STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY IN PROPER CQ^*-ALGEBRAS

Jung Rye Lee, Choonkil Park, and Dong Yun Shin

Abstract. In this paper, we prove the Hyers–Ulam–Rassias stability of the following additive functional inequality:

$$(0.1) \qquad \|f(2x) + f(2y) + 2f(z)\| \quad \leq \quad \|2f(x+y+z)\|.$$

We investigate homomorphisms in proper CQ^*-algebras and derivations on proper CQ^*-algebras associated with the additive functional inequality (0.1).

1. Introduction and preliminaries

Ulam [36] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.

Hyers [13] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Th. M. Rassias [25] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$(1.1) \qquad \|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where θ and p are positive real numbers with $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Th. M. Rassias [26] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [11] following the same approach as in Th. M. Rassias [25], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Th. M. Rassias and Šemrl [31] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [11], as well as of Th. M. Rassias and Šemrl [31] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [12], who among others studied the Hyers–Ulam–Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [25] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [9, 10], D. H. Hyers, G. Isac and Th. M. Rassias [14]).

Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D. H. Hyers and Th. M. Rassias [15], Th. M. Rassias [29] and the references therein).

J. M. Rassias [23] following the spirit of the innovative approach of Th. M. Rassias [25] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [24] for a number of other new results).

Theorem 1.2 ([22, 23, 24]). *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such*

that f satisfies inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}\|x\|^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Several mathematicians have contributed to this subject (see [16]–[20], [27]–[30], [32]).

In this paper, we prove the Hyers–Ulam–Rassias stability of the following additive functional equation:

$$\|f(2x) + f(2y) + 2f(z)\| \quad \leq \quad \|2f(x+y+z)\|.$$

In a series of papers [1]–[8] and [33]–[35], many authors have considered a special class of quasi $*$-algebras, called *proper CQ^*-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;

(ii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(xy)^* = y^*x^*$ for all $x, y \in A$ whenever multiplications xy and y^*x^* are defined;

(iii) $\|y\|_0 = \sup_{x \in A, \|x\| \leq 1} \|xy\|$ for all $y \in A_0$.

Definition 1.3. Let (A, A_0) and (B, B_0) be proper CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $h : A \to B$ is called a *proper CQ^*-algebra homomorphism* if $h(xy) = h(x)h(y)$ for all $x, y \in A$ whenever multiplications xy and $h(x)h(y)$ are defined. In this case, we say that $h(xy) = h(x)h(y)$ for all $x, y \in A$ whenever the multiplication is defined.

(ii) A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$ whenever multiplications $xy, \delta(x)y$, and $x\delta(y)$ are defined. In this case we say $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$ whenever the multiplication is defined.

The purpose of this paper is to investigate the stability of homomorphisms in proper CQ^*-algebras and of derivations on proper CQ^*-algebras associated with the additive functional inequality (0.1). In this paper, we use only the property that a CQ^*-algebra is a Banach space with a partially defined multiplication. Therefore our results are generalized to a Banach space with a partially defined multiplication.

This paper is organized as follows: In Section 2, we prove the stability of $\mathbb{C}$-linear mappings in Banach spaces.

In Section 3, we prove the stability of homomorphisms in proper CQ^*-algebras.

In Section 4, we prove the stability of derivations on proper CQ^*-algebras.

Throughout this paper, we assume that (A, A_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$. We denote that $\mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

2. Stability of $\mathbb{C}$-linear mappings in Banach spaces

We investigate the Hyers–Ulam–Rassias stability of $\mathbb{C}$-linear mappings in Banach spaces associated with the additive functional inequality. In this section, we assume that X, Y are Banach spaces.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying*

$$(2.1) \qquad \|f(2x) + f(2y) + 2f(z)\|_Y \le \|2f(x + y + z)\|_Y$$

for all $x, y, z \in X$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get $\|4f(0)\|_Y \le \|2f(0)\|_Y$ and so $f(0) = 0$.

Letting $z = 0$ and replacing y by $-x$ in (2.1), we get

$$\|f(2x) + f(-2x)\|_Y \le \|2f(0)\|_Y = 0$$

for all $x \in X$. Hence $f(-2x) = -f(2x)$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $y = 0$ and replacing z by $-x$ in (2.1), we get

$$\|f(2x) + 2f(-x)\|_Y \le \|2f(0)\|_Y = 0$$

for all $x \in X$. Thus we have $f(2x) = 2f(x)$ for all $x \in X$.

Letting replacing z by $-x - y$ in (2.1), we get

$$\|f(2x) + f(2y) - 2f(x + y)\|_Y \le \|2f(0)\|_Y = 0$$

for all $x, y \in X$. Thus we have

$$f(2x + 2y) = f(2x) + f(2y)$$

and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. $\qquad \square$

Theorem 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying*

$$(2.2) \qquad \|f(2x) + f(2y) + 2f(z)\|_Y \le \|2f(x + y + z)\|_Y + \varphi(x, y, z),$$

$$(2.3) \qquad \widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right) < \infty$$

for all $x, y, z \in X$, then there exists a unique additive mapping $L : X \to Y$ satisfying

$$(2.4) \qquad \|f(x) - L(x)\|_Y \leq \frac{1}{2}\widetilde{\varphi}(0, -x, x)$$

for all $x \in X$.

Proof. Replacing x, y, z by $0, -(-2)^n x, (-2)^n x$, respectively, and dividing by 2^{n+1} in (2.2), since $f(0) = 0$, we get

$$\left\|\frac{f((-2)^{n+1}x)}{(-2)^{n+1}} - \frac{f((-2)^n x)}{(-2)^n}\right\|_Y \leq \frac{1}{2^{n+1}}\varphi(0, -(-2)^n x, (-2)^n x)$$

for all $x \in X$. From the above inequality, we have

$$\left\|\frac{f((-2)^n x)}{(-2)^n} - \frac{f((-2)^q x)}{(-2)^q}\right\|_Y \leq \sum_{j=q}^{n-1}\left\|\frac{f((-2)^{j+1}x)}{(-2)^{j+1}} - \frac{f((-2)^j x)}{(-2)^j}\right\|_Y$$

$$\leq \sum_{j=q}^{n-1}\frac{1}{2^{j+1}}\varphi(0, -(-2)^j x, (-2)^j x)$$

for all $x \in X$ and all non negative integers q, n with $q < n$. From (2.3), the sequence $\{\frac{f((-2)^n x)}{(-2)^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f((-2)^n x)}{(-2)^n}\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n \to \infty}\frac{f((-2)^n x)}{(-2)^n}$$

for all $x \in X$.

In order to prove that L satisfies (2.4), if we put $q = 0$ and let $n \to \infty$ in the above inequality then we obtain

$$\|f(x) - L(x)\|_Y \leq \sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\varphi(0, -(-2)^j x, (-2)^j x) = \frac{1}{2}\widetilde{\varphi}(0, -x, x)$$

for all $x \in X$.

Replacing x, y, z by $(-2)^n x, (-2)^n y, (-2)^n z$, respectively, and dividing by 2^n in (2.2), we get

$$\left\|\frac{f((-2)^n 2x)}{(-2)^n} + \frac{f((-2)^n 2y)}{(-2)^n} + \frac{2f((-2)^n z)}{(-2)^n}\right\|_Y$$

$$\leq \left\|2\frac{f((-2)^n(x + y + z))}{(-2)^n}\right\|_Y + \frac{1}{2^n}\varphi((-2)^n x, (-2)^n y, (-2)^n z)$$

for all $x, y, z \in X$. Since (2.3) gives that

$$\lim_{n \to \infty}\frac{1}{2^n}\varphi((-2)^n x, (-2)^n y, (-2)^n z) = 0$$

for all $x, y, z \in X$, if we let $n \to \infty$ in the above inequality, then we get

$$\|L(2x) + L(2y) + 2L(z)\|_Y \le \|2L(x + y + z)\|_Y,$$

and so L is additive by Lemma 2.1.

Now to prove the uniqueness of L, let $L' : X \to Y$ be another additive mapping satisfying (2.4). Since both L and L' are additive, we have

$$
\begin{aligned}
\|L(x) - L'(x)\|_Y &= \frac{1}{2^n} \|L(2^n x) - L'(2^n x)\|_Y \\
&\le \frac{1}{2^n} (\|L(2^n x) - f(2^n x)\|_Y + \|L'(2^n x) - f(2^n x)\|_Y) \\
&\le \frac{1}{2^n} \cdot 2\widetilde{\varphi}(0, -2^n x, 2^n x) \\
&= 2 \sum_{j=0}^{\infty} \frac{1}{2^{n+j}} \varphi(0, (-2)^{j+n} x, (-2)^{j+n} x)
\end{aligned}
$$

which goes to zero as $n \to \infty$ for all $x \in X$ by (2.3). Consequently, L is a unique additive mapping satisfying (2.4), as desired. □

Corollary 2.3. *Let $f : X \to Y$ be a mapping. If there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.2) and*

$$(2.5) \qquad \widetilde{\varphi}(x, y, z) := \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{x}{(-2)^i}, \frac{y}{(-2)^i}, \frac{z}{(-2)^i}\right) < \infty$$

for all $x, y, z \in X$, then there exists a unique additive mapping $L : X \to Y$ satisfying

$$(2.6) \qquad \|f(x) - L(x)\|_Y \le \frac{1}{2} \widetilde{\varphi}(0, -x, x)$$

for all $x \in X$.

Proof. Since $\widetilde{\varphi}(0, 0, 0) < \infty$ in (2.5), we have $\varphi(0, 0, 0) = 0$ and so $f(0) = 0$. Replacing x, y, z by $0, -\frac{x}{(-2)^n}, \frac{x}{(-2)^n}$, respectively, and multiplying by 2^{n-1} in (2.2), we get

$$\left\| (-2)^{n-1} f\left(\frac{x}{(-2)^{n-1}}\right) - (-2)^n f\left(\frac{x}{(-2)^n}\right) \right\|_Y \le 2^{n-1} \varphi\left(0, -\frac{x}{(-2)^n}, \frac{x}{(-2)^n}\right)$$

for all $x \in X$. From the above inequality, we have

$$
\begin{aligned}
&\left\| (-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^q f\left(\frac{x}{(-2)^q}\right) \right\|_Y \\
&\le \sum_{i=q+1}^{n} \left\| (-2)^i f\left(\frac{x}{(-2)^i}\right) - (-2)^{i-1} f\left(\frac{x}{(-2)^{i-1}}\right) \right\|_Y \\
&\le \sum_{i=q+1}^{n} 2^{i-1} \varphi\left(0, -\frac{x}{(-2)^i}, \frac{x}{(-2)^i}\right)
\end{aligned}
$$

for all $x \in X$ and all non negative integers q, n with $q < n$. From (2.5), the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n \to \infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in X$.

In order to prove that L satisfies (2.6), if we put $q = 0$ and let $n \to \infty$ in the above inequality, then we obtain

$$\|f(x) - L(x)\|_Y \leq \sum_{i=1}^{\infty} 2^{i-1} \varphi\left(0, -\frac{x}{(-2)^i}, \frac{x}{(-2)^i}\right) = \frac{1}{2}\widetilde{\varphi}(0, -x, x)$$

for all $x \in X$.

Replacing x, y, z by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}$, respectively, and multiplying by 2^n in (2.2), we get

$$\left\|(-2)^n f\left(\frac{2x}{(-2)^n}\right) + (-2)^n f\left(\frac{2y}{(-2)^n}\right) + 2 \cdot (-2)^n f\left(\frac{z}{(-2)^n}\right)\right\|_Y$$
$$\leq \left\|2 \cdot (-2)^n f\left(\frac{x + y + z}{(-2)^n}\right)\right\|_Y + 2^n \varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right)$$

for all $x, y, z \in X$. Since (2.5) gives that

$$\lim_{n \to \infty} 2^n \varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right) = 0$$

for all $x, y, z \in X$, if we let $n \to \infty$ in the above inequality, then we get

$$\|L(2x) + L(2y) + 2L(z)\|_Y \leq \|2L(x + y + z)\|_Y$$

and so L is additive by Lemma 2.1.

The rest of the proof is the same as in the corresponding part of the proof of Theorem 2.2, as desired. $\square$

Lemma 2.4. *Let $f : X \to Y$ be a mapping satisfying*

$$(2.7) \qquad \|f(2x) + \mu f(2y) + 2f(z)\|_Y \leq \|2f(x + \mu y + z)\|_Y$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in X$. Then f is $\mathbb{C}$-linear.

Proof. If we put $\mu = 1$ in (2.7), then f is additive by Lemma 2.1.

Replacing x, y, z by $\mu x, -x, 0$, respectively, we get $f(2\mu x) + \mu f(-2x) = 0$ and so $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in X$. Thus we have $f(\mu x + \bar{\mu} x) = f(\mu x) + f(\bar{\mu} x) = \mu f(x) + \bar{\mu} f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in X$, and so $f(tx) = tf(x)$ for any real number t with $|t| \leq 1$ and all $x \in X$.

On the other hand, since $f(2x) = 2f(x)$, we get $f(2^n x) = 2^n f(x)$ for all $n \in \mathbb{N}$. Since, for any real number t, there is a natural number n with $|t| \leq 2^n$, we have

$$f(tx) = f\left(2^n \cdot \frac{t}{2^n} x\right) = 2^n f\left(\frac{t}{2^n} x\right) = 2^n \cdot \frac{t}{2^n} f(x) = t f(x).$$

Now we consider any $\alpha \in \mathbb{C}$ with $\alpha = t + si$ for some real numbers t, s. Since $f(ix) = if(x)$ holds, we have

$$f(\alpha x) = f(tx) + f(six) = tf(x) + sf(ix) = tf(x) + sif(x) = \alpha f(x)$$

and so f is $\mathbb{C}$-linear, as desired. $\qquad\square$

Theorem 2.5. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.3) and*

$$(2.8) \qquad \|f(2x) + \mu f(2y) + 2f(z)\|_Y \leq \|2f(x + \mu y + z)\|_Y + \varphi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in X$, then there exists a unique $\mathbb{C}$-linear mapping $L : X \to Y$ satisfying (2.4).

Proof. If we put $\mu = 1$ in (2.8), then by Theorem 2.2 there exists a unique additive mapping $L : X \to Y$ defined by

$$L(x) := \lim_{n \to \infty} \frac{f((-2)^n x)}{(-2)^n}$$

for all $x \in X$ which satisfies (2.4). By a similar method to the corresponding part of the proof of Theorem 2.2, L satisfies

$$\|L(2x) + \mu L(2y) + 2L(z)\|_Y \leq \|2L(x + \mu y + z)\|_Y$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in X$. Thus Lemma 2.4 gives that L is $\mathbb{C}$-linear. $\quad\square$

Corollary 2.6. *Let $f : X \to Y$ be a mapping. If there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.5) and (2.8), then there exists a unique $\mathbb{C}$-linear mapping $L : X \to Y$ satisfying (2.6).*

Proof. If we put $\mu = 1$ in (2.8), then by Corollary 2.3, there exists a unique additive mapping $L : X \to Y$ defined by

$$L(x) := \lim_{n \to \infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in X$ which satisfies (2.6).

The rest of the proof is the same as in the corresponding part of the proof of Theorem 2.5, as desired. $\qquad\square$

3. Stability of homomorphisms in proper CQ^*-algebras

We investigate the Hyers–Ulam–Rassias stability of isomorphisms in proper CQ^*-algebras associated with the additive functional inequality. From now on, we suppose that (A, A_0) and (B, B_0) are proper CQ^*-algebras.

Theorem 3.1. *Let $f : A \to B$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying*

$$(3.1) \qquad \|f(2x) + \mu f(2y) + 2f(z)\|_B \leq \|2f(x + \mu y + z)\|_B + \varphi(x, y, z),$$

$$(3.2) \qquad \widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right) < \infty$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$, and, in addition, there exists a function $\phi : A^2 \to [0, \infty)$ satisfying

$$(3.3) \qquad \|f(xy) - f(x)f(y)\|_B \leq \phi(x, y),$$

$$(3.4) \qquad \lim_{n \to \infty} \frac{1}{4^n} \phi((-2)^n x, (-2)^n y) = 0$$

for all $x, y \in A$ whenever the multiplication is defined, then there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ satisfying

$$(3.5) \qquad \|f(x) - h(x)\|_B \leq \frac{1}{2} \widetilde{\varphi}(0, -x, x)$$

for all $x \in A$.

Proof. By Theorem 2.5, we have a unique $\mathbb{C}$-linear mapping $h : A \to B$ defined by

$$h(x) := \lim_{n \to \infty} \frac{f((-2)^n x)}{(-2)^n}$$

for all $x \in A$ which satisfies (3.5).

Now we show that $h(xy) = h(x)h(y)$ for all $x, y \in A$ whenever the multiplication is defined.

Replacing x, y by $(-2)^n x, (-2)^n y$, respectively, and dividing by 4^n in (3.3), we get

$$\left\| \frac{1}{4^n} [f((-2)^n x(-2)^n y) - f((-2)^n x)f((-2)^n y)] \right\|_B \leq \frac{1}{4^n} \phi((-2)^n x, (-2)^n y)$$

for all $x, y \in A$ whenever the multiplication is defined. We have

$$\lim_{n \to \infty} \frac{1}{4^n} f((-2)^n x(-2)^n y) = \lim_{n \to \infty} \frac{f((-2)^{2n} xy)}{(-2)^{2n}} = h(xy)$$

$$\lim_{n \to \infty} \frac{1}{4^n} f((-2)^n x)f((-2)^n y) = \lim_{n \to \infty} \frac{f((-2)^n x)}{(-2)^n} \cdot \lim_{n \to \infty} \frac{f((-2)^n x)}{(-2)^n} = h(x)h(y)$$

for all $x, y \in A$ whenever the multiplication is defined. If we let $n \to \infty$ in the above inequality, then (3.4) gives $h(xy) = h(x)h(y)$ for all $x, y \in A$ whenever the multiplication is defined. $\qquad \square$

Corollary 3.2. *Let θ, p be nonnegative real numbers with $p < 1$ and let $f : A \to B$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_B \le \|2f(x + \mu y + z)\|_B + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f(xy) - f(x)f(y)\|_B \le \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ satisfying

$$\|f(x) - h(x)\|_B \le \frac{2\theta}{2 - 2^p} \|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$. When $p < 1$, we get

$$\begin{aligned}
\widetilde{\varphi}(x, y, z) :&= \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right) \\
&= \sum_{j=0}^{\infty} \frac{2^{pj}}{2^j} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p) \\
&= \frac{2\theta}{2 - 2^p}(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p).
\end{aligned}$$

In addition, let $\phi : A^2 \to [0, \infty)$ be $\phi(x, y) = \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$. When $p < 1$, we have

$$\lim_{n \to \infty} \frac{1}{4^n} \phi((-2)^n x, (-2)^n y) = \lim_{n \to \infty} \frac{2^{2pn}}{4^n} \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0$$

for all $x, y \in A$. By applying Theorem 3.1, there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \le \frac{1}{2} \widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2 - 2^p} \|x\|_A^p$$

for all $x \in A$. $\qquad\qquad\square$

Corollary 3.3. *Let θ, p be nonnegative real numbers with $p < 1$ and let $f : A \to B$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_B \le \|2f(x + \mu y + z)\|_B + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f(xy) - f(x)f(y)\|_B \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$$

for all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ satisfying

$$\|f(x) - h(x)\|_B \le \frac{2\theta}{2 - 2^p} \|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$ and let $\phi : A^2 \to [0, \infty)$ be $\phi(x, y) = \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$. When $p < 1$, we have $\widetilde{\varphi}(x, y, z) < \infty$ and

$$\lim_{n \to \infty} \frac{1}{4^n} \phi((-2)^n x, (-2)^n y) = \lim_{n \to \infty} \frac{2^{2pn}}{4^n} \cdot \theta \cdot \|x\|_A^p \cdot \|y\|_A^p = 0$$

for all $x, y, z \in A$.

By applying Theorem 3.1, there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{1}{2} \widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2 - 2^p} \|x\|_A^p$$

for all $x \in A$. $\square$

Theorem 3.4. *Let $f : A \to B$ be a mapping. If there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying (3.1) and*

$$(3.6) \qquad \widetilde{\varphi}(x, y, z) := \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{x}{(-2)^i}, \frac{y}{(-2)^i}, \frac{z}{(-2)^i}\right) < \infty$$

for all $x, y, z \in A$, and, in addition, there exists a function $\phi : A^2 \to [0, \infty)$ satisfying (3.3) and

$$(3.7) \qquad \lim_{n \to \infty} 4^n \phi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}\right) = 0$$

for all $x, y \in A$ whenever the multiplication is defined, then there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ satisfying

$$(3.8) \qquad \|f(x) - h(x)\|_B \leq \frac{1}{2} \widetilde{\varphi}(0, -x, x)$$

for all $x \in A$.

Proof. By Corollary 2.6, we have a unique $\mathbb{C}$-linear mapping $h : A \to B$ defined by

$$h(x) := \lim_{n \to \infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in A$ which satisfies (3.8).

Now replacing x, y by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}$, respectively, and multiplying by 4^n in (3.3), we get

$$\left\| 4^n \left[f\left(\frac{x}{(-2)^n} \cdot \frac{y}{(-2)^n}\right) - f\left(\frac{x}{(-2)^n}\right) f\left(\frac{y}{(-2)^n}\right) \right] \right\|_B$$
$$\leq 4^n \phi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}\right)$$

for all $x, y \in A$ whenever the multiplication is defined. Since

$$\lim_{n \to \infty} 4^n f\left(\frac{x}{(-2)^n} \cdot \frac{y}{(-2)^n}\right) = \lim_{n \to \infty} (-2)^{2n} f\left(\frac{xy}{(-2)^{2n}}\right) = h(xy)$$

$$\lim_{n \to \infty} 4^n f\left(\frac{x}{(-2)^n}\right) f\left(\frac{y}{(-2)^n}\right)$$

$$= \lim_{n \to \infty} (-2)^n f\left(\frac{x}{(-2)^n}\right) \cdot \lim_{n \to \infty} (-2)^n f\left(\frac{y}{(-2)^n}\right) = h(x)h(y)$$

for all $x, y \in A$ whenever the multiplication is defined. If we let $n \to \infty$ in the above inequality, then (3.7) gives $h(xy) = h(x)h(y)$ for all $x, y \in A$ whenever the multiplication is defined. $\square$

Corollary 3.5. *Let θ, p be nonnegative real numbers with $p > 1$ and let $f : A \to B$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_B \le \|2f(x + \mu y + z)\|_B + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$

$$\|f(xy) - f(x)f(y)\|_B \le \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ satisfying

$$\|f(x) - h(x)\|_B \le \frac{2\theta}{2^p - 2}\|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$. When $p > 1$, we get

$$\widetilde{\varphi}(x, y, z) := \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{x}{(-2)^i}, \frac{y}{(-2)^i}, \frac{z}{(-2)^i}\right)$$

$$= \sum_{i=1}^{\infty} \frac{2^i}{2^{pi}} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$$

$$= \frac{2\theta}{2^p - 2}(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p).$$

In addition, let $\phi : A^2 \to [0, \infty)$ be $\phi(x, y) = \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$. When $p > 1$, we get

$$\lim_{n \to \infty} 4^n \phi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}\right) = \lim_{n \to \infty} \frac{4^n}{2^{2pn}} \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0$$

for all $x, y \in A$.

By applying Theorem 3.4, there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \le \frac{1}{2}\widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2^p - 2}\|x\|_A^p$$

for all $x \in A$. $\square$

Corollary 3.6. *Let θ, p be nonnegative real numbers with $p > 1$ and let $f : A \to B$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_B \le \|2f(x + \mu y + z)\|_B + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f(xy) - f(x)f(y)\|_B \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ satisfying

$$\|f(x) - h(x)\|_B \le \frac{2\theta}{2^p - 2}\|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$ and let $\phi : A^2 \to [0, \infty)$ be $\phi(x, y) = \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$. When $p > 1$, we have $\widetilde{\varphi}(x, y, z) < \infty$ and

$$\lim_{n \to \infty} 4^n \phi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}\right) = \lim_{n \to \infty} \frac{4^n}{2^{2pn}} \cdot \theta \cdot \|x\|_A^p \cdot \|y\|_A^p = 0$$

for all $x, y, z \in A$. By applying Theorem 3.4, there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \le \frac{1}{2}\widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2^p - 2}\|x\|_A^p$$

for all $x \in A$. $\square$

4. Stability of derivations on proper CQ^*-algebras

We investigate the Hyers–Ulam–Rassias stability of derivations on proper CQ^*-algebras associated with the additive functional inequality.

Theorem 4.1. *Let $f : A \to A$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying*

$$(4.1) \qquad \|f(2x) + \mu f(2y) + 2f(z)\|_A \le \|2f(x + \mu y + z)\|_A + \varphi(x, y, z),$$

$$(4.2) \qquad \widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right) < \infty$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$, and, in addition, there exists a function $\psi : A^2 \to [0, \infty)$ satisfying

$$(4.3) \qquad \|f(xy) - f(x)y - xf(y)\|_A \le \psi(x, y),$$

$$(4.4) \qquad \lim_{n \to \infty} \frac{1}{4^n} \psi((-2)^n x, (-2)^n y) = 0$$

for all $x, y \in A$ whenever the multiplication is defined, then there exists a unique derivation δ on A satisfying

$$(4.5) \qquad \|f(x) - \delta(x)\|_A \le \frac{1}{2}\widetilde{\varphi}(0, -x, x)$$

for all $x \in A$.

Proof. By Theorem 2.5, we have a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ defined by

$$\delta(x) := \lim_{n \to \infty} \frac{f((-2)^n x)}{(-2)^n}$$

for all $x \in A$ which satisfies (4.5).

Now we show that $\delta(xy) = \delta(x)\delta(y)$ for all $x, y \in A$ whenever the multiplication is defined.

Replacing x, y by $(-2)^n x, (-2)^n y$, respectively, and dividing by 4^n in (4.3), we get

$$\left\| \frac{1}{4^n} [f((-2)^n x (-2)^n y) - f((-2)^n x)(-2)^n y - (-2)^n x f((-2)^n y)] \right\|_A$$
$$\leq \frac{1}{4^n} \psi((-2)^n x, (-2)^n y)$$

for all $x, y \in A$ whenever the multiplication is defined. We have

$$\lim_{n \to \infty} \frac{1}{4^n} f((-2)^n x (-2)^n y) = \lim_{n \to \infty} \frac{f((-2)^{2n} xy)}{(-2)^{2n}} = \delta(xy),$$

$$\lim_{n \to \infty} \frac{1}{4^n} f((-2)^n x) \cdot (-2)^n y = \lim_{n \to \infty} \frac{f((-2)^n x)}{(-2)^n} \cdot \frac{(-2)^n y}{(-2)^n} = \delta(x)y,$$

$$\lim_{n \to \infty} \frac{1}{4^n} (-2)^n x \cdot f((-2)^n y) = \lim_{n \to \infty} \frac{(-2)^n x}{(-2)^n} \cdot \frac{(-2)^n y}{(-2)^n} = x\delta(y)$$

for all $x, y \in A$ whenever the multiplication is defined. If we let $n \to \infty$ in the above inequality then (4.4) gives $\delta(xy) = \delta(x)y - x\delta(y)$ for all $x, y \in A$ whenever the multiplication is defined. $\square$

Corollary 4.2. *Let θ, p be nonnegative real numbers with $p < 1$ and let $f : A \to A$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_A \leq \|2f(x + \mu y + z)\|_A + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$$

for all $\mu \in \mathbb{T}^1$ and for all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique derivation δ on A satisfying

$$\|f(x) - \delta(x)\|_A \leq \frac{2\theta}{2 - 2^p} \|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$. When $p < 1$, we get

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right)$$

$$= \sum_{j=0}^{\infty} \frac{2^{pj}}{2^j} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$$

$$= \frac{2\theta}{2 - 2^p}(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p).$$

In addition, let $\psi : A^2 \to [0, \infty)$ be $\psi(x, y) = \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$. When $p < 1$, we have

$$\lim_{n\to\infty} \frac{1}{4^n} \psi((-2)^n x, (-2)^n y) = \lim_{n\to\infty} \frac{2^{2pn}}{4^n} \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0$$

for all $x, y \in A$. By applying Theorem 4.1, there exists a unique proper CQ^*-algebra homomorphism $h : A \to B$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{1}{2}\widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2 - 2^p}\|x\|_A^p$$

for all $x \in A$. $\qquad\square$

Corollary 4.3. *Let θ, p be nonnegative real numbers with $p < 1$ and let $f : A \to B$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_A \leq \|2f(x + \mu y + z)\|_A + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$$

for all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique derivation δ on A satisfying

$$\|f(x) - \delta(x)\|_A \leq \frac{2\theta}{2 - 2^p}\|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$ and let $\psi : A^2 \to [0, \infty)$ be $\psi(x, y) = \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$. When $p < 1$, we have $\widetilde{\varphi}(x, y, z) < \infty$ and

$$\lim_{n\to\infty} \frac{1}{4^n} \phi((-2)^n x, (-2)^n y) = \lim_{n\to\infty} \frac{2^{2pn}}{4^n} \cdot \theta \cdot \|x\|_A^p \cdot \|y\|_A^p = 0$$

for all $x, y, z \in A$.

By applying Theorem 4.1, there exists a unique proper CQ^*-algebra homomorphism $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{1}{2}\widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2 - 2^p}\|x\|_A^p$$

for all $x \in A$. $\qquad\square$

Theorem 4.4. *Let $f : A \to A$ be a mapping. If there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying (4.1) and*

$$(4.6) \qquad \widetilde{\varphi}(x, y, z) := \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty,$$

and, in addition, there exists a function $\psi : A^2 \to [0, \infty)$ satisfying (4.3) and

$$(4.7) \qquad \lim_{n\to\infty} 4^n \psi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}\right) = 0$$

for all $x, y \in A$, then there exists a unique derivation δ on A satisfying

$$(4.8) \qquad \|f(x) - \delta(x)\|_A \leq \frac{1}{2}\widetilde{\varphi}(0, -x, x)$$

for all $x \in A$.

Proof. By Corollary 2.6, we have a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ defined by

$$\delta(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in A$ which satisfies (4.8).

Now replacing x, y by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}$, respectively, and multiplying by 4^n in (4.3), we get

$$\left\| 4^n \left[f\left(\frac{x}{(-2)^n} \cdot \frac{y}{(-2)^n}\right) - f\left(\frac{x}{(-2)^n}\right) f\left(\frac{y}{(-2)^n}\right) \right] \right\|_B$$
$$\leq 4^n \psi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}\right)$$

for all $x, y \in A$ whenever the multiplication is defined. We have

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{(-2)^n} \cdot \frac{y}{(-2)^n}\right) = \lim_{n\to\infty} (-2)^{2n} f\left(\frac{xy}{(-2)^{2n}}\right) = \delta(xy),$$

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{(-2)^n}\right) \cdot \frac{y}{(-2)^n} = \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right) \cdot \lim_{n\to\infty} \frac{(-2)^n y}{(-2)^n} = \delta(x)y,$$

$$\lim_{n\to\infty} 4^n \frac{x}{(-2)^n} \cdot f\left(\frac{y}{(-2)^n}\right) = \lim_{n\to\infty} \frac{(-2)^n x}{(-2)^n} \cdot \lim_{n\to\infty} (-2)^n f\left(\frac{y}{(-2)^n}\right) = x\delta(y)$$

for all $x, y \in A$ whenever the multiplication is defined. If we let $n \to \infty$ in the above inequality, then (4.7) gives $\delta(xy) = \delta(x)y - x\delta(y)$ for all $x, y \in A$ whenever the multiplication is defined. $\qquad\square$

Corollary 4.5. *Let θ, p be nonnegative real numbers with $p > 1$ and let $f : A \to A$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_B \leq \|2f(x + \mu y + z)\|_A + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique derivation δ on A satisfying

$$\|f(x) - \delta(x)\|_A \leq \frac{2\theta}{2^p - 2}\|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$. When $p > 1$, we get

$$\widetilde{\varphi}(x, y, z) := \sum_{i=1}^{\infty} 2^i \varphi \left(\frac{x}{(-2)^i}, \frac{y}{(-2)^i}, \frac{z}{(-2)^i} \right)$$

$$= \sum_{i=1}^{\infty} \frac{2^i}{2^{pi}} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$$

$$= \frac{2\theta}{2^p - 2}(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p).$$

In addition, let $\psi : A^2 \to [0, \infty)$ be $\phi(x, y) = \theta(\|x\|_A^{2p} + \|y\|_A^{2p})$. When $p > 1$, we get

$$\lim_{n \to \infty} 4^n \psi \left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n} \right) = \lim_{n \to \infty} \frac{4^n}{2^{2pn}} \theta(\|x\|_A^{2p} + \|y\|_A^{2p}) = 0$$

for all $x, y \in A$.

By applying Theorem 4.4, there exists a unique proper CQ^*-algebra homomorphism $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{1}{2} \widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2^p - 2} \|x\|_A^p$$

for all $x \in A$. $\qquad\square$

Corollary 4.6. *Let θ, p be nonnegative real numbers with $p > 1$ and let $f : A \to B$ a mapping satisfying*

$$\|f(2x) + \mu f(2y) + 2f(z)\|_A \le \|2f(x + \mu y + z)\|_A + \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f(xy) - f(x)y - xf(y)\|_A \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ whenever the multiplication is defined. Then there exists a unique derivation δ on A satisfying

$$\|f(x) - \delta(x)\|_A \le \frac{2\theta}{2^p - 2} \|x\|_A^p$$

for all $x \in A$.

Proof. Let $\varphi : A^3 \to [0, \infty)$ be $\varphi(x, y, z) = \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$ and let $\psi : A^2 \to [0, \infty)$ be $\psi(x, y) = \theta \cdot \|x\|_A^p \cdot \|y\|_A^p$. When $p > 1$, we have $\widetilde{\varphi}(x, y, z) < \infty$ and

$$\lim_{n \to \infty} 4^n \phi \left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n} \right) = \lim_{n \to \infty} \frac{4^n}{2^{2pn}} \cdot \theta \cdot \|x\|_A^p \cdot \|y\|_A^p = 0$$

for all $x, y, z \in A$. By applying Theorem 3.4, there exists a unique proper CQ^*-algebra homomorphism $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{1}{2} \widetilde{\varphi}(0, -x, x) = \frac{2\theta}{2^p - 2} \|x\|_A^p$$

for all $x \in A$. $\qquad\square$

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Jung Rye Lee
Department of Mathematics
Daejin University
Kyeonggi 487-711, Korea
E-mail address: jrlee@daejin.ac.kr

Choonkil Park
Department of Mathematics
Research Institute for Natural Sciences
Hanyang University
Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

Dong Yun Shin
Department of Mathematics
University of Seoul
Seoul 130-743, Korea
E-mail address: dyshin@uos.ac.kr