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Orthogonal Stability of an Additive-Quadratic Functional Equation

Choonkil Park

Correspondence: baak@hanyang.ac.kr
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea

Abstract

Using the fixed point method and using the direct method, we prove the Hyers-Ulam stability of an orthogonally additive-quadratic functional equation in orthogonality spaces.

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1. Introduction and Preliminaries

Assume that X is a real inner product space and $f : X \rightarrow \mathbb{R}$ is a solution of the orthogonally Cauchy functional equation $f(x + y) = f(x) + f(y)$, $\langle x, y \rangle = 0$. By the Pythagorean theorem, $f(x) = ||x||^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonally Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

Pinsker [1] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [2] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonally Cauchy functional equation

$$f(x + y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [3]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [4] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [5] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of Rätz; cf. [4].

Suppose X is a real vector space (algebraic module) with dim $X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

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(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, ||\cdot||)$ defined by $x \perp y$ if and only if $||x + \lambda y|| \geq ||x||$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [6-12]).

The stability problem of functional equations originated from the following question of Ulam [13]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* In 1941, Hyers [14] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [15] extended the theorem of Hyers by considering the unbounded Cauchy difference $||f(x + y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p)$, ($\varepsilon > 0$, $p \in [0, 1)$). The result of Th.M. Rassias has provided a lot of influence in the development of what we now call *generalized Hyers-Ulam stability* or *Hyers-Ulam stability* of functional equations. During the last decades, several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [16-20] and references therein for detailed information on stability of functional equations.

Ger and Sikorska [21] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $||f(x + y) - f(x) - f(y)|| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $||f(x) - g(x)|| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was Skof [22] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $||f(x + y) + f(x - y) - 2f(x) - 2f(y)|| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $||f(x) - g(x)|| \leq \frac{\varepsilon}{2}$. Cholewa [23] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by Czerwik [24] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [25-28]).

The orthogonally quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), x \perp y$$

was first investigated by Vajzović [29] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [30], Fochi [31], Moslehian [32,33], Szabó [34], Moslehian and Th.M. Rassias [35] and Paganoni and Rätz [36] have investigated the orthogonal stability of functional equations.

Let X be a set. A function $m : X \times X \to [0, \infty]$ is called a *generalized metric* on X if m satisfies

(1) $m(x, y) = 0$ if and only if $x = y$;
(2) $m(x, y) = m(y, x)$ for all $x, y \in X$;
(3) $m(x, z) \leq m(x, y) + m(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [37,38]*Let (X, m) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$m(J^n x, \ J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $m(J^n x, J^{n+1} x) < \infty, \ \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) y^* *is the unique fixed point of J in the set $Y = \{y \in X | m(J^{n_0} x, y) < \infty\}$;*
(4) $m(y, y^*) \leq \frac{1}{1-\alpha} m(y, Jy)$*for all $y \in Y$.*

In 1996, Isac and Th.M. Rassias [39] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [40-46]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the following orthogonally additive-quadratic functional equation

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = \frac{3f(x)}{2} - \frac{f(-x)}{2} + \frac{f(y)}{2} + \frac{f(-y)}{2} \tag{1.1}$$

in orthogonality spaces by using the fixed point method. In Section 3, we prove the Hyers-Ulam stability of the orthogonally additive-quadratic functional equation (1.1) in orthogonality spaces by using the direct method.

Throughout this paper, assume that $(X, \perp)$ is an orthogonality space and that $(Y, ||.||_Y)$ is a real Banach space.

2. Hyers-Ulam Stability of the Orthogonally Additive-Quadratic Functional Equation (1.1): Fixed Point Method

For a given mapping $f : X \to Y$, we define

$$
\begin{aligned}
Df(x, y) : \ &= \ 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) \\
&- \ \frac{3f(x)}{2} + \frac{f(-x)}{2} - \frac{f(y)}{2} - \frac{f(-y)}{2}
\end{aligned}
$$

for all $x, y \in X$ with $x \perp y$, where $\perp$ is the orthogonality in the sense of Rätz.

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (1.1). Then, f is a quadratic mapping, i.e., $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$ holds.

Using the fixed point method and applying some ideas from [18,21], we prove the Hyers-Ulam stability of the additive-quadratic functional equation $Df(x, y = 0)$ in orthogonality spaces.

Theorem 2.1. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \le 4\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.1}$$

for all $x, y \in X$ with $x \perp Y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\| Df(x, y)\|_Y \le \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{\alpha}{1-\alpha}\varphi(x, 0) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.2), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\|_Y \le \varphi(x, 0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. Thus

$$\left\| f(x) - \frac{1}{4}f(2x) \right\|_Y \le \frac{1}{4}\varphi(2x, 0) \le \frac{4\alpha}{4}\varphi(x, 0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$m(g, h) = \inf\{\mu \in \mathbb{R}_+ : \| g(x) - h(x)\|_Y \le \mu\varphi(x, 0), \ \forall x \in X\},$$

where, as usual, inf φ = +∞. It is easy to show that (S, m) is complete (see [[47], Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $m(g, h) = \varepsilon$. Then,

$$\| g(x) - h(x)\|_Y \le \varphi(x, 0)$$

for all $x \in X$. Hence

$$\| Jg(x) - Jh(x)\|_Y = \left\| \frac{1}{4}g(2x) - \frac{1}{4}h(2x) \right\|_Y \le \alpha\varphi(x, 0)$$

for all $x \in X$. So $m(g, h) = \varepsilon$ implies that $m(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$m(Jg, Jh) \leq \alpha m(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that $m(f, Jf) \leq \alpha$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(2x) = 4Q(x) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : m(h, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\|_Y \quad \leq \quad \mu\varphi(x, 0)$$

for all $x \in X$;

(2) $m(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} \frac{1}{4^n} f(2^n x) = Q(x)$$

for all $x \in X$;

(3) $m(f, Q) \leq \frac{1}{1-\alpha} m(f, Jf)$, which implies the inequality

$$m(f, Q) \leq \frac{\alpha}{1 - \alpha}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$
\begin{aligned}
\left\| DQ(x, y) \right\|_Y &= \lim_{n \to \infty} \frac{1}{4^n} \| Df(2^n x, 2^n y) \|_Y \\
&\leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \leq \lim_{n \to \infty} \frac{4^n \alpha^n}{4^n} \varphi(x, y) = 0
\end{aligned}
$$

for all $x, y \in X$ with $x \perp y$. So $DQ(x, y) = 0$ for all $x, y \in X$ with $x \perp y$. Hence $Q : X \to Y$ is an orthogonally quadratic mapping, as desired. □

Corollary 2.2. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\| Df(x, y)\|_Y \leq \theta(\| x\|^p + \| y\|^p) \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x)\|_Y \quad \leq \quad \frac{2^p \theta}{4 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{p-2}$ in Theorem 2.1, we get the desired result. □

Theorem 2.3. *Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \le \frac{\alpha}{4}\varphi(2x, 2y)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique orthogonally quadratic mapping $Q : X \rightarrow Y$ such that

$$\| f(x) - Q(x)\|_Y \le \frac{1}{1 - \alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, m) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $m(f, Jf) \le 1$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then, there exists a unique orthogonally quadratic mapping $Q : X \rightarrow Y$ such that*

$$\| f(x) - Q(x)\|_Y \le \frac{2^p\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{2-p}$ in Theorem 2.3, we get the desired result. □

Let $f : X \rightarrow Y$ be an odd mapping satisfying (1.1). Then, f is an additive mapping, i.e., $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = 2f(x)$ holds.

Theorem 2.5. *Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \le 2\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \rightarrow Y$ be an odd mapping satisfying (2.2). Then, there exists a unique orthogonally additive mapping $A : X \rightarrow Y$ such that

$$\| f(x) - A(x)\|_Y \le \frac{\alpha}{2 - 2\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.2), we get

$$\left\|4f\left(\frac{x}{2}\right) - 2f(x)\right\|_Y \le \varphi(x, 0) \tag{2.8}$$

for all $x \in X$, since $x \perp 0$. Thus,

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \le \frac{1}{4}\varphi(2x, 0) \le \frac{2\alpha}{4}\varphi(x, 0) \tag{2.9}$$

for all $x \in X$.

Let (S, m) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

It follows from (2.9) that $m(f, Jf) \leq \frac{\alpha}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \|_Y \leq \frac{2^p \theta}{2(2 - 2^p)} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{p-1}$ in Theorem 2.5, we get the desired result. □

Theorem 2.7. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq \frac{\alpha}{2}\varphi(2x, 2y)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\| f(x) - A(x) \|_Y \leq \frac{1}{2 - 2\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, m) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.8) that $m(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \|_Y \leq \frac{2^p \theta}{2(2^p - 2)} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{1-p}$ in Theorem 2.7, we get the desired result. □

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (1.1). Let $f_e(x) := \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then, f_e is an even mapping satisfying (1.1) and f_o is an odd mapping satisfying (1.1) such that $f(x) = f_e(x) + f_o(x)$. So we obtain the following.

Theorem 2.9. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a positive real number with $p \neq 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then, there exist an orthogonally additive mapping $A : X \to Y$ and an orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\| f(x) - A(x) - Q(x) \|_Y \leq \left(\frac{2^p}{2|2 - 2^p|} + \frac{2^p}{|4 - 2^p|} \right) \theta \|x\|^p$$

for all $x \in X$.

3. Hyers-Ulam Stability of the Orthogonally Additive-Quadratic Functional Equation (1.1): Direct Method

In this section, using the direct method and applying some ideas from [18,21], we prove the Hyers-Ulam stability of the additive-quadratic functional equation $Df(x, y) = 0$ in orthogonality spaces.

Theorem 3.1. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^2 \to [0, \infty)$ satisfying (2.2) and*

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 4^j \varphi \left(\frac{x}{2^j}, \frac{y}{2^j} \right) < \infty \tag{3.1}$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \|_Y \leq \tilde{\varphi}(x, 0) \tag{3.2}$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\left\| 4^l f \left(\frac{x}{2^l} \right) - 4^m f \left(\frac{x}{2^m} \right) \right\|_Y \leq \sum_{j=1}^{m-1} 4^j \varphi \left(\frac{x}{2^j}, 0 \right) \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.1) and (3.3) that the sequence $\{4^n f (\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f (\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} 4^n f \left(\frac{x}{2^n} \right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.1, one can show that the mapping $Q : X \to Y$ is an orthogonally quadratic mapping satisfying (3.2).

Now, let $Q' : X \to Y$ be another orthogonally quadratic mapping satisfying (3.2). Then, we have

$$\begin{aligned}
\| Q(x) - Q'(x) \|_Y &= 4^n \left\| Q \left(\frac{x}{2^n} \right) - Q' \left(\frac{x}{2^n} \right) \right\|_Y \\
&\leq 4^n \left(\left\| Q \left(\frac{x}{2^n} \right) - f \left(\frac{x}{2^n} \right) \right\|_Y + \left\| Q' \left(\frac{x}{2^n} \right) - f \left(\frac{x}{2^n} \right) \right\|_Y \right) \\
&\leq 2 \cdot 4^n \tilde{\varphi} \left(\frac{x}{2^n}, 0 \right),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. □

Corollary 3.2. *Assume that $(X, \perp)$ is an orthogonality space. Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then, there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\| f(x) - Q(x) \|_Y \le \frac{2^p \theta}{2^p - 4} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$, and applying Theorem 3.1, we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 3.3. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^2 \to [0, \infty)$ satisfying (2.2) and*

$$\tilde{\varphi}(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \|_Y \le \tilde{\varphi}(x, 0)$$

for all $x \in X$.

Corollary 3.4. *Assume that $(X, \perp)$ is an orthogonality space. Let θ be a positive real number and p a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then, there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \le \frac{2^p \theta}{4 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$, and applying Theorem 3.3, we get the desired result. □

Theorem 3.5. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\phi : X^2 \to [0, \infty)$ satisfying (2.2) and*

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.4}$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{1}{2} \tilde{\varphi}(x, 0) \tag{3.5}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_Y \le \frac{1}{2} \varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.5 and 3.1. □

Corollary 3.6. *Assume that $(X, \perp)$ is an orthogonality space. Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \|_Y \leq \frac{2^p \theta}{2(2^p - 2)} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$, and applying Theorem 3.5, we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 3.7. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\phi : X^2 \to [0, \infty)$ satisfying (2.2) and*

$$\tilde{\varphi}(x, y) : \quad = \quad \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\| f(x) - A(x) \|_Y \leq \frac{1}{2} \tilde{\varphi}(x, 0)$$

for all $x \in X$.

Corollary 3.8. *Assume that $(X, \perp)$ is an orthogonality space. Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then, there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \|_Y \leq \frac{2^p \theta}{2(2 - 2^p)} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$, and applying Theorem 3.7, we get the desired result. □

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