(x) = A(2x) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g,\ h) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x),\ \mu t) \geq \frac{t}{t + \varphi(x, 2x, x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$.

(3) $d(f,\ A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$ which implies the inequality

$$d(f,\ A) \leq \frac{1}{2 - 2L}.$$

This implies that the inequality (2.7) holds.

The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 2.4. *Let* $\theta \geq 0$ *and let p be a real number with* $0 < p < \frac{1}{3}$. *Let X be a normed vector space with norm* $|| \, . \, ||$. *Let* $f : X \to Y$ *be a mapping satisfying*

$$N\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z),\ t\right) \geq \frac{t}{t + \theta(\| x\|^p . \| y\|^p . \| z\|^p)}$$

for all x, y, z ∈ *X and all t* > 0. *Then*

$$A(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for each x ∈ X *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x),\ t) \geq \frac{(2^{3p} - 1)t}{(2^{3p} - 1)t + 2^{3p-1}\theta \, \| x\|^{3p}}.$$

for all x ∈ *X*.

Proof. The proof follows from Theorem 2.3 by taking $\phi(x, y, z) := \theta(||x||^p \cdot ||y||^p \cdot ||z||^p)$ for all *x*, *y*, *z* ∈ *X*. Then we can choose $L = 2^{-3p}$ and we get the desired result. □

2.2. **Direct method**. In this subsection, using direct method, we prove the Hyers-Ulam stability of the functional Eq. (0.1) in fuzzy Banach spaces.

Throughout this subsection, we assume that *X* is a linear space, (*Y*, *N*) is a fuzzy Banach space and (*Z*, *N*′) is a fuzzy normed spaces. Moreover, we assume that *N*(*x*,.) is a left continuous function on ℝ.

Theorem 2.5. *Assume that a mapping* $f : X \to Y$ *satisfies the inequality*

$$N\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \geq N'(\varphi(x, y, z), t) \tag{2.9}$$

for all x, y, z ∈ *X*, *t* > 0 *and* $\phi : X^3 \to Z$ *is a mapping for which there is a constant r* ∈ ℝ *satisfying* $0 < | r | < \frac{1}{2}$ *and*

$$N'(\varphi(x, y, z), t) \geq N'\left(\varphi(2x, 2y, 2z), \frac{t}{| r |}\right) \tag{2.10}$$

for all x, y, z ∈ *X and all t* > 0. *Then there exist a unique additive mapping* $A : X \to Y$ *satisfying (0.1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, 2x, x), \frac{(1 - 2 | r |)t}{| r |}\right) \tag{2.11}$$

for all x ∈ *X and all t* > 0.

Proof. It follows from (2.10) that

$$N'\left(\varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right), t\right) \geq N'\left(\varphi(x, y, z), \frac{t}{| r|^j}\right). \tag{2.12}$$

So

$$N'\left(\varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right), | r|^j t\right) \geq N'(\varphi(x, y, z), t)$$

for all $x, y, z \in X$ and all $t > 0$. Substituting $y = 2x$ and $z = x$ in (2.9), we obtain

$$N(f(2x) - 2f(x), t) \geq N'(\varphi(x, 2x, x), t) \tag{2.13}$$

So

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq N'\left(\varphi\left(\frac{x}{2}, x, \frac{x}{2}\right), t\right) \tag{2.14}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{2^j}$ in (2.14), we have

$$\begin{aligned} N\left(2^{j+1}f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right), 2^j t\right) &\geq N'\left(\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^j}, \frac{x}{2^{j+1}}\right), t\right) \\ &\geq N'\left(\varphi(x, 2x, x), \frac{t}{|r|^{j+1}}\right) \end{aligned} \tag{2.15}$$

for all $x \in X$, all $t > 0$ and any integer $j \geq 0$. So

$$\begin{aligned} &N\left(f(x) - 2^n f\left(\frac{x}{2^n}\right), \sum_{j=0}^{n-1} 2^j |r|^{j+1} t\right) \\ &= N\left(\sum_{j=0}^{n-1}\left[2^{j+1}f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right)\right], \sum_{j=0}^{n-1} 2^j |r|^{j+1} t\right) \\ &\geq \min_{0\leq j\leq n-1}\left\{N\left(2^{j+1}f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right), 2^j |r|^{j+1} t\right)\right\} \\ &\geq N'(\varphi(x, 2x, x), t). \end{aligned} \tag{2.16}$$

Replacing x by $\frac{x}{2^p}$ in the above inequality, we find that

$$\begin{aligned} N\left(2^{n+p}f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{j=0}^{n-1} 2^j |r|^{j+1} t\right) &\geq N'\left(\varphi\left(\frac{x}{2^p}, \frac{2x}{2^p}, \frac{x}{2^p}\right), t\right) \\ &\geq N'\left(\varphi(x, 2x, x), \frac{t}{|r|p}\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and all integers $n \geq 0$, $p \geq 0$. So

$$N\left(2^{n+p}f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{j=0}^{n-1} 2^{j+p} |r|^{j+p+1} t\right) \geq N'(\varphi(x, 2x, x), t)$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. Hence, one obtains

$$N\left(2^{n+p}f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), t\right) \geq N'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} 2^{j+p} |r|^{j+p+1}}\right) \tag{2.17}$$

for all $x \in X$, $t > 0$ and all integers $n > 0$, $p \geq 0$. Since the series $\sum_{j=0}^{\infty} 2^j |r|^j$ is convergent, by taking the limit $p \to \infty$ in the last inequality, we know that a sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N) and so it converges in Y. Therefore, a mapping $A: X \to Y$ defined by

$$A(x) := N - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

is well defined for all $x \in X$. It means that

$$\lim_{n\to\infty} N\left(A(x) - 2^n f\left(\frac{x}{2^n}\right), t\right) = 1 \tag{2.18}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (2.17) that

$$N\left(2^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \geq N'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} 2^j \mid r|^{j+1}}\right)$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned} N(f(x) - A(x), t) &\geq \min\left\{N\left(f(x) - 2^n f\left(\frac{x}{2^n}\right), (1-\varepsilon)t\right), N\left(A(x) - 2^n f\left(\frac{x}{2^n}\right), \varepsilon t\right)\right\} \\ &\geq N'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} 2^j \mid r|^{j+1}}\right) \\ &\geq N'\left(\varphi(x, 2x, x), \frac{(1 - 2 \mid r \mid)\varepsilon t}{\mid r \mid}\right) \end{aligned}$$

for sufficiently large n and for all $x \in X$, $t > 0$ and N with $0 < N < 1$. Since N is arbitrary and N' is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, 2x, x), \frac{(1 - 2 \mid r \mid)t}{\mid r \mid}\right)$$

for all $x \in X$ and $t > 0$. It follows from (2.9) that

$$\begin{aligned} &N\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), t\right) \\ &\quad \geq N'\left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), \frac{t}{2^n}\right) \\ &\quad \geq N'\left(\varphi(x, y, z), \frac{t}{2^n \mid r|^n}\right) \end{aligned}$$

for all $x, y, z \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} N'\left(\varphi(x, y, z), \frac{t}{2^n \mid r|^n}\right) = 1$$

and so

$$N\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), t\right) \to 1$$

for all $x, y, z \in X$ and all $t > 0$. Therefore, we obtain in view of (2.18)

$$\begin{aligned} &N\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z), t\right) \\ &\quad \geq \min\left\{N(A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z) - 2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right)\right. \\ &\quad -2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), \frac{t}{2}\Big), \\ &\quad \left. N\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), \frac{t}{2}\right)\right\} \\ &\quad = N\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), \frac{t}{2}\right) \\ &\quad \geq N'\left(\varphi(x, y, z), \frac{t}{2^{n+1} \mid r|^n}\right) \to 1 \text{ as } n \to \infty \end{aligned}$$

which implies

$$2A\left(\frac{x+y+z}{2}\right) = A(x) + A(y) + A(z)$$

for all $x, y, z \in X$. Thus, A: $X \to Y$ is a mapping satisfying the Eq. (0.1) and the inequality (2.11).

To prove the uniqueness, assume that there is another mapping $L : X \to Y$ which satisfies the inequality (2.11). Since $L(x) = 2^n L(\frac{x}{2^n})$ for all $x \in X$, we have

$$\begin{aligned} N(A(x) - L(x), t) &= \left(2^n A\left(\frac{x}{2^n}\right) - 2^n L\left(\frac{x}{2^n}\right), t\right) \\ &\geq \min\left\{N\left(2^n A\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right), \frac{t}{2}\right), N\left(2^n f\left(\frac{x}{2^n}\right) - 2^n L\left(\frac{x}{2^n}\right), \frac{t}{2}\right)\right\} \\ &\geq N'\left(\varphi\left(\frac{x}{2^n}, \frac{2x}{2^n}, \frac{x}{2^n}\right), \frac{(1-2\mid r\mid)t}{\mid r\mid 2^{n+1}}\right) \\ &\geq N\left(\varphi(x, 2x, x), \frac{(1-2\mid r\mid)t}{\mid r\mid^{n+1} 2^{n+1}}\right) \to 1 \text{ as } n \to \infty \end{aligned}$$

for all $t > 0$. Therefore, $A(x) = L(x)$ for all $x \in X$, this completes the proof. □

Corollary 2.6. *Let X be a normed spaces and* $(\mathbb{R}, N')$ *a fuzzy Banach space. Assume that there exist real numbers* $\theta \geq 0$ and $0 < p < 1$ *such that a mapping* $f : X \to Y$ *satisfies the following inequality*

$$N\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \geq N'(\theta(\| x\|^p + \| y\|^p + \| z\|^p), t)$$

for all $x, y, z \in X$ *and* $t > 0$. *Then there is a unique additive mapping* $A : X \to Y$ *satisfying (0.1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\theta \| x\|^p, \frac{2t}{2^r + 2}\right)$$

Proof. Let $\phi(x, y, z) := \theta(||x||^p + ||y||^p + ||z||^p)$ and $\mid r \mid = \frac{1}{4}$. Applying Theorem 2.5, we get the desired result. □

Theorem 2.7. *Assume that a mapping* $f : X \to Y$ *satisfies (2.9) and* $\phi : X^2 \to Z$ *is a mapping for which there is a constant* $r \in \mathbb{R}$ satisfying $0 < |r| < 2$ *and*

$$N'(\varphi(2x, 2y, 2z), \mid r \mid t) \geq N'(\varphi(x, y, z), t) \tag{2.19}$$

for all $x, y, z \in X$ *and all* $t > 0$. *Then there is a unique additive mapping* $A : X \to Y$ *satisfying (0.1) and the following inequality*

$$N(f(x) - A(x), t) \geq N'(\varphi(x, 2x, x), (2 - \mid r \mid)t). \tag{2.20}$$

for all $x \in X$ *and all* $t > 0$.

Proof. It follows from (2.13) that

$$N\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right) \geq N'(\varphi(x, 2x, x), t) \tag{2.21}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^n x$ in (2.21), we obtain

$$N\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{t}{2^{n+1}}\right) \geq N'(\varphi(2^n x, 2^{n+1}x, 2^n x), t) \geq N'\left(\varphi(x, 2x, x), \frac{t}{|r|^n}\right).$$

So

$$N\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{|r|^n t}{2^{n+1}}\right) \geq N'(\varphi(x, 2x, x), t) \tag{2.22}$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 2.5, we obtain that

$$N\left(f(x) - \frac{f(2^n x)}{2^n}, \sum_{j=0}^{n-1} \frac{|r|^j t}{2^{j+1}}\right) \geq N'(\varphi(x, 2x, x), t)$$

for all $x \in X$, all $t > 0$ and all integers $n > 0$. So

$$N\left(f(x) - \frac{f(2^n x)}{2^n}, t\right) \geq N'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} \frac{|r|^j}{2^{j+1}}}\right) \geq N'(\varphi(x, 2x, x), (2 - |r|)t).$$

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 2.8. *Let X be a normed spaces and* $(\mathbb{R}, N')$ *a fuzzy Banach space. Assume that there exist real numbers* $\theta \geq 0$ *and* $0 < p < \frac{1}{3}$ *such that a mapping* $f : X \to Y$ *satisfies the following inequality*

$$N\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \geq N'(\theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p), t)$$

for all $x, y, z \in X$ *and* $t > 0$. *Then there is a unique additive mapping* $A : X \to Y$ *satisfying (0.1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\theta \|x\|^p, \frac{t}{2^r + 2}\right)$$

Proof. Let $\varphi(x, y, z) := \theta(\|x\|^{p_1} \cdot \|y\|^{p_2} \cdot \|z\|^{p_3})$ and $|r| = 1$. Applying Theorem 2.7, we get the desired result. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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