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Trends, probability functions and fuzzy right ideals for *d*-algebras

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ABSTRACT

In this paper, we introduce the notion of a trend (probability function) on a *d*-algebra, and obtain an equivalent condition defining a trend π_0 with condition (j) on a standard *BCK*-algebra. We obtain an example of *d*-algebra such that any function $\varphi : X \to (0, 1)$ with $\int_0^\infty \varphi(z)dz < \infty$ is a fuzzy co-right ideal.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras: *BCK*-algebras and *BCI*-algebras [1,2]. Dvurečenskij and Graziano [3], Hoo [4] and Font et al. [5] have discussed *BCK*-algebras in connection with the areas of lattice ordered groups, *MV*-algebras and Wajsberg algebras. Mundici [6] proved that *MV*-algebras are categorically equivalent to bounded commutative *BCK*-algebras, and Meng [7] proved that implicative commutative semigroups are equivalent to a class of *BCK*-algebras. Georgescu and Iorgulescu [8] introduced the notion of pseudo-*BCK* algebras as an extension of *BCK*-algebras. Neggers and Kim introduced the notion of *d*-algebras which is another useful generalization of *BCK*-algebras, and then investigated several relations between *d*-algebras and *BCK*-algebras as well as several other relations between *d*-algebras and oriented digraphs [9]. After that several further aspects were studied [10–12,9]. Recently, Han et al. [13] defined several special varieties of *d*-algebras, such as strong *d*-algebras, (weakly) selective *d*-algebras and others, and they dealt with a generalization of *d*-algebras such as pre-*d*-algebras. Much of their discussion involved the associative groupoid product $(X, \square) = (X, *)\square(X, \circ)$ where $x\square y = (x * y) \circ (y * x)$. One assertion is that the squared algebra $(X, \square, 0)$ of a pre-*d*-algebra $(X, *, 0)$ is a strong *d*-algebra if and only if $(X, *, 0)$ is strong.

In this paper, we introduce the notion of a trend (probability function) on a *d*-algebra, and obtain an equivalent condition for a trend π_0 with condition (j) on a standard *BCK*-algebra, and we obtain an example of a *d*-algebra such that any function $\varphi: X \to (0, 1)$ with $\int_0^\infty \varphi(z)dz < \infty$ is a fuzzy co-right ideal.

2. Preliminaries

A *d-algebra* [9] is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

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Fig. 1.

For brevity we also call X a *d-algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

A *BCK*-algebra is a *d*-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Let $(X, \leq)$ be a poset with minimal element 0. Define a binary operation "$*$" on X by

$$x * y := \begin{cases} 0 & \text{if} x \leq y, \\ x & \text{otherwise.} \end{cases}$$

Then $(X, *, 0)$ is a *BCK*-algebra, called a *standard BCK-algebra*.

Theorem 2.1 ([11]). *Let $(X, *, 0)$ be a BCK-algebra.*

(i) *if $x * y = 0 = y * z$, then $x * z = 0$,*
(ii) *if $x * y = 0$, then $(z * y) * (z * x) = 0$,*
(iii) *if $x * y = 0$, then $(x * z) * (y * z) = 0$.*

Definition 2.2 ([14]). Let $(X, *, 0)$ be a *d*-algebra and $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* of X if it satisfies:

(D_0) $0 \in I$,
(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

It is known that, in a *d*-algebra X, a *BCK*-ideal need not be a *d*-subalgebra, and also a *d*-subalgebra need not be a *BCK*-ideal. Moreover, $\{0\}$ is a *d*-subalgebra of any *d*-algebra X and every *d*-ideal of X is a *d*-subalgebra, but the converse need not be true. Note that every *d*-ideal of a *d*-algebra is a *BCK*-ideal, but the converse need not be true [14].

Given a poset $P(\leq)$ it is *A-free* if there is no full-subposet $X(\leq)$ of $P(\leq)$ which is order isomorphic to the poset $A(\leq)$. If C_n denotes a chain of length n and if $\underline{n}$ denotes an antichain of cardinal number n, while $+$ denotes the disjoint union of posets, then the poset $(C_2 + \underline{1})$ (or $C_2 + C_1$) has the Hasse-diagram (Fig. 1), and may be represented as $\{p \leq q, p \| r, q \| r\}$, where $a \parallel b$ denotes the relation of not being comparable (i.e., $a \parallel b$ if and only if $a \leq b$ and $b \leq a$ are both false) (see [15]).

Let μ be a fuzzy set in a *d*-algebra X. Then a fuzzy set μ is called a *fuzzy d-subalgebra* [1] of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$. A fuzzy set μ is called a *fuzzy BCK-ideal* of X [16] if

(F_0) $\mu(0) \geq \mu(x)$,
(F_1) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,

for all $x, y \in X$. A fuzzy set μ is called a *fuzzy d-ideal* [1] of X if it satisfies (F_1) and

(F_2) $\mu(x * y) \geq \mu(x)$ for all $x, y \in X$.

It is known that, in a *d*-algebra, a fuzzy *BCK*-ideal need not be a fuzzy *d*-subalgebra, and also a fuzzy *d*-subalgebra need not be a fuzzy *BCK*-ideal. Moreover, every fuzzy *d*-ideal of a *d*-algebra X is a fuzzy *d*-subalgebra, but the converse need not be true. Note that every fuzzy *d*-ideal of a *d*-algebra is a fuzzy *BCK*-ideal, but the converse need not be true.

Lemma 2.3 ([1]). *If μ is a fuzzy d-ideal of a d-algebra X, then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Theorem 2.4 ([1]). *A fuzzy subset μ of a d-algebra X is a fuzzy d-subalgebra (d-ideal, resp.) of X if and only if, for every $\lambda \in [0, 1]$, $\mu_\lambda = \{x \in X | \mu(x) \geq \lambda\}$ is a d-subalgebra (d-ideal, resp.) of X, where $\mu_\lambda \neq \emptyset$.*

3. Trends and probability functions

Let X be a non-empty set and let $\pi : X \times X \to [0, 1]$ be a mapping. We say $\pi(x, y)$ that "*x is less than or equal to y*. Among the rules π may be expected to obey the following:

(a) $x * y = 0$ implies $\pi(x, y) = 1$;
(b) $x * y \neq 0$ implies $\pi(x, y) + \pi(y, x) = 1$;
(c) $y * z = 0$ implies $\pi(x, y) \leq \pi(x, z)$;

(d) $\pi(x, z) \le \pi(x * y, z)$;
(e) $\pi(x, y * z) \le \pi(x, y)$

for any $x, y, z \in X$.

Let $(X, *)$ be any groupoid. We say π is a *trend* on X if it satisfies (a) and (b). A trend π on X is said to be a *probability function* on X if it satisfies (c).

Proposition 3.1. (i) *Let $(X, *)$ be a groupoid with axiom (I). If π is a trend on X, then $\pi(x, x) = 1$ for any $x \in X$.*
(ii) *Let $(X, *)$ be a groupoid with axiom (II). If π is a trend on X, then $\pi(0, x) = 1$ for any $x \in X$.*
(iii) *If $(X, *, 0)$ is a d-algebra and π is a trend on X, then $\pi(x, 0) = 0$ for any $x \in X - \{0\}$.*
(iv) *If $\pi_i(i = 1, 2)$ are trends on a groupoid $(X, *, 0)$ and $0 \le \alpha \le 1$, then $\pi = \alpha\pi_1 + (1 - \alpha)\pi_2$ is also a trend on $(X, *, 0)$.*

Proof. (i). Given $x \in X, x*x = 0$. Since π is a trend, we have $\pi(x, x) = 1$. (ii). Since $0*x = 0, \pi(0, x) = 1$. (iii). Let $(X, *, 0)$ be a d-algebra. We claim that $x*0 \neq 0$ for any $x \in X-\{0\}$. Assume that there is an x in $X-\{0\}$ such that $x*0 = 0$. Since $0*x = 0$, by (III) we obtain $x = 0$, a contradiction. Since π is a trend, $1 = \pi(x, 0) + \pi(0, x) = \pi(x, 0) + 1$ and hence $\pi(x, 0) = 0$ for any $x \in X - \{0\}$. (iv). If $x * y = 0$, then $\pi_i(x, y) = 1$ and hence $(\alpha\pi_1 + (1 - \alpha)\pi_2)(x, y) = \alpha\pi_1(x, y) + (1 - \alpha)\pi_2(x, y) = \alpha + (1 - \alpha) = 1$. If $x * y \neq 0$, then $\pi_i(x, y) + \pi_i(y, x) = 1$ and hence $(\alpha\pi_1 + (1 - \alpha)\pi_2)(x, y) + (\alpha\pi_1 + (1 - \alpha)\pi_2)(y, x) = \alpha[\pi_1(x, y) + \pi_1(y, x)] + (1 - \alpha)[\pi_2(x, y) + \pi_2(y, x)] = 1$. □

Proposition 3.2. *Let $(X, *, 0)$ be a d-algebra. If $T(X) := \{ \pi \mid \pi \text{ is a trend on } X \}$, then $T(X) \neq \emptyset$.*

Proof. Define a map $\pi_0 : X^2 \to [0, 1]$ by

$$\pi_0(x, y) := \begin{cases} 0 & \text{if } y * x = 0,\ y \neq x, \\ 1 & \text{if } x * y = 0, \\ \dfrac{1}{2} & \text{if } x * y \neq 0 \neq y * x. \end{cases}$$

Then it is easy to see that π_0 is a trend on X. □

Let $(X, *, 0)$ be a d-algebra and let $P(X) := \{\pi \mid \pi \text{ is a probability function on } X \}$. Then the trend π_0 need not be a probability function on X. See the following example.

Example 3.3. Consider a d-algebra $X := \{0, a, b, c\}$ with the following table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Since $b*c = 0, a*c = a, c*a = c, a*b = 0, b*a = b$, we have $\pi_0(a, c) = 1/2, \pi_0(a, b) = 1$, i.e., $\pi_0(a, b) > 1/2 = \pi_0(a, c)$ proving that $\pi_0 \notin P(X)$.

Given a d-algebra $(X, *, 0)$, we classify 3 cases: (i) $P(X) = \emptyset$; (ii) $P(X) \neq \emptyset, \pi_0 \notin P(X)$; (iii) $\pi_0 \in P(X)$.

A d-algebra $(X, *, 0)$ is said to be *d-transitive* [9] if $x * y = 0 = y * z$ implies $x * z = 0$.

Theorem 3.4. *If $(X, *, 0)$ is a d-transitive d-algebra, then $\pi_0 \in P(X)$.*

Proof. Assume that π_0 is not a probability function on X. Then there exist $x, y, z \in X$ such that $y * z = 0$, but $\pi_0(x, y) > \pi_0(x, z)$. *Case* 1: $\pi_0(x, y) = 1, \pi_0(x, z) = 1/2$. Then $x * y = 0, x * z \neq 0 \neq z * x$. Since $y * z = 0$, we obtain $x * z = 0$, a contradiction. *Case* 2: $\pi_0(x, y) = 1, \pi_0(x, z) = 0$. Then $x * y = 0 = z * x$, and hence $z * y = 0$. Since $y * z = 0$, we obtain $y = z$, a contradiction. *Case* 3: $\pi_0(x, y) = 1/2, \pi_0(x, z) = 0$. Then $x * y \neq 0 \neq y * x, z * x = 0$. Since $y * z = 0$, we obtain $y * x = 0$, a contradiction. □

Corollary 3.5. *Let $(X, *, 0)$ be a BCK-algebra. Then $\pi_0 \in P(X)$, i.e., π_0 is a probability function on X.*

Proof. By Theorem 2.1-(i), every *BCK*-algebra is a d-transitive d-algebra. □

Proposition 3.6. *Let π be a trend on a BCK-algebra $(X, *, 0)$. Then*

(f) $x * y = 0$ *implies* $\pi(z * y, z * x) = 1$.

Proof. If $x * y = 0$, then by Theorem 2.1-(ii), $(z * y) * (z * x) = 0$. Since π is a trend on X, we obtain $\pi(z * y, z * x) = 1$. □

Proposition 3.7. *Let π be a trend on a BCK-algebra $(X, *, 0)$. Then*

(g) $x * y = 0$ *implies* $\pi(x * z, y * z) = 1$.

Proof. If $x * y = 0$, then by Theorem 2.1-(iii), $(x * z) * (y * z) = 0$. Since π is a trend on X, we obtain $\pi(x * z, y * z) = 1$. □

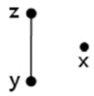

Fig. 2.

Fig. 3.

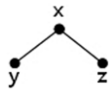

Fig. 4.

Proposition 3.8. *Let* $(X, *, 0)$ *be a d-algebra. If* π *is a probability function on* X*, then*

(h) $x * y = 0 = y * z$ *implies* $\pi(x, z) = 1$.

Proof. Since π is a probability function on X, if $x * y = 0 = y * z$, then $1 = \pi(x, y) \leq \pi(x, z)$, proving $\pi(x, z) = 1$. □

Proposition 3.9. *Let* $(X, *, 0)$ *be a standard BCK-algebra. Then the trend* π_0 *satisfies condition* (d).

Proof. If $x * y = x$, then $\pi_0(x * y, z) = \pi_0(x, z)$. If $x * y = 0$, then by Proposition 3.1-(ii), $\pi_0(x * y, z) = \pi_0(0, z) = 1 \geq \pi_0(x, z)$. □

Let X be a non-empty set and let $\pi : X \times X \to [0, 1]$ be a mapping. We have additional rules as follows:

(j) $\pi(x * y, z) \geq \frac{1}{2}[\pi(x, z) + \pi(y, z)]$;
(k) $\pi(x * y, z) \geq \pi(y, z)$.

Condition (j) is a special case of the condition (j_α):

(j_α) $\pi(x * y, z) \geq \alpha\pi(x, z) + (1 - \alpha)\pi(y, z)$

where $0 \leq \alpha \leq 1$. Note that if $\alpha = 1$, then $(j_1) = (d)$, and if $\alpha = 1/2$, then $(j_{1/2}) = (j)$.

Lemma 3.10. *Let* $(X, *, 0)$ *be a standard BCK-algebra associated with a poset* $(X, \leq)$ *with minimal element 0, and let* π_0 *be a trend on* X *with condition* (j). *Then the poset* $(X, \leq)$ *is* $(C_2 + \underline{1})$*-free.*

Proof. Assume that $(X, \leq)$ has a full subposet $(C_2 + \underline{1})$, i.e., there exist $x, y, z \in X$ such that we obtain the diagram in Fig. 2.

Then $x * y = x$ and $\pi_0(x * y, z) = \pi_0(x, z) = 1/2$. Since $\pi_0(y, z) = 1$, we have $\pi_0(x * y, z) = 1/2 < 3/4 = \frac{1}{2}[\pi_0(x, z) + \pi_0(y, z)]$, which shows that π_0 does not satisfy condition (j), a contradiction. □

Lemma 3.11. *Let* $(X, *, 0)$ *be a standard BCK-algebra associated with a poset* $(X, \leq)$ *with minimal element 0, and let* π_0 *be a trend on* X *with condition* (j). *Then the poset* $(X, \leq)$ *is* C_3*-free.*

Proof. Assume that $(X, \leq)$ has a full subposet C_3, i.e., there exist $x, y, z \in X$ such that we obtain the diagram in Fig. 3.

Then $x * y = x$ and hence $\pi_0(x * y, z) = \pi_0(x, z) = 0$. Since $\pi_0(y, z) = 1$, we obtain $\pi_0(x * y, z) = 0 < 1/2 = \frac{1}{2}[\pi_0(x, z) + \pi_0(y, z)]$, which shows that π_0 does not satisfy the condition (j), a contradiction. □

Lemma 3.12. *Let* $(X, *, 0)$ *be a standard BCK-algebra associated with a poset* $(X, \leq)$ *with minimal element 0, and let* π_0 *be a trend on* X *with condition* (j). *Then the poset* $(X, \leq)$ *is* $\underline{2} \oplus \underline{1}$*-free.*

Proof. Assume that $(X, \leq)$ has a full subposet $\underline{2} \oplus \underline{1}$, i.e., there exist $x, y, z \in X$ such that we obtain the diagram in Fig. 4.

Then $x * y = x$ and hence $\pi_0(x * y, z) = \pi_0(x, z) = 0$. Since $\pi_0(y, z) = 1/2$, we obtain $\pi_0(x * y, z) = 0 < 1/2 = \frac{1}{2}[\pi_0(x, z) + \pi(y, z)]$, which shows that π_0 does not satisfy condition (j), a contradiction. □

Theorem 3.13. *Let* $(X, *, 0)$ *be a standard BCK-algebra associated with a poset* $(X, \leq)$ *with minimal element 0, and let* π_0 *be a trend on* X *with condition* (j). *Then the poset* $(X, \leq)$ *is free of* $(C_2 + \underline{1})$, C_3 *and* $\underline{2} \oplus \underline{1}$.

Proof. It follows from Lemmas 3.10–3.12 □

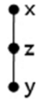

Fig. 5.

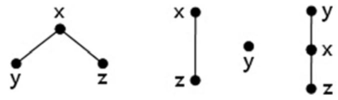

Fig. 6.

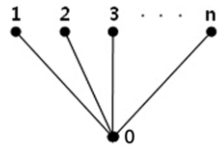

Fig. 7.

It is natural to have a question. Is the converse of Theorem 3.13 also true ? The answer is yes.

Theorem 3.14. *Let* $(X, *, 0)$ *be a standard BCK-algebra associated with a poset* $(X, \leq)$ *with minimal element 0 which is free of* $(C_2 + \underline{1})$, C_3 *and* $\underline{2} \oplus \underline{1}$. *Then the trend* π_0 *satisfies condition* (j).

Proof. Assume the trend π_0 does not satisfy condition (j). Then there exist $x, y, z \in X$ such that

$$\pi_0(x * y, z) < \frac{\pi_0(x, z) + \pi_0(y, z)}{2} \cdots\cdots (**)$$

Case 1. $\pi_0(x * y, z) = 1/2$: By the above inequality $(**)$ we obtain $\pi_0(x, z) = 1 = \pi_0(y, z)$ and hence $x * z = 0 = y * z$. Since $(x * y) * z \neq 0 \neq z * (x * y)$ and $(X, *, 0)$ is a standard BCK-algebra, we obtain $x * y = x$. In fact, if $x * y = 0$, then $0 \neq (x * y) * z = 0$, a contradiction. Hence $0 = x * z = (x * y) * z \neq 0$, a contradiction. *Case* 2. $\pi_0(x * y, z) = 0$, i.e., $z * (x * y) = 0, z \neq x * y$: To satisfy the inequality $(**)$, at least one of $\pi_0(x, z)$ or $\pi_0(y, z)$ should not be zero. Subcase (2.1). $\pi_0(x, z) \neq 0$: We obtain either $x * z = 0$ or $x * z \neq 0 \neq z * x$. Subcase (2.1.1) $x * z = 0$: We claim that $x * y = 0$. In fact, if $x * y = x$, then $0 = z * (x * y) = z * x$, since $\pi_0(x * y, z) = 0$. By assumption $x * z = 0$, we obtain $x = z$, which leads to $z \neq x * y = x$, a contradiction, proving the claim. Hence $0 = z * (x * y) = z * 0 = z \neq x * y = 0$, a contradiction. Hence the subcase (2.1.1) cannot happen. Subcase (2.1.2). $x * z \neq 0 \neq z * x$: This means that x and z are incomparable. Since $\pi_0(x * y, z) = 0$, we have $z * (x * y) = 0, z \neq x * y$. We claim that $x * y = x$. In fact, if $x * y = 0$, then $0 = z * (x * y) = z * 0 = z \neq x * y = 0$, a contradiction. Hence $0 = z * (x * y) = z * x \neq 0$, a contradiction. Hence this case cannot happen. Subcase (2.2). $\pi_0(y, z) \neq 0$: Then we have either $y * z = 0$ or $y * z \neq 0 \neq z * y$. Subcase (2.2.1). $y * z = 0$: Since $\pi_0(x * y, z) = 0$, we have $z * (x * y) = 0, z \neq x * y$. We claim that $x * y = x$. In fact, if $x * y = 0$, then $0 = z * (x * y) = z * 0 = z \neq x * y = 0$, a contradiction. Hence $0 = z * (x * y) = z * x$. Thus we have as a subposet (Fig. 5), a contradiction. Subcase (2.2.2). $y * z \neq 0 \neq z * y$: This means that y and z are incomparable. We claim that $x * y = x$. In fact, if $x * y = 0$, then $0 = z * (x * y) = z * 0 = z \neq x * y = 0$, a contradiction. Hence $z * x = z * (x * y) = 0$, i.e., $z \leq x$. Since $(X, *, 0)$ is a standard BCK-algebra, we have

$$y * x := \begin{cases} y & \text{if } y > x \text{ or } y \parallel x, \\ 0 & \text{if } y \leq x. \end{cases}$$

In any case, we have one of the diagrams in Fig. 6, which is a contradiction, proving the theorem. □

Example 3.15. Let $(X, *, 0)$ be a standard BCK-algebra associated with a poset $\underline{1} \oplus A$, where A is an antichain (see Fig. 7). Then the trend π_0 satisfies condition (j).

4. Fuzzy right ideals and fuzzy co-right ideals

Let $(X, *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is called a *right ideal* of X if it satisfies conditions (D_0) and (D_2).

Example 4.1. In Example 3.3, if we let $J := \{0, a, c\}$, then J is a right ideal of X, but not a d/BCK-ideal, since $b * c = 0 \in J$, $c \in J$, but $b \notin J$.

We fuzzify the notion of right ideals in d-algebras as follows. Let X be a d-algebra. A fuzzy subset μ of X is said to be a *fuzzy right ideal* of X if it satisfies conditions (F_0) and (F_2). Then we obtain the following proposition. The proof is similar to Theorem 2.4, and we omit it.

Proposition 4.2. *A fuzzy subset μ of a d-algebra X is a fuzzy right ideal of X if and only if, for every $\lambda \in [0, 1]$, $\mu_\lambda = \{x \in X | \mu(x) \geq \lambda\}$ is a right ideal of X, where $\mu_\lambda \neq \emptyset$.*

Note that every fuzzy right ideals of a d-algebra $(X, *, 0)$ is a fuzzy d-subalgebra of X.

Proposition 4.3. *Let $(X, *, 0)$ be a d-algebra. If μ is a fuzzy d-ideal of X, then it is a fuzzy right ideal of X.*

Proof. It follows immediately from the definitions of fuzzy d/right-ideals and Lemma 2.3. □

Note that the converse of Proposition 4.3 need not be true in general.

Example 4.4. Consider a d-algebra $(X, *, 0)$ [1] with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Define a map $\mu : X \to [0, 1]$ by $\mu(a) = t_1$, $\mu(0) = \mu(b) = \mu(c) = t_2$, $t_1 < t_2$. Then it is a fuzzy right ideal of X, but not a fuzzy d-ideal of X, since $\mu(a) = t_1 < t_2 = \min\{\mu(b), \mu(c)\} = \min\{\mu(a * c), \mu(c)\}$.

Proposition 4.5. *Let $(X, *, 0)$ be a d-algebra with $|X| = n$. Let $\pi : X^2 \to [0, 1]$ be a trend on X with condition (d). If we define a map $\mu_\pi(x) := \frac{1}{n}\sum_{y\in X} \pi(x, y)$, then it is a fuzzy right ideal of X.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned}\mu_\pi(x * y) &= \frac{1}{n}\sum_{z\in X} \pi(x * y, z)\\ &\geq \frac{1}{n}\sum_{z\in X} \pi(x, z)\\ &= \mu_\pi(x),\end{aligned}$$

proving the proposition. □

The map μ_π discussed in Proposition 4.5 is also a fuzzy d-algebra of X. We discuss the map μ_π in the case that π has condition (j).

Proposition 4.6. *Let $(X, *, 0)$ be a d-algebra with $|X| = n$. Let $\pi : X^2 \to [0, 1]$ be a trend on X with condition (j). If we define a map $\mu_\pi(x) := \frac{1}{n}\sum_{y\in X} \pi(x, y)$, then it is a fuzzy d-subalgebra of X.*

Proof. Let $t \in [0, 1]$ with $(\mu_\pi)_t \neq \emptyset$ and let $x, y \in (\mu_\pi)_t$. Then

$$\begin{aligned}\mu_\pi(x * y) &= \frac{1}{n}\sum_{z\in X} \pi(x * y, z)\\ &\geq \frac{1}{n}\sum_{z\in X} \frac{\pi(x, z) + \pi(y, z)}{2}\\ &= [\mu_\pi(x) + \mu_\pi(y)]/2\\ &\geq t,\end{aligned}$$

proving that $x * y \in (\mu_\pi)_t$. By Theorem 2.4, we proved that μ_π is a fuzzy d-subalgebra of X. □

Let $(X, *, 0)$ be a d-algebra with $|X| = n$. Let π be a trend on X and $\varphi : X \to (0, \infty)$ be a map. Define a map $\mu_\pi^\varphi : X \to [0, 1]$ by $\mu_\pi^\varphi(x) := \sum_{z\in X} \pi(x, z)\varphi(z)$. The map φ is said to be a *fuzzy co-d-subalgebra* (*fuzzy co-right ideal*, resp.) of X if μ_π^φ is a fuzzy d-subalgebra (fuzzy right ideal, resp.) of X. In continuous cases, we define the map $\mu_\pi^\varphi : X \to [0, 1]$ by $\mu_\pi^\varphi(x) := \int_0^\infty \pi(x, z)\varphi(z)\mathrm{d}z$.

Proposition 4.7. *Let $(X, *, 0)$ be a d-algebra with $|X| = n$. Let π be a trend on X with condition (d). Then the map $\varphi : X \to (0, \infty)$ is a fuzzy co-right ideal of X.*

Proof. Assume that π has condition (d). Since $\varphi(z) > 0$, $\pi(x * y, z)\varphi(z) \geq \pi(x, z)\varphi(z)$ and hence $\mu_\pi^\varphi(x * y) = \sum_{z\in X} \pi(x * y, z)\varphi(z) \geq \sum_{z\in X} \pi(x, z)\varphi(z) = \mu_\pi^\varphi(x)$. □

Theorem 4.8. *Let $X := [0, \infty)$. Define a binary operation "$*$" on X by*

$$x * y := \begin{cases} 0 & \text{if } x \le y, \\ x - y & \text{otherwise.} \end{cases}$$

If we define a map $\pi : X^2 \to [0, 1]$ by

$$\pi(x, y) := \begin{cases} 1 & \text{if } x \le y, \\ 0 & \text{otherwise,} \end{cases}$$

then any map $\varphi : X \to (0, 1)$ with $\int_0^\infty \varphi(x)\mathrm{d}x < \infty$ is a fuzzy co-right ideal of X.

Proof. It is easy to see that π is a trend and also a probability function on X. Given $x \in X$, we have

$$\begin{aligned} \mu_\pi^\varphi(x) &= \int_0^\infty \pi(x, z)\varphi(z)\mathrm{d}z \\ &= \int_x^\infty \varphi(z)\mathrm{d}z. \end{aligned}$$

Using Leibniz's rule, we obtain $\frac{\mathrm{d}}{\mathrm{d}x}[\mu_\pi^\varphi(x)] = \frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \varphi(z)\mathrm{d}z = -\varphi(x)$. Let $x, y \in [0, \infty)$ with $y < x$. Then

$$\begin{aligned} \mu_\pi^\varphi(x * y) &= \int_{x-y}^\infty \varphi(z)\mathrm{d}z \\ &= \int_{x-y}^x \varphi(z)\mathrm{d}z + \int_x^\infty \varphi(z)\mathrm{d}z \\ &\ge \int_x^\infty \varphi(z)\mathrm{d}z = \mu_\pi^\varphi(x). \end{aligned}$$

Let $x, y \in [0, \infty)$ with $x \le y$. Then

$$\begin{aligned} \mu_\pi^\varphi(x * y) &= \int_0^x \varphi(z)\mathrm{d}z + \int_x^\infty \varphi(z)\mathrm{d}z \\ &\ge \mu_\pi^\varphi(x). \end{aligned}$$

Hence μ_π^φ is a fuzzy right ideal of X for any positive map φ with $\int_0^\infty \varphi(z)\mathrm{d}z < \infty$, proving the theorem. □

Corollary 4.9. *Let $F : [0, \infty) \to [0, 1]$ be a map with $\frac{\mathrm{d}F}{\mathrm{d}x} = -\varphi(x)$, where $\varphi(x)$ is a positive function. If $\lim_{x\to\infty} F(x) = 0$, then it is a fuzzy right ideal of $([0, \infty), *, 0)$ as defined in Theorem 4.8.*

Proof. For any α with $x \le \alpha$, we have $\int_x^\alpha \varphi(z)\mathrm{d}z = [-F(z)]_x^\alpha = F(x) - F(\alpha)$. Since $\lim_{z\to\infty} F(z) = 0$, obtain $F(x) = \int_x^\infty \varphi(z)\mathrm{d}z = \mu_\pi^\varphi(x)$ is a fuzzy right ideal by Theorem 4.8. □

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