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Nakajima monomials, Young walls and Kashiwara embedding for $U_q(A_n^{(1)})$

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ABSTRACT

In this paper, we realize the crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ of level 1 for $U_q(A_n^{(1)})$ using Nakajima monomials satisfying some conditions. Also, from this monomial realization, we obtain the image of Kashiwara embedding $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbb{Z}^\infty \otimes R_\lambda$, where ι is some infinite sequence from the index set of simple roots. Finally, we give a $U_q(A_n^{(1)})$-crystal isomorphism between Young wall realization and monomial realization, and so we can understand the image of Kashiwara embedding $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbb{Z}^\infty \otimes R_\lambda$ using the combinatorics of Young walls.

Introduction

The *quantum groups* $U_q(\mathfrak{g})$ introduced by Drinfel'd and Jimbo, independently are deformations of the universal enveloping algebras $U(\mathfrak{g})$ of Kac–Moody algebras $\mathfrak{g}$ [2,5]. The important feature of quantum groups $U_q(\mathfrak{g})$ is that their representation theory is the same as that of $U(\mathfrak{g})$. The *crystal bases*, introduced by Kashiwara [15], can be viewed as bases at $q = 0$ for the integrable modules over quantum groups and they are given as a structure of colored oriented graphs, called the *crystal graphs*, reflecting the combinatorial structure of integrable modules.

In [16], Kashiwara introduced the embedding of crystal $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}^\infty$, where ι is some infinite sequence from the index set of simple roots. But, in general, it is not easy to find the image $\operatorname{Im}\Psi_\iota$.

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In [1], Cliff described the image of the Kashiwara embedding for the classical Lie algebras, and for more general types, Zelevinsky and Nakashima obtained the image of the embedding by a unified method, called the *polyhedral realization* [28]. Moreover, in [27], for a dominant integral weight λ, Nakashima gave the embedding of crystal $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbb{Z}^\infty \otimes R_\lambda$, and described the explicit form of $\operatorname{Im}\Psi_\iota^\lambda$. Recently, in [30,31], the second author extended their theory to the quantum generalized Kac–Moody algebras. That is, he gave the polyhedral realizations of the crystals $B(\infty)$ and $B(\lambda)$ over the quantum generalized Kac–Moody algebras.

In [6], Kang introduced an affine combinatorial object called the *Young wall*, and showed that the crystal bases for the irreducible highest weight modules of level 1 for quantum affine algebras are realized as the sets of *reduced proper Young walls*. In [14], Kang and Lee extended his theory to the realization of the crystal basis for the irreducible highest weight representations of higher level for quantum affine algebras. Also, in [10], Kang, Lee and the authors used Young walls to realize the crystal bases of irreducible highest weight modules over classical quantum finite algebras.

In [17,26], Kashiwara and Nakajima independently defined a crystal structure on the set $\mathcal{M}$ of some monomials which we call the *Nakajima monomials*. Moreover, they showed that the connected component $C(M)$ containing a maximal vector M of a dominant integral weight λ is isomorphic to the irreducible highest weight crystal $B(\lambda)$. The explicit description of this connected component $C(M)$ was given for $U_q(\mathfrak{g})$ ($\mathfrak{g} = A_n, B_n, C_n, D_n, G_2$ and $A_n^{(1)}$) by Kang, Kim, and Shin [11,12], Kim and Shin [19,29]. In [13], Kang and the authors introduced the notion of *modified Nakajima monomials*, defined a crystal structure on the set of modified Nakajima monomials, and showed that the connected component containing **1** is isomorphic to the crystal $B(\infty)$. Also, in [21], the authors described explicitly the connected component containing **1** for quantum finite algebras. Moreover, Jeong, Kang, and the authors extended their theory to the construction of the crystals $B(\infty)$ and $B(\lambda)$ over quantum generalized Kac–Moody algebras [4,22].

Besides above mentioned realizations of the crystals, there are several well-known descriptions, e.g., Young tableaux realization for classical Lie algebras [18,20], path realization using perfect crystals for quantum affine algebras [7–9], Littelmann's path realization for symmetrizable Kac–Moody algebras [23,24]. Therefore, it is natural to consider the connection between several realizations. However, except for some cases, e.g., path realization and Young wall realization, tableau realization and monomial realization, it is not well known.

In this paper, we give a new monomial realization of crystal bases for the irreducible highest weight representations of level 1 over quantum affine algebra $U_q(A_n^{(1)})$, which is different from the one given in [19]. One of the important advantages of this realization is that it has a natural 1–1 correspondence with the image of the Kashiwara embedding Ψ_ι^λ. In addition, we give a crystal isomorphism from Young wall realization to monomial realization. Therefore, by combining two crystal isomorphisms, we have a crystal isomorphism between the set of all reduced Young walls and the image of Ψ_ι^λ. In other words, we can understand the image of Ψ_ι^λ from the combinatorics of Young walls.

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1. Crystals

1.1. Quantum affine algebras

Let I be a finite index set and let $A = (a_{ij})_{i,j\in I}$ be a *Cartan matrix* of affine type. Let $P^\vee = (\bigoplus_{i\in I} \mathbb{Z}h_i) \oplus \mathbb{Z}d$ and $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$ be the *dual weight lattice* and the *weight lattice*, respectively. Let $\Pi^\vee = \{h_i \mid i \in I\}$ and $\Pi = \{\alpha_i \mid i \in I\}$ be the sets of *simple coroots* and *simple roots*, respectively. Then the quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ is called an *affine Cartan datum*, and we denote by $U_q(\mathfrak{g})$ the *quantum affine algebra* associated with the affine Cartan datum.

We denote by $P^+ = \{\lambda \in P \mid \langle h_i, \lambda\rangle \geqslant 0$ for all $i \in I\}$ the set of *dominant integral weights*. For instance, the *fundamental weights* Λ_i ($i \in I$) defined by

$$\langle h_j, \Lambda_i \rangle = \delta_{ij} \quad \text{and} \quad \langle d, \Lambda_i \rangle = 0 \quad (j \in I)$$

are dominant integral weights.

1.2. Abstract crystals

An *abstract crystal* for $U_q(\mathfrak{g})$ or a $U_q(\mathfrak{g})$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ $(i \in I)$ such that for all $i \in I$ and $b \in B$,

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$,
(ii) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq 0$,
(iii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq 0$,
(iv) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \neq 0$,
(v) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \neq 0$,
(vi) $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$ for $b, b' \in B$,
(vii) $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

Let B_1 and B_2 be $U_q(\mathfrak{g})$ crystals. A *morphism of crystals* or *crystal morphism* is a map $\psi : B_1 \to B_2$ satisfying the following conditions:

(i) for $b \in B_1$, $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, and $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $i \in I$,
(ii) if $b \in B_1$ and $\tilde{f}_i b \in B_1$, then $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

1.3. Examples of crystals

The crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ with $\lambda \in P^+$ is a $U_q(\mathfrak{g})$-crystal, where the maps ε_i, φ_i $(i \in I)$ are given by

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \qquad \varphi_i(b) = \max\{k \geqslant 0 \mid \tilde{f}_i^k b \neq 0\}.$$

Let $U_q^-(\mathfrak{g})$ be the negative part of the quantum group $U_q(\mathfrak{g})$. Then the crystal basis $B(\infty)$ of $U_q^-(\mathfrak{g})$ is also a $U_q(\mathfrak{g})$-crystal, where

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \qquad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle.$$

Moreover, the singleton $R_\lambda = \{r_\lambda\}$ $(\lambda \in P)$ is a $U_q(\mathfrak{g})$-crystal with

$$\mathrm{wt}(r_\lambda) = \lambda, \qquad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \qquad \varphi_i(r_\lambda) = 0, \qquad \tilde{e}_i r_\lambda = \tilde{f}_i r_\lambda = 0.$$

1.4. Tensor product of crystals

Let B_1 and B_2 be crystals. Then their tensor product $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1,\ b_2 \in B_2\}$ is also a crystal with the maps $\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i$ given by

$$\begin{aligned}
&\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
&\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1) \rangle\}, \\
&\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2) \rangle\}, \\
&\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
&\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}
\end{aligned}$$

2. Young walls

2.1. Young walls for $U_q(A_n^{(1)})$

The *Young walls* for $U_q(A_n^{(1)})$ are built of colored blocks with shape unit width, unit height, unit thickness. Given a dominant integral weight λ of level 1 for $U_q(A_n^{(1)})$, we build the walls on the frame Y_λ, called the *ground-state wall* of weight λ, following the building rules given below.

(i) The walls must be built on top of the ground-state wall.
(ii) The colored blocks should be stacked in the patterns given below.
On Y_{Λ_i}:

					0	1
					n	0
					$n-1$	n
					$\vdots$	$\vdots$
0	1	2	3	$\cdots$	i	$i+1$
n	0	1	2	$\cdots$	$i-1$	i

(iii) Except for the right-most column, there should be no free space to the right of any block.

A Young wall is said to be *reduced* if

$$\#(k) - \#(k+1) \leqslant n \quad \text{for all } k \geqslant 1.$$

Here, $\#(k)$ is the number of blocks in the kth column of Y from the right.

2.2. Crystal structure on $\mathcal{F}(\lambda)$

For a dominant integral wight λ of level 1, let $\mathcal{F}(\lambda)$ be the set of all Young walls on Y_λ. Given a Young wall $Y \in \mathcal{F}(\lambda)$, a colored i-block is called a *removable i-block* if the wall remains a Young wall after removing the block, and a column in Y is called *i-removable* if the top of that column is a removable i-block. A place in Y where one may add an i-block to obtain another Young wall is called an *admissible i-slot*, and a column in Y is called *i-admissible* if the top of that column is an admissible i-slot.

We now define the action of Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$ on $\mathcal{F}(\lambda)$. Fix $i \in I$ and let $Y = (y_k)_{k=1}^{\infty} \in \mathcal{F}(\lambda)$ be a reduced Young wall. To each column y_k of Y, we assign its *i-signature $sgn_i(y_k)$* as $-$ (resp. $+$) if y_k is i-removable (resp. i-admissible). From the sequence of $+$'s and $-$'s, we obtain a finite sequence of $-$'s followed by $+$'s, reading from left to right by removing every $(+,-)$-pair, which is called the *i-signature* of Y. We define $\tilde{e}_i Y$ to be the Young wall obtained from Y by removing the i-block corresponding to the right-most $-$ in the i-signature of Y. If there exists no $-$ in the i-signature of Y, we define $\tilde{e}_i Y = 0$. We define $\tilde{f}_i Y$ to be the Young wall obtained from Y by adding an i-block to the column corresponding to the left-most $+$ in the i-signature of Y. If there exists no $+$ in the i-signature of Y, we define $\tilde{f}_i Y = 0$. We also define the maps

$$\mathrm{wt} : \mathcal{F}(\lambda) \to P, \qquad \varepsilon_i : \mathcal{F}(\lambda) \to \mathbb{Z}, \qquad \varphi_i : \mathcal{F}(\lambda) \to \mathbb{Z}$$

by

$$\begin{aligned} \mathrm{wt}(Y) &= -\sum_{i\in I} k_i\alpha_i, \\ \varepsilon_i(Y) &= \text{the number of $-$'s in the i-signature of Y}, \\ \varphi_i(Y) &= \text{the number of $+$'s in the i-signature of Y}, \end{aligned}$$

where k_i is the number of i-colored blocks in Y. Then $(\mathcal{F}(\lambda), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ becomes a $U_q(A_n^{(1)})$-crystal. Moreover, if we let $\mathcal{Y}(\lambda)$ be the set of all reduced Young walls on Y_λ, then it is not difficult to see that $\mathcal{Y}(\lambda)$ is a subcrystal of $\mathcal{F}(\lambda)$ [3,6].

2.3. Young wall realization of $B(\lambda)$

Theorem 2.1. *(See [3,6].) Let λ be a dominant integral weight of level* 1*, and let $\mathcal{Y}(\lambda)$ be the set of all reduced Young walls on Y_λ. Then there exists a crystal isomorphism $\phi : \mathcal{Y}(\lambda) \to B(\lambda)$ sending Y_λ to the highest weight vector u_λ in $B(\lambda)$.*

3. Embedding of crystals

3.1. Embedding of $B(\infty)$

Let

$$\mathbb{Z}^\infty := \{(\ldots, x_k, \ldots, x_1) \mid x_k \in \mathbb{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$

and let $\iota = (\ldots, i_k, \ldots, i_1)$ be an infinite sequence such that

$$i_k \neq i_{k+1} \quad \text{and} \quad \#\{k \mid i_k = i\} = \infty \quad \text{for any } i \in I. \tag{3.1}$$

Now, we associate to ι a crystal structure on $\mathbb{Z}^\infty$. Let $\vec{x} = (\ldots, x_k, \ldots, x_1)$ be an element of $\mathbb{Z}^\infty$. For $k \geqslant 1$, we set

$$\sigma_k(\vec{x}) = x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j.$$

Let $\sigma^{(i)}(\vec{x}) = \max_{k:\ i_k=i}\{\sigma_k(\vec{x})\}$, and

$$n_f = \min\{k \mid i_k = i,\ \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x})\}, \qquad n_e = \max\{k \mid i_k = i,\ \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x})\}.$$

Now, we define

$$\tilde{f}_i\vec{x} = (x_k + \delta_{k,n_f})_{k\geqslant 1}, \qquad \tilde{e}_i\vec{x} = \begin{cases} (x_k - \delta_{k,n_e})_{k\geqslant 1} & \text{if } \sigma^{(i)}(\vec{x}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We also define

$$\mathrm{wt}(\vec{x}) = -\sum_{j=1}^{\infty} x_j\alpha_{i_j}, \qquad \varepsilon_i(\vec{x}) = \sigma^{(i)}(\vec{x}), \qquad \varphi_i(\vec{x}) = \langle h_i, \mathrm{wt}(\vec{x})\rangle + \varepsilon_i(\vec{x}).$$

Then it is easy to see that $\mathbb{Z}^\infty$ is a crystal [28]. We denote this crystal by $\mathbb{Z}_\iota^\infty$.

The following is the Kashiwara embedding theorem.

Theorem 3.1. *(See [16,28].) Let ι be an infinite sequence satisfying* (3.1)*. Then there exists a unique strict embedding of crystals*

$$\Psi_i : B(\infty) \hookrightarrow \mathbb{Z}_\iota^\infty \quad \textit{such that} \quad u_\infty \mapsto (\ldots, 0, \ldots, 0),$$

where u_∞ is the highest weight vector in $B(\infty)$.

3.2. Embedding of $B(\lambda)$

Let $R_\lambda = \{r_\lambda\}$ $(\lambda \in P)$ be the crystal given in Section 1.3. Then we have the strict embedding of crystals

$$\Omega_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes R_\lambda.$$

Therefore, by Theorem 3.1, we have

Theorem 3.2. *(See [27].) Let ι be an infinite sequence satisfying* (3.1)*. Then there exists the unique strict embedding of crystals*

$$\Psi_i^\lambda : B(\lambda) \overset{\Omega_\lambda}{\hookrightarrow} B(\infty) \otimes R_\lambda \overset{\Psi_\iota \otimes \mathrm{id}}{\hookrightarrow} \mathbb{Z}_\iota^\infty \otimes R_\lambda,$$

such that the highest weight vector u_λ sends to $(\ldots, 0, \ldots, 0) \otimes r_\lambda$.

Remark 3.3. By Theorem 3.1 and Theorem 3.2, the crystals $B(\infty)$ and $B(\lambda)$ are isomorphic to the connected components of $\mathbb{Z}_\iota^\infty$ and $\mathbb{Z}_\iota^\infty \otimes R_\lambda$ containing $(\ldots, 0, \ldots, 0)$ and $(\ldots, 0, \ldots, 0) \otimes r_\lambda$, respectively.

4. Nakajima monomials

4.1. Nakajima monomials

Let $\mathfrak{M}$ be the set of monomials in the commuting variables $Y_i(n)$ $(i \in I, n \in \mathbb{Z})$ of the form

$$\mathfrak{m} = \prod_{i \in I, n \in \mathbb{Z}} Y_i(n)^{y_i(n)} \quad (y_i(n) \in \mathbb{Z}).$$

The monomials in $\mathfrak{M}$ are called the *Nakajima monomials*.

4.2. Crystal structure of $\mathfrak{M}$

Let $\mathfrak{m}$ be a monomial in $\mathfrak{M}$. For each $i \in I$, we define

$$\begin{aligned}
\mathrm{wt}(\mathfrak{m}) &= \sum_i \Bigl(\sum_n y_i(n)\Bigr) \Lambda_i, \\
\varphi_i(\mathfrak{m}) &= \max\Bigl\{\sum_{k \leqslant n} y_i(k) \Bigm| n \in \mathbb{Z}\Bigr\}, \\
\varepsilon_i(\mathfrak{m}) &= \max\Bigl\{-\sum_{k > n} y_i(k) \Bigm| n \in \mathbb{Z}\Bigr\}.
\end{aligned} \tag{4.1}$$

Let $C = (c_{ij})_{i \neq j}$ be a set of nonnegative integers such that $c_{ij} + c_{ji} = 1$, and for each $i \in I$, $n \in \mathbb{Z}$, $A_i(n)$ is defined by

$$A_i(n) = Y_i(n) Y_i(n+1) \prod_{j \neq i} Y_j(n + c_{ji})^{a_{ji}}.$$

We set

$$n_f = \min\left\{ n \in \mathbb{Z} \;\middle|\; \varphi_i(\mathfrak{m}) = \sum_{k \leqslant n} y_i(k) \right\}, \qquad n_e = \max\left\{ n \in \mathbb{Z} \;\middle|\; \varphi_i(\mathfrak{m}) = \sum_{k \leqslant n} y_i(k) \right\}.$$

Now, we define the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ by

$$\tilde{f}_i \mathfrak{m} = \begin{cases} \mathfrak{m} \cdot A_i(n_f)^{-1} & \text{if } \varphi_i(\mathfrak{m}) > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{e}_i \mathfrak{m} = \begin{cases} \mathfrak{m} \cdot A_i(n_e) & \text{if } \varepsilon_i(\mathfrak{m}) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Then $\mathfrak{M}$ becomes a $U_q(\mathfrak{g})$-crystal with the maps wt, $\varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i$ $(i \in I)$ defined in (4.1) and (4.2).

4.3. Monomial realization of $B(\lambda)$

Theorem 4.1. *(See [17,26].) Let $\mathfrak{m}$ be a maximal vector in $\mathfrak{M}$ of weight λ, and let $C(\mathfrak{m})$ be the connected component of $\mathfrak{M}$ containing $\mathfrak{m}$. Then there exists a $U_q(\mathfrak{g})$-crystal isomorphism $\psi : C(\mathfrak{m}) \to B(\lambda)$ sending $\mathfrak{m}$ to the highest weight vector u_λ in $B(\lambda)$.*

5. Characterization of monomials in $C(\mathfrak{m})$

We know that the description of the monomials in the connected component $C(\mathfrak{m})$ containing a maximal vector $\mathfrak{m}$ depends on the choice of the set $C = (c_{ij})_{i \neq j}$ given in Section 4. In [19], the first author gave the characterization of the connected component $C(\mathfrak{m})$ when the set $C = (c_{ij})_{i \neq j}$ is given below.

$$c_{ij} = \begin{cases} 0 & \text{if } i > j, \text{ or } (i,j) = (0,n), \\ 1 & \text{if } i < j, \text{ or } (i,j) = (n,0). \end{cases}$$

Also, she gave the crystal isomorphism between the set $\mathcal{Y}(\lambda)$ of all reduced Young walls on the ground state wall Y_λ and the set $C(\mathfrak{m})$. However, she could not give information for the relation between $C(\mathfrak{m})$ and the image of crystal embedding $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbb{Z}_{\geqslant 0,\iota}^\infty \otimes R_\lambda$.

In this section, we select another set $C = (c_{ij})_{i \neq j}$ given by

$$c_{ij} = 0 \quad \text{if } i > j, \quad \text{and} \quad 1 \quad \text{if } i < j,$$

and we characterize the connected component $C(\mathfrak{m})$ using $A_i(k)$ $(i \in I,\ k \in \mathbb{Z})$. Moreover, in the next section, we give crystal isomorphisms not only between $\mathcal{Y}(\lambda)$ and $C(\mathfrak{m})$, but also $\operatorname{Im} \Psi_\iota^\lambda$ and $C(\mathfrak{m})$.

5.1. Characterization of monomial realization of $B(\lambda)$

For each $i = 0, 1, \ldots, n$, and $k \in \mathbb{Z}_{\geqslant 0}$, we denote by S_i the infinite sequences of the integers as follows:

$$\begin{aligned}
S_i = \big(s_i(k)\colon k \geqslant 0\big) = (&1, 2, \ldots, n-i, n-i+2, n-i+3, \ldots, 2n-i+1, \\
&2n-i+3, 2n-i+4, \ldots, 3n-i+2, \\
&3n-i+4, 3n-i+5, \ldots) \quad (i \neq n), \\
S_n = \big(s_n(k)\colon k \geqslant 0\big) = (&2, 3, \ldots, n+1, n+3, n+4, \ldots, 2n+2, \\
&2n+4, 2n+5, \ldots, 3n+3, \\
&3n+5, 3n+6, \ldots).
\end{aligned}$$

For each $p = 0, 1, \ldots, n$, we set

$$s_{i,p}(k) = \begin{cases} s_{i-p}(k-1) & \text{if } 0 \leqslant i \leqslant p-1, \\ s_{i-p}(k) & \text{if } p \leqslant i \leqslant n, \end{cases}$$

where $s_i(-1) = 0$ for all $i \in I$. Now, we are in a position to state our main theorem.

Theorem 5.1. *For each $p = 0, 1, \ldots, n$, the connected component $C(Y_p(0))$ which is isomorphic to $B(\Lambda_p)$ for $U_q(A_n^{(1)})$ is the set $\mathfrak{M}(\Lambda_p)$ of the monomials of the following form*

$$M = Y_p(0) \cdot \prod_{i \in I, k \geqslant 0} A_i(k)^{-a_i(k)}$$

satisfying the following conditions:

(i) *For each $k \geqslant 0$, $a_i(k) \leqslant s_{i,p}(k)$.*
(ii) *For (i,k) with $n \mid (i+k-p)$, if $1 \leqslant a_i(k) < s_{i,p}(k)$, then*

$$a_i(k) \leqslant \begin{cases} \min\{a_0(k), a_{n-1}(k)+1\} & \text{for } i = n, \\ \min\{a_1(k-1), a_n(k-1)+1\} & \text{for } i = 0, \\ \min\{a_{i+1}(k-1), a_{i-1}(k)+1\} & \text{otherwise,} \end{cases}$$

and if $a_i(k) = s_{i,p}(k)$, then

$$a_i(k) = \begin{cases} a_0(k)+1 = a_{n-1}(k)+1 & \text{for } i = n, \\ a_1(k-1)+1 = a_n(k-1)+1 & \text{for } i = 0, \\ a_{i+1}(k-1)+1 = a_{i-1}(k)+1 & \text{otherwise.} \end{cases}$$

(iii) *For (i,k) with $n \nmid (i+k-p)$, if $1 \leqslant a_i(k) < s_{i,p}(k)$, then*

$$a_i(k) \leqslant \begin{cases} \min\{a_0(k), a_{n-1}(k)\} & \text{for } i = n, \\ \min\{a_1(k-1), a_n(k-1)\} & \text{for } i = 0, \\ \min\{a_{i+1}(k-1), a_{i-1}(k)\} & \text{otherwise,} \end{cases}$$

and if $a_i(k) = s_{i,p}(k)$, then

$$a_i(k) = \begin{cases} a_0(k)+1 = a_{n-1}(k) & \text{for } i = n, \\ a_1(k-1)+1 = a_n(k-1) & \text{for } i = 0, \\ a_{i+1}(k-1)+1 = a_{i-1}(k) & \text{otherwise.} \end{cases}$$

Let $\iota = (\ldots, 0, n, \ldots, 1, 0, n, \ldots, 1, 0)$, and let $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbb{Z}_{\geqslant 0,\iota}^\infty \otimes R_\lambda$ be the crystal embedding given in Section 3. Then there should be a crystal isomorphism between $\mathfrak{M}(\Lambda_p)$ and $\operatorname{Im}\Psi_\iota^{\Lambda_p}$. The following theorem gives an explicit crystal isomorphism from $\mathfrak{M}(\Lambda_p)$ to $\operatorname{Im}\Psi_\iota^{\Lambda_p}$.

Theorem 5.2. *For each $p = 0, 1, \ldots, n$, the function $\phi : \mathfrak{M}(\Lambda_p) \to \operatorname{Im}\Psi_\iota^{\Lambda_p}$ defined by*

$$\begin{aligned}
&\phi\Biggl(Y_p(0) \cdot \prod_{i \in I, k \geqslant 0} A_i(k)^{-a_i(k)}\Biggr) \\
&\quad = \bigl(\ldots, a_0(2), a_n(1), \ldots, a_1(1), a_0(1), a_n(0), \ldots, a_1(0), a_0(0)\bigr) \otimes r_{\Lambda_p}
\end{aligned}$$

is a $U_q(A_n^{(1)})$-crystal isomorphism.

Proof. By the similar arguments given in the proof of Theorem 5.1 in [21], we can easily show that ϕ is a crystal isomorphism. $\square$

5.2. The proof of Theorem 5.1

First, we show that $\mathfrak{M}(\Lambda_p)$ is closed under the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$. Let $M = Y_p(0) \prod_{i \in I, k \in \mathbb{Z}_{\geqslant 0}} A_i(k)^{-a_i(k)} \in \mathfrak{M}(\Lambda_p)$. For the condition (i), suppose that $a_i(k) = s_{i,p}(k) = s_{i-p}(k)$ for some $k \in \mathbb{Z}_{\geqslant 0}$, and

$$\begin{aligned}
\tilde{f}_i M &= M \cdot A_i(k)^{-1} \\
&= M_1 \cdot \bigl(A_{i-1}(k)^{-a_{i-1}(k)} A_i(k-1)^{-a_i(k-1)} A_i(k)^{-a_i(k)} A_{i+1}(k-1)^{-a_{i+1}(k-1)}\bigr) \cdot A_i(k)^{-1}.
\end{aligned}$$

Then in M,

$$y_i(k) = a_{i-1}(k) - a_i(k-1) - a_i(k) + a_{i+1}(k-1) > 0.$$

Since almost the same arguments are available for all $i \in I$, we only treat the case when $p+1 \leqslant i < n$, and $n \mid (i+k-p)$. Now, by the condition (ii), $a_{i-1}(k) = a_{i+1}(k-1) = a_i(k) - 1$, and so

$$a_i(k) > a_i(k-1) + 2. \tag{5.1}$$

Also, since $s_{i-p}(k) = a_i(k) = a_{i-1}(k) + 1$, by the definitions of the sequences $(s_i(k) \colon k \geqslant 0)$ $(i \in I)$, we have $a_{i-1}(k) = s_{i-p-1}(k)$. Moreover, by the condition (iii) and the definitions of the sequences $(s_i(k) \colon k \geqslant 0)$ $(i \in I)$, we have $a_i(k-1) = s_{i-p}(k-1)$. Thus, (5.1) is rewritten as

$$s_{i-p}(k) > s_{i-p}(k-1) + 2,$$

which contradicts the definitions of the sequences $(s_i(k) \colon k \geqslant 0)$ $(i \in I)$. Similarly, we can prove that $\tilde{f}_i M$ satisfies the condition (i) for the remaining case when $a_i(k) = s_{i,p}(k) = s_{i-p}(k-1)$.

For the condition (ii), suppose that $a_i(k) = \min\{a_{i+1}(k-1), a_{i-1}(k) + 1\}$ in M and $\tilde{f}_i M = M \cdot A_i(k)^{-1}$. Since the proofs are similar, we only consider the case when $i = 1, \ldots, n-1$. By the definition of the Kashiwara operator $\tilde{f}_i$, in M,

$$y_i(k) = a_{i-1}(k) - a_i(k-1) - a_i(k) + a_{i+1}(k-1) > 0.$$

First, if $a_{i+1}(k-1) = a_{i-1}(k) + 1$, then $a_i(k) = a_{i+1}(k-1)$, and so

$$a_{i-1}(k) > a_i(k-1).$$

But, since $a_i(k) = a_{i-1}(k) + 1$ and $a_i(k) \neq s_{i,p}(k)$, we have $a_{i-1}(k) \neq s_{i-1,p}(k)$. Thus, the condition (iii) implies $a_{i-1}(k) \leqslant a_i(k-1)$, which is a contradiction. Second, if $a_{i-1}(k) + 1 < a_{i+1}(k-1)$, then $a_i(k) = a_{i-1}(k) + 1$, and so

$$a_{i+1}(k-1) > a_i(k-1) + 1.$$

But, since $n \mid ((i+1) + (k-1) - p)$, by the condition (ii), we have $a_{i+1}(k-1) \leqslant a_i(k-1) + 1$. It is a contradiction. Finally, if $a_{i-1}(k) + 1 > a_{i+1}(k-1)$, then $a_i(k) = a_{i+1}(k-1)$, and so

$$a_{i-1}(k) > a_i(k-1).$$

But, since $n \nmid ((i-1) + k - p)$, by the condition (iii), we have $a_{i-1}(k) \leqslant a_i(k-1) + 1$. Thus, $a_i(k-1) < a_{i-1}(k) \leqslant a_i(k-1) + 1$, and so

$$a_{i-1}(k) = a_i(k-1) + 1 \quad \text{and} \quad a_{i-1}(k) = s_{i-1,p}(k).$$

Since $a_{i-1}(k) \leqslant a_{i-2}(k)$, we have $a_{i-2}(k) = s_{i-2,p}(k)$, and by the similar argument, we have $a_{i+1}(k-1) = s_{i+1,p}(k-1)$. Therefore, by the definitions of the sequences $(s_i(k)\colon k \geqslant 0)$, in $\tilde{f}_i M$,

$$a_i(k) = s_{i,p}(k) \quad \text{and} \quad a_i(k) = a_{i+1}(k-1) + 1 = a_{i-1}(k) + 1.$$

That is, $\tilde{f}_i M$ satisfies the condition (ii). Similarly, we can show that $\tilde{f}_i M$ satisfies the condition (iii). Moreover, by the similar arguments, we can prove that $\mathfrak{M}(\Lambda_p)$ is closed under the Kashiwara operators $\tilde{e}_i$ for all $i \in I$.

Finally, suppose that $M \neq Y_p(0)$ and $\tilde{e}_i M = 0$ for all $i \in I$. Let (i_0, k_0) be the largest pair among the set $\{(i,k) \in I \times \mathbb{Z}_{\geqslant 0} \mid a_i(k) > 0\}$ under the ordering

$$(i_1, k_1) > (i_2, k_2) \quad \text{if}\, k_1 > k_2,\ \text{or}\, k_1 = k_2 \text{ and } i_1 > i_2.$$

Then since

$$A_{i_0}(k_0)^{-a_{i_0}(k_0)} = Y_{i_0}(k_0)^{-a_{i_0}(k_0)} Y_{i_0}(k_0+1)^{-a_{i_0}(k_0)} Y_{i_0+1}(k_0)^{-a_{i_0}(k_0)} Y_{i_0-1}(k_0+1)^{-a_{i_0}(k_0)},$$

we have $\varepsilon_{i_0}(M) > 0$, which is a contradiction.

6. Correspondence between $\mathcal{Y}(\lambda)$ and $\mathfrak{M}(\lambda)$

In previous section, we gave a new realization $\mathfrak{M}(\Lambda_i)$ $(i \in I)$, the connected component containing $Y_i(0)$, of the irreducible highest weight crystal $B(\Lambda_i)$ in terms of Nakajima monomials, which is different from the one given in [19]. In this section, we discuss the relation between the set $\mathcal{Y}(\Lambda_i)$ of all reduced Young walls on Y_{Λ_i} and $\mathfrak{M}(\Lambda_i)$.

6.1. *Characterization of* $\mathfrak{M}(\lambda)$*-I*

Let $Y \in \mathcal{Y}(\Lambda_i)$ be a reduced Young wall. Then we associate a monomial M_Y as follows:

(a) First, for the stacked blocks in Y, we associate the monomials following the rules given below.
  (i) The rightmost bottom block corresponds to $A_i(0)^{-1}$.
  (ii) Assume that a j-block ($j \in I$) corresponds to $A_j(k)^{-1}$. Then the above $(j+1)$-block corresponds to the monomial

$$A_{j+1}(k)^{-1} \quad \text{if } j \neq n, \qquad A_{j+1}(k+1)^{-1} \quad \text{if } j = n,$$

and the left $(j-1)$-block corresponds to the monomial

$$A_{j-1}(k+1)^{-1} \quad \text{if } j \neq 0, \qquad A_{j-1}(k)^{-1} \quad \text{if } j = 0.$$

(b) Finally, the monomial M_Y is defined by the product of $Y_i(0)$ and the monomials corresponding to the blocks stacked in Y.

Remark 6.1. The following pictures (a) and (b) represent the pattern of the blocks in Young walls in $\mathcal{Y}(\Lambda_2)$ and the pattern of indices k in $A_i(k)^{-1}$ corresponding to the blocks in Young walls for $U_q(A_5^{(1)})$, respectively.

(a)

⋮	⋮	⋮	⋮	⋮
2	3	4	5	0
1	2	3	4	5
0	1	2	3	4
5	0	1	2	3
4	5	0	1	2
3	4	5	0	1
2	3	4	5	0
1	2	3	4	5
0	1	2	3	4
5	0	1	2	3
4	5	0	1	2

(b)

⋮	⋮	⋮	⋮	⋮
5	4	3	2	2
5	4	3	2	1
5	4	3	2	1
4	4	3	2	1
4	3	3	2	1
4	3	2	2	1
4	3	2	1	1
4	3	2	1	0
4	3	2	1	0
3	3	2	1	0
3	2	2	1	0

Example 6.2. Let

$$Y = \begin{array}{|c|c|c|} \hline & & 0 \\ \hline & 2 & 3 \\ \hline & 1 & 2 \\ \hline & 0 & 1 \\ \hline 2 & 3 & 0 \\ \hline \end{array} \in \mathcal{Y}(\Lambda_0)$$

for $U_q(A_3^{(1)})$. Then the first, second, third columns (from right to left) in Y correspond to the monomials

$$A_0(0)^{-1}A_1(0)^{-1}A_2(0)^{-1}A_3(0)^{-1}A_0(1)^{-1},$$
$$A_3(0)^{-1}A_0(1)^{-1}A_1(1)^{-1}A_2(1)^{-1},$$
$$A_2(1)^{-1},$$

respectively, and so

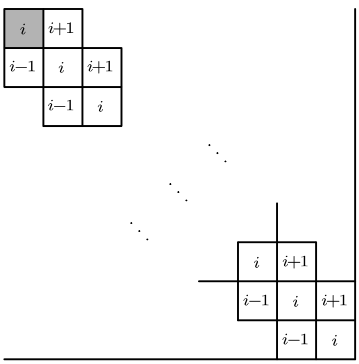
$$M_Y = Y_0(0)\cdot\big(A_0(0)^{-1}A_1(0)^{-1}A_2(0)^{-1}A_3(0)^{-1}A_0(1)^{-1}\big)$$
$$\cdot\big(A_3(0)^{-1}A_0(1)^{-1}A_1(1)^{-1}A_2(1)^{-1}\big)\cdot A_2(1)^{-1}.$$

Theorem 6.3. *For each $p = 0, 1, \ldots, n$, the map $\psi : \mathcal{Y}(\Lambda_p) \to \mathfrak{M}(\Lambda_p)$ defined by $\psi(Y) = M_Y$ is a $U_q(A_n^{(1)})$-crystal isomorphism sending Y_{Λ_p} to $Y_p(0)$.*

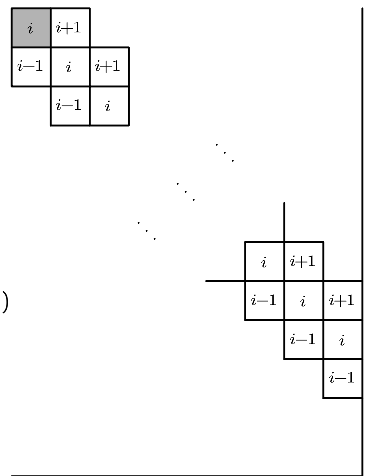
Proof. By the definition of ψ, clearly $\psi(Y_{\Lambda_p}) = Y_p(0)$. Let Y be a reduced Young wall in $\mathcal{Y}(\Lambda_p)$. Then from the coloring of the blocks in Y and the rules for building the walls, it is not difficult to see that $\psi(Y)$ satisfies the conditions (i)–(iii) in Theorem 5.1. Also, since Y is reduced, it is easy to see that ψ is a 1–1 correspondence.

Now, consider the subparts of Y consisting of only j-blocks ($j = i-1, i, i+1$). Then they are divided into three cases with respect to the existence of removable i-blocks and admissible i-slots. First, consider the case when the removable i-block exists as follows.

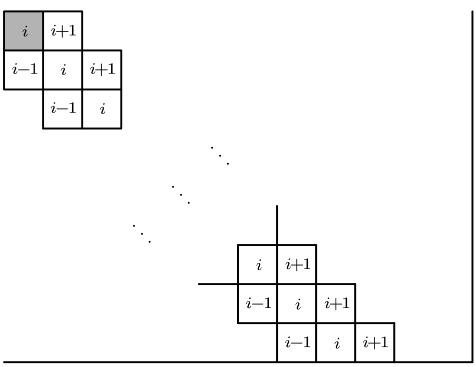
(i)



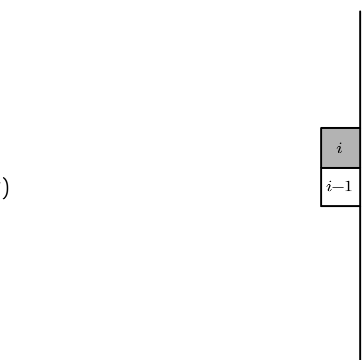
(ii)



(iii)



(iv)



Here, the shaded blocks are the removable i-blocks. Then in any case, if the leftmost removable i-block corresponds to $A_i(m)^{-1}$, we have

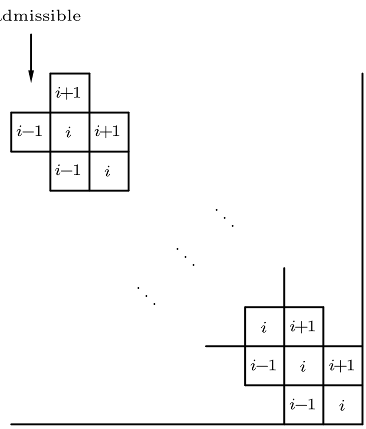
$$y_i(k) = 0 \quad \text{if } k \neq m+1, \qquad -1 \quad \text{if } k = m+1. \tag{6.1}$$

Indeed, the subwall in case (i) corresponds to the monomial

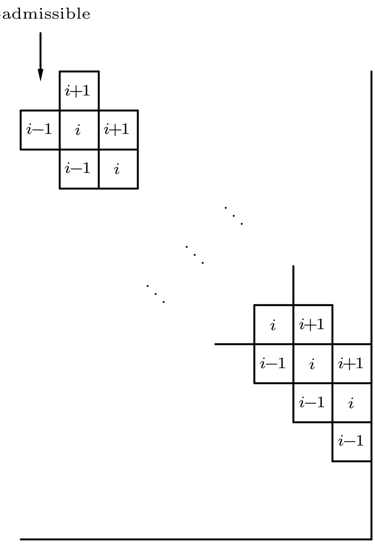
$$A_i(m)^{-1}A_{i-1}(m)^{-1}\left(A_{i+1}(m-1)^{-1}A_i(m-1)^{-1}A_{i-1}(m-1)^{-1}\right)$$
$$\cdots\left(A_{i+1}(1)^{-1}A_i(1)^{-1}A_{i-1}(1)^{-1}\right)A_{i+1}(0)^{-1}A_i(0)^{-1}Y_i(0),$$

and so by simple calculation, we have (6.1). Second, consider the case when the admissible i-slot exists as follows.

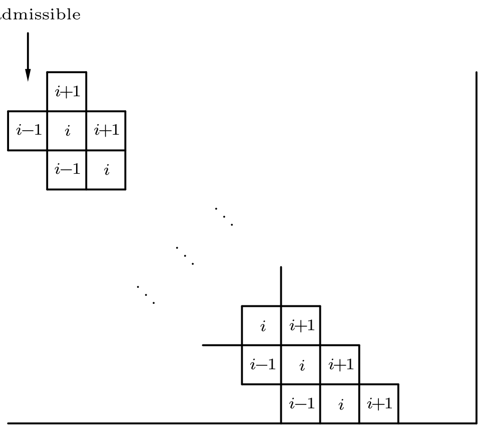
(i)



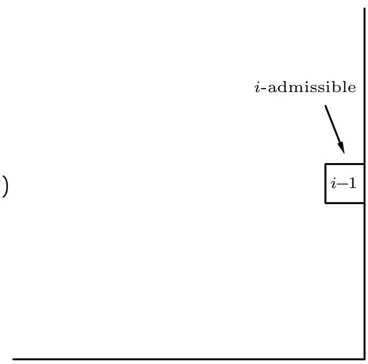
(ii)



(iii)



(iv)



Then in any case, if the leftmost admissible i-slot corresponds to $A_i(m)^{-1}$, we have

$$y_i(k) = 0 \quad \text{if } k \neq m, \qquad 1 \quad \text{if } k = m.$$

By the similar arguments, it is easy to see that if there is neither admissible i-slot nor removable i-block, $y_i(k) = 0$ for all $k \geqslant 0$. Therefore, by the definition of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ given in the sets $\mathcal{Y}(\Lambda_p)$ and $\mathfrak{M}(\Lambda_p)$, ψ is a crystal morphism for $U_q(A_n^{(1)})$. $\square$

By Theorem 6.3 and Theorem 5.2, we have

Corollary 6.4. *For each $p = 0, 1, \ldots, n$, the function $\psi \circ \phi : \mathcal{Y}(\Lambda_p) \to \operatorname{Im} \Psi_\iota^{\Lambda_p}$ is a $U_q(A_n^{(1)})$-crystal isomorphism sending Y_{Λ_p} to $(\ldots, 0, 0, 0) \otimes r_{\Lambda_p}$.*

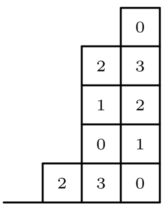
Example 6.5. Let Y be the Young wall for $U_q(A_3^{(1)})$ given in Example 6.2, and let $\iota = (\ldots, 0, 3, 2, 1, 0, 3, 2, 1, 0)$. Then

$$\begin{aligned}\psi(Y) = M_Y &= Y_0(0) \cdot \big(A_0(0)^{-1} A_1(0)^{-1} A_2(0)^{-1} A_3(0)^{-1} A_0(1)^{-1}\big)\\ &\quad \cdot \big(A_3(0)^{-1} A_0(1)^{-1} A_1(1)^{-1} A_2(1)^{-1}\big) \cdot A_2(1)^{-1}\\ &= Y_0(0) \cdot A_0(0)^{-1} A_1(0)^{-1} A_2(0)^{-1} A_3(0)^{-2} A_0(1)^{-2} A_1(1)^{-1} A_2(1)^{-2}.\end{aligned}$$

Moreover, we have

$$\phi\big(\psi(Y)\big) = (\ldots, 0, 0, 2, 1, 2, 2, 1, 1, 1) \otimes r_{\Lambda_0} \in \operatorname{Im} \Psi_\iota^{\Lambda_0}.$$

That is,



$$\begin{aligned}&\xleftrightarrow{\psi} Y_0(0) \cdot A_0(0)^{-1} A_1(0)^{-1} A_2(0)^{-1} A_3(0)^{-2} A_0(1)^{-2} A_1(1)^{-1} A_2(1)^{-2}\\ &\xleftrightarrow{\phi} (\ldots, 0, 0, 2, 1, 2, 2, 1, 1, 1) \otimes r_{\Lambda_0}.\end{aligned}$$

6.2. Characterization of $\mathfrak{M}(\lambda)$-II

For $i \in \{0, 1, \ldots, n\}$ and $m \in \mathbb{Z}$, we introduce new variables as follows.

$$X_i(m) = \begin{cases} Y_n(m)^{-1} Y_0(m) & \text{for } i = 0,\\ Y_{i-1}(m+1)^{-1} Y_i(m) & \text{otherwise.}\end{cases}$$

Then we have

$$\begin{aligned} X_0(m) A_0(m)^{-1} &= X_1(m),\\ X_1(m) A_1(m)^{-1} &= X_2(m),\\ &\vdots\\ X_{n-1}(m) A_{n-1}(m)^{-1} &= X_n(m),\end{aligned}$$

$$X_n(m)A_n(m)^{-1} = X_0(m+1),$$
$$X_0(m+1)A_0(m+1)^{-1} = X_1(m+1),$$
$$\vdots$$

Thus, the bijection between $\mathcal{Y}(\Lambda_p)$ and $\mathfrak{M}(\Lambda_p)$ given in Section 6.1 can be rewritten as follows. Let $Y \in \mathcal{Y}(\lambda)$ be a reduced Young wall. Then the associated monomial M_Y is rewritten as follows:

(a) First, to each column y_k, we associate a monomial as follows.
  (i) If $Y = Y_{\Lambda_i}$, the columns y_k $(k \geqslant 0)$ correspond to the monomials

$$X_i(0), X_{i-1}(1), \ldots, X_0(i), X_n(i), X_{n-1}(i+1), \ldots,$$
$$X_0(i+n), X_n(i+n), X_{n-1}(i+n+1), \ldots,$$

  respectively.
  (ii) Assume that $Y \neq Y_{\Lambda_i}$, the top block of y_k corresponds to $A_j(p)^{-1}$ by the rule given in Section 6.1. Then the kth column y_k corresponds to the monomial

$$X_{j+1}(p) \quad \text{if } j \neq n, \qquad X_{j+1}(p+1) \quad \text{if } j = n.$$

(b) Finally, the monomial M_Y is determined by the product of the monomials corresponding to the columns y_k in Y.

By above 1–1 correspondence, we have another characterization of the connected component $C(Y_p(0))$.

Theorem 6.6. *The connected component $C(Y_p(0)) = \mathfrak{M}(\Lambda_p)$ containing $Y_p(0)$ is the set of the monomials of the following form*

$$M = Y_{i_r}(k_r) \cdot \prod_{j=0}^{r-1} X_{i_j}(k_j) \quad (r \geqslant 1,\ k_j \geqslant 0)$$

satisfying the following conditions:

(i) $r + s \equiv p \pmod{n+1}$.
(ii) *For each* $j = 0, 1, \ldots, r-1$, $0 \leqslant k_{j+1} - k_j \leqslant 1$.
(iii) *For each* $j = 0, 1, \ldots, r-1$, *if* $k_{j+1} = k_j + 1$, *then* $i_{j+1} < i_j$, *and if* $k_{j+1} = k_j$, *then* $i_{j+1} \geqslant i_j$.

Finally, from Theorem 6.6, we have the following explicit description of $\mathfrak{M}(\Lambda_p)$ in terms of $Y_i(k)$'s $(i \in I,\ k \geqslant 0)$. Indeed, the Nakajima monomials in the commuting variables $Y_i(k)$ $(i \in I,\ k \in \mathbb{Z})$ were introduced by Nakajima while studying the structure of quiver varieties [25,26]. Because of the relation of quiver varieties, the characterization of $\mathfrak{M}(\Lambda_p)$ in terms of $Y_i(k)$ is important.

Theorem 6.7. *The connected component $C(Y_p(0)) = \mathfrak{M}(\Lambda_p)$ containing $Y_p(0)$ is the set of the monomials of the following form*

$$M = Y_{c_{k+1}}(d_{k+1}) \cdot \prod_{j=0}^{k} Y_{a_j}(b_j)^{-1} Y_{c_j}(d_j) \quad (k \geqslant 0,\ b_j \geqslant d_j)$$

satisfying the following conditions:

(i) $a_j + b_j \equiv c_j + d_j \pmod{n}$.
(ii) *For each* $j = 0, 1, \ldots, k$,

$$0 \leqslant b_j - d_{j+1} \leqslant 1.$$

Moreover, if $b_j = d_{j+1}$, *then* $c_{j+1} < a_j$, *and if* $b_j = d_{j+1} + 1$, *then* $c_{j+1} > a_j$.
(iii) $c_{k+1} + \sum_{j=0}^{k}(c_j - a_j) \equiv p \pmod{n+1}$.

Proof. Even though it is derived from Theorem 6.6, we will prove directly. For a monomial $M = Y_{c_{k+1}}(d_{k+1}) \cdot \prod_{j=0}^{k} Y_{a_j}(b_j)^{-1} Y_{c_j}(d_j) \in \mathfrak{M}(\Lambda_p)$, suppose that $\tilde{f}_i M = M \cdot A_i(m)^{-1}$. By the definition of the Kashiwara operator $\tilde{f}_i$, $i = c_j$ and $m = d_j$ for some j, and so $M = M' \cdot Y_{a_j}(b_j)^{-1} Y_{c_j}(d_j)$ and

$$\begin{aligned}\tilde{f}_i M &= M' \cdot Y_{a_j}(b_j)^{-1} Y_{c_j}(d_j) \cdot A_{c_j}(d_j)^{-1} \\ &= M' \cdot Y_{a_j}(b_j)^{-1} Y_{c_j - 1}(d_j + 1) Y_{c_j}(d_j + 1)^{-1} Y_{c_j + 1}(d_j).\end{aligned}$$

Thus, it is clear that $\tilde{f}_i M$ satisfies the conditions (i)–(iii). Similarly, we can show that $\tilde{e}_i M$ satisfies the conditions (i)–(iii). Moreover, by the definition of Kashiwara operator $\tilde{e}_i$, it is easy to see that $\tilde{e}_i M = 0$ for all $i \in I$ implies $M = Y_p(0)$. □

Example 6.8. Let Y be a reduced Young wall given in Example 6.2. Then we have

$$\begin{aligned}M_Y &= \bigl(\cdots X_1(5) X_2(4) X_3(3) X_0(3) X_1(2)\bigr) \cdot X_3(1) X_3(1) X_1(1) \\ &= Y_1(2) X_3(1)^2 X_1(1) \\ &= Y_1(2) Y_2(2)^{-2} Y_3(1)^2 Y_0(2)^{-1} Y_1(1).\end{aligned}$$

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