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AN UPPER BOUND ON STICK NUMBER OF KNOTS

YOUNGSIK HUH

Department of Mathematics, College of Natural Sciences,
Hanyang University, Haengdang-1-dong,
Seongdong-gu, Seoul 133-791, Korea
yshuh@hanyang.ac.kr

SEUNGSANG OH

Department of Mathematics, Korea University,
Anam-dong, Sungbuk-ku, Seoul 136-701, Korea
seungsang@korea.ac.kr

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ABSTRACT

In 1991, Negami found an upper bound on the stick number $s(K)$ of a nontrivial knot K in terms of crossing number $c(K)$ which is $s(K) \leq 2c(K)$. In this paper we give a new upper bound in terms of arc index, and improve Negami's upper bound to $s(K) \leq \frac{3}{2}(c(K)+1)$. Moreover if K is a nonalternating prime knot, then $s(K) \leq \frac{3}{2}c(K)$.

Keywords: Knot; stick number; upper bound.

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1. Introduction

A simple closed curve embedded in Euclidean 3-space is called a *knot*. Two knots K and K' are said to be *equivalent*, if there exists an orientation preserving homeomorphism isotopy of $\mathbb{R}^3$ which sends K to K'. The equivalence class of K is called the *knot type of* K. A knot equivalent to another knot in a plane of 3-space is said to be *trivial*.

A *stick knot* is a knot which consists of finite line segments, called *sticks*. This presentation of knots can be considered to be a reasonable mathematical model of cyclic molecules or molecular chains because such physical objects have rigidity.

Concerning stick knots, one natural problem may be to determine stick numbers. The *stick number* $s(K)$ of a knot K is defined to be the minimal number of sticks required to realize the knot type as a stick knot. This quantity was investigated for some specific knots including knots with crossing number below 10 [6, 19], torus knots [13], 2-bridge knots [17] and knots with 1, 2 or 3-integer Conway notation [11] (see [6, 1.1] for a brief survey). On the other hand Negami's work is also in need of mention [18]. He showed that for any nontrivial knot K with crossing number $c(K)$,

$$\frac{5+\sqrt{25+8(c(K)-2)}}{2} \leq s(K) \leq 2c(K).$$

Although Negami's bounds are applicable to all nontrivial knots, they are not likely to be quite strict. Furthermore, among 53 prime knots with crossing number below 10, the trefoil knot is the only one for which Negami's upper bound is tight. Therefore we may raise questions on the tightness of these bounds [1, 11]. To be explicit,

Q1. Is there any knot satisfying $2s(K) = 5 + \sqrt{25+8(c(K)-2)}$?
Q2. Is there any knot satisfying $s(K) = 2c(K)$ other than trefoil knot?

Negami's lower bound was slightly improved by Calvo [5]. And recently Elifai showed that the answer for Q1 is negative for knots with $c(K) \leq 26$ [10].

In this paper we establish a new upper bound on stick number which is described by arc index (Theorem 2.2), and improve Negami's upper bound (Theorem 1.1). Here we remark that question Q2 is answered by our improved bound.

Theorem 1.1. *Let K be a nontrivial knot. Then $s(K) \leq \frac{3}{2}(c(K)+1)$. Moreover if K is a nonalternating prime knot, then $s(K) \leq \frac{3}{2}c(K)$.*

Note that $c(K) \geq 3$ for any nontrivial knot K. If $\frac{3}{2}(c(K)+1) \geq 2c(K)$, then $c(K) \leq 3$. It is known that the trefoil knot is the only nontrivial knot with crossing number 3. And its stick number is exactly 6. Therefore, by Theorem 1.1, we can conclude that the answer to Q2 is negative.

This theorem follows from the inequality $s(K) \leq \frac{3}{2}(a(K)-1)$ of Theorem 2.2. This upper bound of stick number in terms of arc index is sometimes relatively effective as a general bound. For example, we consider torus knots. It is known that, for (p,q)-torus knots, arc index is $p+q$ [16], and stick number is $2q$ when $2 \leq p < q \leq 2p$ [13]. For the best case of this upper bound, $(p, 2p-1)$-torus knots have stick number $4p-2$, while Theorem 2.2 yields an upper bound of $4.5p-3$. But for the worst case, $(p, p+1)$-torus knots have stick number $2p+2$, while Theorem 2.2 yields an upper bound of $3p$.

Furthermore we discuss the efficacy of this upper bound of stick number in terms of crossing number. The (p,q)-torus knots with $2 \leq p < q \leq 2p$ have crossing number $(p-1)q$. Thus Theorem 1.1 yields an upper bound of $\frac{3}{2}(p-1)q$, while their exact stick number is $2q$. So the upper bound provided by Theorem 1.1 grows quadratically with q because p must be at least half q, whereas the actual value grows linearly.

2. Arc Index, Stick Number and Crossing Number

In this section, we prove Theorem 1.1 by establishing a new upper bound on stick number which is described by another minimality invariant, *arc index*.

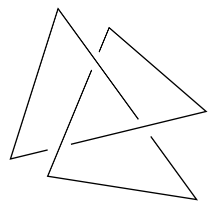

Fig. 1. A stick presentation of right-handed trefoil.

First we consider a specific type of knot diagram which is obtained by drawing n chords $l_1, \ldots, l_n$ on a 2-dimensional circular disk B according to the following rules:

(1) The end points of each l_i lie on the boundary of B.
(2) If l_i and l_j share a crossing in the interior of B and $i < j$, then l_i passes under l_j.

If a diagram of such a type represents a knot K, it is called an *arc presentation* of K. And the *arc index* $a(K)$ of a knot K is defined to be the minimal number of chords among all possible arc presentations of its knot type. In fact our definition of arc presentation is a little modified from the original one, but essentially identical [4, 7]. The left of Fig. 2 shows an arc presentation of trefoil knot.

Bae and Park established an upper bound on arc index in terms of crossing number. [3, Corollary 4 and Theorem 9] provide the following:

Theorem 2.1 [3]. *Let K be any nontrivial knot. Then $a(K) \leq c(K)+2$. Moreover if K is a nonalternating prime knot, then $a(K) \leq c(K) + 1$.*

Therefore, if we prove Theorem 2.2, then the proof of our main theorem is completed.

Theorem 2.2. *Let K be any nontrivial knot. Then $s(K) \leq \frac{3}{2}(a(K) - 1)$.*

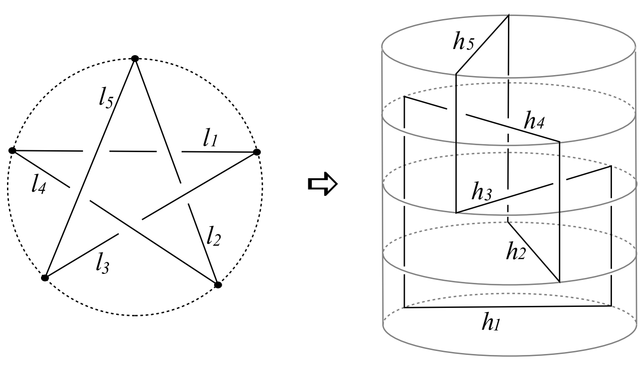

Fig. 2. A stick knot in cylinder constructed from an arc presentation.

Proof. Let K be a nontrivial knot with $a(K) = n$ and D be an its arc presentation with n chords $l_1, \ldots, l_n$. $\overline{D}$ denotes the projection of D. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by $\pi(x, y, z) = (x, y)$. From D we construct a stick knot K_1 in the cylinder $B \times [1, n]$ so that $\pi(K_1) = \overline{D}$. For each integer $i \in [1, n]$, put a line segment h_i into $B \times \{i\}$ so that $\pi(h_i) = l_i$. If $l_i \cap l_j \cap \partial B = \{p\}$, then connect $\pi^{-1}(p) \cap h_i$ to $\pi^{-1}(p) \cap h_j$ by a vertical line segment v_{ij} so that $\pi(v_{ij}) = p$. Note that we do not distinguish v_{ij} from v_{ji}. By adding all such vertical sticks, we obtain a stick knot K_1 with $2n$ sticks which is equivalent to K. Figure 2 shows an example of a stick knot constructed from an arc presentation of the right-handed trefoil.

A horizontal stick h_i is said be *type-I* (*respectively*, *type-III*), if the indices of the two chords adjacent to l_i in D are greater (*respectively*, less) than i. If neither type-I nor type-III, then h_i is *type-II*. Notice that h_1 and h_n should be type-I and type-III, respectively. If h_{n-1} is type-II, then we can modify K_1 as illustrated in Fig. 3(a), so that the number of horizontal sticks is reduced by one, which is contradictory to the minimality of the number of chords. Since K_1 is a nontrivial knot, h_{n-1} cannot be type-I. Hence h_{n-1} should be type-III and similarly h_2 should be type-I.

From K_1 we construct another stick knot K_2 in which the z-coordinate of each h_i may be changed into some integer z_i, while its xy-coordinates are preserved. Concretely, if we denote the ith horizontal stick of K_2 by h'_i, then $\pi(h_i) = \pi(h'_i)$ and $h'_i \subset B \times \{z_i\}$ in $B \times [0, \infty)$. The height z_i will be determined in an inductive manner. Firstly, set $z_1 = 1$ and $z_2 = 2$. For $3 \le i \le n$, if h_i is type-I, then z_i is set to be $z_{i-1} + 1$. If h_i is type-II, there is a vertical stick v_{ij} with $j < i$ which is adjacent to h_i. Then put h'_i into $B \times \{z_i\}$ for some large enough z_i and connect h'_i to h'_j via the vertical stick v'_{ij} between $B \times \{z_i\}$ and $B \times \{z_j\}$, so that the interior of the triangle determined by $h'_i \cup v'_{ij}$ has no intersection with any other horizontal stick h'_k, $k < i$. Here, such a triangle will be called a *reducible triangle* of h'_i. If h_i is type-III, that is, h_i is adjacent to some v_{ij} and v_{ik} with $i > j > k$, then similarly the height of h'_i is determined so that the triangle whose boundary contains $h'_i \cup v'_{ij}$ is reducible.

Now we modify K_2 for the purpose of decreasing the number of sticks. For each i from 3 to $n - 1$, if h'_i is type-II or III, replace $h'_i \cup v'_{ij}$ with the other edge of the reducible triangle (see Fig. 3(b)). Since the interior of the triangle has no intersection with any other part of the knot, such a replacement preserves the knot type. And the number of sticks is reduced by one, after each modification. For h'_n, we modify the knot in another way. Let v'_{ni} and v'_{nj} be the sticks adjacent to h'_n. The other stick adjacent to v'_{ni} (*respectively*, v'_{nj}) is denoted by e_i (*respectively*, e_j). Extend e_i and e_j toward the end points $e_i \cap v'_{ni}$ and $e_j \cap v'_{nj}$, respectively, long enough so that the two extended line segments are connected by a line segment outside of $B \times [1, z_n]$. Replace $e_i \cup v'_{ni} \cup h'_n \cup v'_{nj} \cup e_j$ with these three line segments (see Fig. 3(c) for example). Then the knot type is preserved, but the number of sticks is reduced by two. Let K_3 be the resulting stick knot.

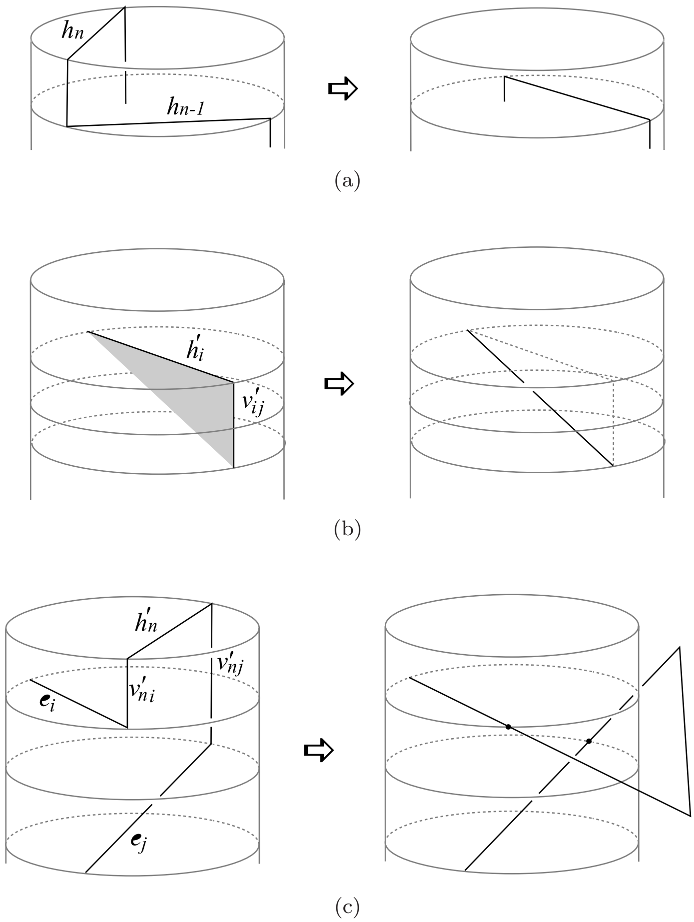

Fig. 3. (a) Reduction when h_{n-1} is type-II, (b) reduction along a reducible triangle and (c) reduction near h'_n.

Let $\beta_1(K_1)$, $\beta_2(K_1)$ and $\beta_3(K_1)$ be the numbers of type-I, type-II and type-III horizontal sticks of K_1, respectively. Note that $\beta_1(K_1) = \beta_3(K_1)$. Since $n = \beta_1(K_1) + \beta_2(K_1) + \beta_3(K_1)$, the number of sticks of K_3 is equal to

$$2n - \beta_2(K_1) - (\beta_3(K_1) - 1) - 2 = n + \beta_1(K_1) - 1.$$

Therefore,

$$s(K) \leq n + \beta_1(K_1) - 1.$$

Now we consider an upper bound of $\beta_1(K_1)$. If n is odd, then $\beta_1(K_1) \leq (n-1)/2$. If n is even, then $\beta_1(K_1) \leq n/2$ in which the equality holds only when $\beta_2(K_1) = 0$. In that case, let v_{i1} and v_{j1} be the horizontal sticks adjacent to h_1 in K_1. And replace $v_{i1} \cup h_1 \cup v_{j1}$ with $v_{i(n+1)} \cup h_{n+1} \cup v_{j(n+1)}$, where h_{n+1} is the horizontal line segment in $B \times \{n+1\}$ satisfying $\pi(h_1) = \pi(h_{n+1})$. Then the resulting stick knot K_1' in $B \times [2, n+1]$ is equivalent to K_1. Because $\beta_2(K_1) = 0$ and $\beta_2(K_1') = 2$, we have

$$\beta_1(K_1') = \beta_1(K_1) - 1 \leq \frac{n}{2} - 1 < \frac{n-1}{2}.$$

To summarize, for a nontrivial knot K with $a(K) = n$, there exists an equivalent stick knot K' with $2n$ sticks in the cylinder satisfying

$$\beta_1(K') \leq \frac{n-1}{2}$$

and therefore

$$s(K) \leq n + \beta_1(K') - 1 \leq a(K) + \frac{a(K)-1}{2} - 1 = \frac{3}{2}(a(K) - 1). \qquad \square$$

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