Applied Mathematics Letters 24 (2011) 1312–1316

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Hyers–Ulam stability of additive set-valued functional equations

Gang Lu [a], Choonkil Park [b,*,1]

[a] Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea

[b] Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea

ARTICLE INFO

Article history:
Received 8 September 2010
Received in revised form 21 February 2011
Accepted 23 February 2011

Keywords:
Hyers–Ulam stability
Additive set-valued functional equation
Closed and convex subset
Cone

ABSTRACT

In this paper, we define the following additive set-valued functional equations

$$f(\alpha x+\beta y)=rf(x)+sf(y), \tag{1}$$

$$f(x+y+z)=2f\left(\frac{x+y}{2}\right)+f(z) \tag{2}$$

for some real numbers $\alpha>0$, $\beta>0$, $r,s\in\mathbb{R}$ with $\alpha+\beta=r+s\neq 1$, and prove the Hyers–Ulam stability of the above additive set-valued functional equations.

1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the past decades. The pioneering papers by Aumann [1] and Debreu [2] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [3], McKenzie [4], the monographs by Hindenbrand [5], Aubin and Frankow [6], Castaing and Valadier [7], Klein and Thompson [8] and the survey by Hess [9].

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [12] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [13] has provided a lot of influence in the development of what we call *Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach (see [15–24]).

It is easy to show that if $f:\mathbb{R}\to\mathbb{R}$ is a solution of the inequality

$$|f(\alpha x+\beta y)-rf(x)-sf(y)|<\varepsilon \tag{1.1}$$

for some $\varepsilon>0$ then there exists a linear function $g(x)=mx$, $m\in\mathbb{R}$, such that $|f(x)-g(x)|<\varepsilon$ for all $x\in\mathbb{R}$.

The inequality (1.1) can be written in the form

$$f(\alpha x+\beta y)-rf(x)-sf(y)\in B(0,\varepsilon),$$

where $B(0,\varepsilon):=(-\varepsilon,\varepsilon)$. Hence we have

$$f(\alpha x+\beta y)+B(0,\varepsilon)\subseteq rf(x)+B(0,\varepsilon)+sf(y)+B(0,\varepsilon)$$

* Corresponding author. Tel.: +82 2 2220 0892; fax: +82 2 2281 0019.
E-mail addresses: lvgang1234@hanmail.net (G. Lu), baak@hanyang.ac.kr (C. Park).

1 The second author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2009-0070788).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.02.024

and denoting by $F(x) = f(x) + B(0, \varepsilon)$, $x \in \mathbb{R}$, we get

$$F(\alpha x + \beta y) \subseteq rF(x) + sF(y), \quad x, y \in \mathbb{R} \text{ (if } r, s \geq 1)$$

and

$$g(x) \in F(x).$$

Let Y be a real normed space. The family of all closed and convex subsets, containing 0, of Y will be denoted by $ccz(Y)$.

Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$A + B = \{x \in X : x = a + b,\ a \in A, b \in B\},$$
$$\lambda A = \{x \in X : x = \lambda a,\ a \in A\}.$$

Lemma 1.1 ([25]). *Let λ and μ be real numbers. If A and B are nonempty subset of a real vector space X, then*

$$\lambda(A + B) = \lambda A + \lambda B,$$
$$(\lambda + \mu)A \subseteq \lambda A + \mu B.$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

A subset $A \subseteq X$ is said to be a *cone* if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero*.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [26–29]).

2. Stability of the set-valued functional equation (1)

In this section, let X be a real vector space, $A \subseteq X$ a cone with zero and Y a Banach space.

The following theorem is similar to the results of [30,31].

Theorem 2.1. *If $F : A \to ccz(Y)$ is a set-valued map satisfying*

$$F(\alpha x + \beta y) \subseteq rF(x) + sF(y) \tag{2.1}$$

and

$$\sup\{\mathrm{diam} F(x) : x \in A\} < +\infty$$

for all $x, y \in A$ and some $\alpha > 0$, $\beta > 0$, $r, s \in \mathbb{R}$ with $\alpha + \beta = r + s \neq 1$, then there exists a unique additive map $g : A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Proof. For $x \in A$, replacing y by x in (2.1), we get

$$F((\alpha + \beta)x) \subseteq rF(x) + sF(x) = (r + s)F(x) \tag{2.2}$$

and if we replace x by $(\alpha + \beta)^n x$, $n \in \mathbb{N}$, in (2.2), then we obtain

$$F((\alpha + \beta)^{n+1} x) \subseteq (r + s)F((\alpha + \beta)^n x)$$

and

$$\frac{F((\alpha + \beta)^{n+1} x)}{(\alpha + \beta)^{n+1}} \subseteq \frac{(r + s)}{(\alpha + \beta)} \frac{F((\alpha + \beta)^n x)}{(\alpha + \beta)^n}.$$

Thus we get

$$\frac{F((\alpha + \beta)^{n+1} x)}{(\alpha + \beta)^{n+1}} \subseteq \frac{F((\alpha + \beta)^n x)}{(\alpha + \beta)^n}.$$

Case 1. Let $\alpha + \beta > 1$.

Denoting by $F_n(x) = \frac{F((\alpha+\beta)^n x)}{(\alpha+\beta)^n}$, $x \in A$, $n \in \mathbb{N}$, we obtain that $(F_n(x))_{n \geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$\mathrm{diam} F_n(x) = \frac{1}{(\alpha + \beta)^n} \mathrm{diam} F((\alpha + \beta)^n x).$$

Since $\alpha + \beta > 1$ and $\sup\{\mathrm{diam}(F(x)) : x \in A\} < \infty$, we get that $\lim_{n \to +\infty} \mathrm{diam}(F_n(x)) = 0$ for all $x \in A$.

Using the Cantor theorem for the sequence $(F_n(x))_{n\geq 0}$, we obtain that the intersection $\bigcap_{n\geq 0} F_n(x)$ is a singleton set and we denote this intersection by $g(x)$ for all $x \in A$. Thus we obtain a map $g : A \to Y$. Then $g(x) \in F_0(x) = F(x)$ for all $x \in A$.

Now we show that g is additive. We have

$$\begin{aligned} F_n(\alpha x + \beta y) &= \frac{F((\alpha+\beta)^n(\alpha x + \beta y))}{(\alpha+\beta)^n} = \frac{F(\alpha(\alpha+\beta)^n x + \beta(\alpha+\beta)^n y)}{(\alpha+\beta)^n} \\ &\subseteq \frac{rF((\alpha+\beta)^n x) + sF((\alpha+\beta)^n y)}{(\alpha+\beta)^n} = r\frac{F((\alpha+\beta)^n x)}{(\alpha+\beta)^n} + s\frac{F((\alpha+\beta)^n y)}{(\alpha+\beta)^n} \\ &= rF_n(x) + sF_n(y). \end{aligned}$$

By definition of g, we can get for all $x, y \in A$,

$$g(\alpha x + \beta y) = \bigcap_{n=0}^{\infty} F_n(\alpha x + \beta y) \subseteq \bigcap_{n=0}^{\infty} \left(rF_n(x) + sF_n(y)\right). \tag{2.3}$$

On the other hand, it is easy to show that, for all $x, y \in A$,

$$rg(x) + sg(y) \in rF_n(x) + sF_n(y). \tag{2.4}$$

Now, we fix $n \in \mathbb{N}$ and $x, y \in A$. Then it follows from (2.3) and (2.4) that

there exist $a_1 \in F_n(x)$ and $b_1 \in F_n(y)$ such that $g(\alpha x + \beta y) = ra_1 + sb_1$,
there exist $a_2 \in F_n(x)$ and $b_2 \in F_n(y)$ such that $rg(x) + sg(y) = ra_2 + sb_2$.

Thus we obtain

$$g(\alpha x + \beta y) - (rg(x) + sg(y)) = r(a_1 - a_2) + s(b_1 - b_2).$$

We know that $ra_1, ra_2 \in rF_n(x)$ and $sb_1, sb_2 \in sF_n(y)$. So we get

$$\begin{aligned} \|g(\alpha x + \beta y) - (rg(x) + sg(y))\| &\leq \|r(a_1 - a_2)\| + \|s(b_1 - b_2)\| \\ &\leq r \cdot \mathrm{diam}F_n(x) + s \cdot \mathrm{diam}F_n(y), \end{aligned}$$

which tends to zero as n tends to ∞. Thus

$$g(\alpha x + \beta y) = rg(x) + sg(y). \tag{2.5}$$

Let $x = y = 0$ in (2.5). Then we get $g(0) = (r+s)g(0)$. So $g(0) = 0$. Letting $y = 0$ and $x = 0$ in (2.5), respectively, we obtain

$$g(\alpha x) = rg(x) \quad \text{and} \quad g(\beta y) = sg(y). \tag{2.6}$$

Replacing x by $\frac{x}{\alpha}$ and y by $\frac{y}{\beta}$ in (2.6), respectively, we get $g(x) = rg\left(\frac{x}{\alpha}\right)$ and $g(y) = sg\left(\frac{y}{\beta}\right)$ for all $x, y \in A$. By (2.5),

$$g(x+y) = g\left(\alpha \cdot \frac{x}{\alpha} + \beta \cdot \frac{y}{\beta}\right) = rg\left(\frac{x}{\alpha}\right) + sg\left(\frac{y}{\beta}\right) = g(x) + g(y)$$

for all $x, y \in A$. Thus g is additive.

Case 2. Let $0 < \alpha + \beta < 1$.

Replacing x in (2.2) by $\frac{x}{(\alpha+\beta)^{n+1}}$, $n \in \mathbb{N}$, and multiplying the resulting relation by $(\alpha+\beta)^n$, we obtain

$$(\alpha+\beta)^n F\left(\frac{x}{(\alpha+\beta)^n}\right) \subseteq \frac{r+s}{\alpha+\beta}(\alpha+\beta)^{n+1} F\left(\frac{x}{(\alpha+\beta)^{n+1}}\right).$$

Since $\frac{r+s}{\alpha+\beta} > 0$, we get

$$(\alpha+\beta)^n F\left(\frac{x}{(\alpha+\beta)^n}\right) \subseteq (\alpha+\beta)^{n+1} F\left(\frac{x}{(\alpha+\beta)^{n+1}}\right).$$

Let

$$F'_n(x) = (\alpha+\beta)^n F\left(\frac{x}{(\alpha+\beta)^n}\right).$$

The sequence $(F'_n(x))_{n\geq 0}$ is increasing and the sequence of positive numbers $(\mathrm{diam}F'_n(x))_{n\geq 0}$ is increasing, too. Hence we have

$$\mathrm{diam}F'_n(x) = (\alpha+\beta)^n \mathrm{diam}F\left(\frac{x}{(\alpha+\beta)^n}\right)$$

and so

$$\lim_{n\to\infty} \mathrm{diam} F'_n(x) = 0.$$

Thus $F'_n(x)$ is single valued for all $x \in A$. The set-valued map F is single valued and

$$F(\alpha x + \beta y) = rF(x) + sF(y)$$

for all $x, y \in A$. Using the same method as in Case 1, we can show the additivity of F.

Therefore, we conclude that there exists an additive map $g : A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Next, let us prove the uniqueness of g.

Suppose that F have two additive selections $g_1, g_2 : A \to Y$. We have

$$ng_i(x) = g_i(nx) \in F(nx)$$

for all $n \in \mathbb{N}, x \in A, i \in \{1, 2\}$. Then we get

$$n\|g_1(x) - g_2(x)\| = \|ng_1(x) - ng_2(x)\| = \|g_1(nx) - g_2(nx)\| \le \mathrm{diam} F(nx)$$

for all $x \in A, n \in \mathbb{N}$. It follows from $\sup\{\mathrm{diam} F(x) : x \in A\} < \infty$ that $g_1(x) = g_2(x)$ for all $x \in A$, as desired. □

Remark 2.2. The stability problem for a singled-valued functional equation is whether, for a map satisfying almost a given functional equation, there exists an exact solution of the functional equation near the given almost map. On the other hand, the stability problem for a set-valued functional equation is whether, for a set-valued map satisfying almost a given set-valued functional equation, there exists an exact solution, in the set related to the set-valued functional equation, of a functional equation related to the set-valued functional equation.

3. Stability of the set-valued functional equation (2)

In this section, let X be a real vector space, $A \subseteq X$ a cone with zero and Y a Banach space.

Theorem 3.1. *If $F : A \to ccz(Y)$ is a set-valued map satisfying*

$$F(x + y + z) \subseteq 2F\left(\frac{x+y}{2}\right) + F(z) \tag{3.1}$$

and

$$\sup\{\mathrm{diam} F(x) : x \in A\} < +\infty$$

for all $x, y, z \in A$, then there exists a unique additive map $g : A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Proof. Letting $x = y = z$ in (3.1), we get

$$F(3x) \subseteq 3F(x). \tag{3.2}$$

Replacing x by $3^n x$, $n \in \mathbb{N}$, in (3.2), we obtain

$$F(3 \cdot 3^n x) \subseteq 3F(3^n x)$$

and

$$\frac{F(3^{n+1}x)}{3^{n+1}} \subseteq \frac{F(3^n x)}{3^n}.$$

Denoting by $F_n(x) = \frac{F(3^n x)}{3^n}$, $x \in A$, $n \in \mathbb{N}$, we obtain that $(F_n(x))_{n\ge 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$\mathrm{diam} F_n(x) = \frac{1}{3^n} \mathrm{diam} F(3^n x).$$

Taking account of $\sup\{\mathrm{diam} F(x) : x \in A\} < +\infty$, we get

$$\lim_{n\to\infty} \mathrm{diam} F_n(x) = 0.$$

Using the Cantor theorem for the sequence $(F_n(x))_{n\ge 0}$, we obtain that the intersection $\cap_{n\ge 0} F_n(x)$ is a singleton set and we denote this intersection by $g(x)$ for all $x \in A$. Thus we get a map $g : A \to Y$ and $g(x) \in F_0(x) = F(x)$ for all $x \in A$.

We now show that g is additive. For all $x, y, z \in A$ and $n \in \mathbb{N}$,

$$F_n(x+y+z) = \frac{F(3^n(x+y+z))}{3^n} = \frac{F(3^n x + 3^n y + 3^n z)}{3^n} \subseteq \frac{2F\left(\frac{3^n x + 3^n y}{2}\right)}{3^n} + \frac{F(3^n z)}{3^n} = 2F_n\left(\frac{x+y}{2}\right) + F_n(z).$$

By definition of g, we obtain

$$g(x+y+z) = \bigcap_{n=0}^{\infty} F_n(x+y+z) \subseteq \bigcap_{n=0}^{\infty}\left(2F_n\left(\frac{x+y}{2}\right) + F_n(z)\right),$$

$g\left(\frac{x+y}{2}\right) \in F_n\left(\frac{x+y}{2}\right)$ and $g(z) \in F_n(z)$. Thus we get

$$\left\| g(x+y+z) - 2g\left(\frac{x+y}{2}\right) - g(z)\right\| \le 2 \cdot \mathrm{diam} F_n\left(\frac{x+y}{2}\right) + \mathrm{diam} F_n(z),$$

which tends to zero as n tends to ∞. Thus

$$g(x+y+z) = 2g\left(\frac{x+y}{2}\right) + g(z). \tag{3.3}$$

Letting $x = y = z = 0$ in (3.3), we have $g(0) = 2g(0) + g(0)$. Thus $g(0) = 0$. Letting $y = z = 0$ and $x = z = 0$ in (3.3), respectively, we obtain

$$g(x) = 2g\left(\frac{x}{2}\right) \quad \text{and} \quad g(y) = 2g\left(\frac{y}{2}\right)$$

for all $x, y \in A$. So we get

$$\|g(x+y+z) - g(x) - g(y) - g(z)\| = \left\| 2g\left(\frac{x+y}{2}\right) - 2g\left(\frac{x}{2}\right) - 2g\left(\frac{y}{2}\right)\right\| = 0$$

for all $x, y, z \in A$. Thus g is additive.

The rest of the proof is similar to the proof of Theorem 2.1. □

References

[1] R.J. Aumann, Integrals of set-valued functions, J. Math. Anal. Appl. 12 (1965) 1–12.
[2] G. Debreu, Integration of correspondences, in: Proceedings of Fifth Berkeley Symposium on Mathematical Statistics and Probability, vol. II, part. I, 1966, pp. 351–372.
[3] K.J. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica 22 (1954) 265–290.
[4] L.W. McKenzie, On the existence of general equilibrium for a competitive market, Econometrica 27 (1959) 54–71.
[5] W. Hindenbrand, Core and Equilibria of a Large Economy, Princeton Univ. Press, Princeton, 1974.
[6] J.P. Aubin, H. Frankow, Set-Valued Analysis, Birkhäuser, Boston, 1990.
[7] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, in: Lect. Notes in Math., vol. 580, Springer, Berlin, 1977.
[8] E. Klein, A. Thompson, Theory of Correspondence, Wiley, New York, 1984.
[9] C. Hess, Set-valued integration and set-valued probability theory: an overview, in: Handbook of Measure Theory, vol. I–II, North-Holland, Amsterdam, 2002.
[10] S.M. Ulam, Problems in Modern Mathematics, science ed., Wiley, New York, 1940, Chapter VI.
[11] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222–224.
[12] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64–66.
[13] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[14] P. Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.
[15] J. Aczel, J. Dhombres, Functional Equations in Several Variables, Cambridge Univ. Press, Cambridge, 1989.
[16] P.W. Cholewa, Remarks on the stability of functional equations, Aequationes Math. 27 (1984) 76–86.
[17] P. Găvruta, L. Găvruta, A new method for the generalized Hyers–Ulam–Rassias stability, Internat. J. Nonlinear Anal. Appl. 1 (2010) 11–18.
[18] M.E. Gordji, M.B. Savadkouhi, Stability of a mixed type cubic–quartic functional equation in non-Archimedean spaces, Appl. Math. Lett. 23 (2010) 1198–1202.
[19] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.
[20] G. Isac, Th.M. Rassias, On the Hyers–Ulam stability of ψ-additive mappings, J. Approx. Theory 72 (1993) 131–137.
[21] H. Khodaei, Th.M. Rassias, Approximately generalized additive functions in several variables, Internat. J. Nonlinear Anal. Appl. 1 (2010) 22–41.
[22] Th.M. Rassias (Ed.), Functional Equations and Inequalities, Kluwer Academic, Dordrecht, 2000.
[23] Th.M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284.
[24] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000) 23–130.
[25] K. Nikodem, K-convex and K-concave set-valued functions, Z. K. Nr. 559 (1989) Lodz.
[26] T. Cardinali, K. Nikodem, F. Papalini, Some results on stability and characterization of K-convexity of set-valued functions, Ann. Polon. Math. 58 (1993) 185–192.
[27] K. Nikodem, On quadratic set-valued functions, Publ. Math. Debrecen 30 (1984) 297–301.
[28] K. Nikodem, On Jensen's functional equation for set-valued functions, Rad. Mat. 3 (1987) 23–33.
[29] K. Nikodem, Set-valued solutions of the Pexider functional equation, Funkcial. Ekvac. 31 (1988) 227–231.
[30] Y.J. Piao, The existence and uniqueness of additive selection for (α, β)–(β, α) type subadditive set-valued maps, Journal of Northeast Normal University 41 (2009) 38–40.
[31] D. Popa, Additive selections of (α, β)-subadditive set-valued maps, Glas. Mat. Ser. III 36 (56) (2001) 11–16.