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An efficient variable stepsize rational method for stiff, singular and singularly perturbed problems

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65L70

Abstract In this article, a new iterative method of the rational type having fifth-order of accuracy is proposed to solve initial value problems. The method is self-starting, stable, consistent, and convergent, whereas local truncation error analysis has also been discussed. Furthermore, the method has been analyzed with a variable stepsize approach that increases performance while taking fewer steps with acceptable local errors. The method is also tested against some existing fifth-order methods having rational structure. The proposed one outperforms concerning maximum absolute error, final absolute error, average error, and norm, while CPU time computed in seconds is comparable. Furthermore, stiff, singular, and singularly perturbed problems for single and system of differential equations chosen for simulations yielded minor errors when solved with the new rational method.

1. Introduction

Let us assume the following initial value problem (IVP):

$$\frac{dy(t)}{dt} = f(t, y(t)),\; y(t_0) = y_0, \tag{1}$$

where $t \in [t_0, t_n]$ and $y, f \in \mathbb{R}$. It is also assumed that the problem (1) has a continuously differentiable solution $y(t) (\in C^1$ class); that is, it is a well-posed IVP. Moreover, the analytical solution is denoted by $y(t_n)$ whereas the approximate one is y_n at the mesh points $t_n = t_0 + nh; n = 0, 1, 2, \ldots N,$ where

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$h = \frac{t_n - t_0}{N}$ is known as the step-length or stepsize which may either be a constant or variable along the integration interval $[t_0, t_n]$. Ordinary differential equations are substantially used in many fields of Science and Engineering, including Epidemiology, Mathematical Biology, Cell Biology, Physics, Chemistry, Nuclear Energy, Fluid Dynamics, Petroleum and Natural Gas, Econometrics, Electronics, Mechatronics, and many more as can be found in [1–6]. It is almost impossible to obtain analytical solutions of a differential equation model in hand in many circumstances. It appears especially when the model is non-linear, stiff in nature, or possesses blow-up solutions with singularities and has singularly perturbed solutions. In these situations, we resort to the discrete type of solutions rather than looking for continuous ones [7–10].

There are numerous iterative methods to compute approximate solutions of the problem of the type (1). Among well-known methods are the methods called classical single-step Runge Kutta [11], multi-step in explicit and implicit forms [12], rational (nonlinear) methods [13–19], and block methods [20–25]. Some of these methods have lower order of convergence, are computationally expensive, difficult to code, or are unsuitable for a stiff and singular type of IVPs. From the available literature, it is found that there are not many versions of rational methods having fifth-order of accuracy. Therefore, this article contributes a fifth-order iterative method of rational type, which is observed to be useful to deal with IVPs, including singular and singularly perturbed solutions. The motivation of the present research work comes from a recently published paper [26] wherein authors have proposed rational methods from second to the fourth-order of accuracy. We will consider both formulations, namely constant and variable stepsize, for the new rational iterative method suitable for handling stiff, singular, and singularly perturbed IVPs.

2. Formulation and derivation

Traditional numerical methods derived from the local representation of a polynomial via Taylor series fail at/or near the singularities in the analytical solution of an IVP defined along with an integration interval $[t_0, t_n]$ containing some singular point(s) within the interval. In such situations, various researchers have worked on the development of rational methods which originate from a rational approximation for the analytical solution to an IVP. Such methods smoothly cross singularities without being failed thereat. Based on the structure proposed in [26–31], we present the following rational type of approximation y_{n+1} for the analytical solution of the form $y(t_{n+1})$:

$$y_{n+1} = G + hHy'_n\left[1 + \sum_{i=1}^{j}\phi_i h^i\right]^{-1}, \quad \sum_{i=1}^{j}\phi_i h^i \neq -1, \tag{2}$$

where G, H and $\phi_i\,(i = 1, 2, 3, \cdots, j)$ are unknown parameters to be determined. For the formation introduced above, the linear difference operator $\mathscr{L}$ can be associated with it as follows:

$$\mathscr{L}[y(t); h]_{\mathrm{RMSSP}_{j+1}} = [y(t_n + h) - G] \times \left[1 + \sum_{i=1}^{j}\phi_i h^i\right] - hHy'_n, \tag{3}$$

where $(j+1)$ denotes order of the method. Expanding $y(t_n + h)$ via Taylor's expansion around t_n and collecting the terms in powers of h, we obtain the following form of the above linear difference operator:

$$\mathscr{L}[y(t); h]_{\mathrm{RMSSP}_{m+1}} = A_0 + A_1 h + A_2 h^2 + \cdots + A_m h^m + A_{m+1}h^{m+1} + \cdots, \tag{4}$$

where it must be noted that the constants $A_k\,(k = 0, 1, 2, \cdots, m, m+1)$ would be computed in such a way that $A_0 = A_1 = A_2 = \ldots A_m = A_{m+1} = 0$, $A_{m+2} \neq 0$ for the method to be of order $r = m + 1$. To derive the new fifth order rational method, we put the value of $j = 4$ in (2) to obtain the following form

$$y_{n+1} = G + hHy'_n\left(1 + \phi_1 h + \phi_2 h^2 + \phi_3 h^3 + \phi_4 h^4\right)^{-1}, \tag{5}$$

and the associated linear difference operator of (5) is given as follows:

$$\mathscr{L}[y(t); h]_{\mathrm{RMSSP}_5} = (y(t_n + h) - G) \times (1 + \phi_1 h + \phi_2 h^2 + \phi_3 h^3 + \phi_4 h^4) - hHy'(t_n). \tag{6}$$

Expanding $y(t_n + h)$ via Taylor's expansion around t_n and collecting the powers of h, we obtain the following

$$\begin{aligned}\mathscr{L}[y(t); h]_{\mathrm{RMSSP}_5} = \; & (-G + y(t_n)) + (-Hy'(t_n) - G\phi_1 + y(t_n)\phi_1 + y'(t_n))h, \\ & + \left(\tfrac{1}{2}y''(t_n) + y(t_n)\phi_2 + y'(t_n)\phi_1 - G\phi_2\right)h^2 \\ & + \left(\tfrac{1}{6}y'''(t_n) + y(t_n)\phi_3 + y'(t_n)\phi_2 + \tfrac{1}{2}y''(t_n)\phi_1 - G\phi_3\right)h^3 \\ & + \left(\tfrac{1}{24}y^{(iv)}(t_n) + y(t_n)\phi_4 + y'(t_n)\phi_3 + \tfrac{1}{2}y''(t_n)\phi_2 + \tfrac{1}{6}y'''(t_n)\phi_1 - G\phi_4\right)h^4 \\ & + \left(\tfrac{1}{120}y^{(v)}(t_n) + y'(t_n)\phi_4 + \tfrac{1}{2}y''(t_n)\phi_3 + \tfrac{1}{6}y'''(t_n)\phi_2 + \tfrac{1}{24}y^{(iv)}(t_n)\phi_1\right)h^5 \\ & + \left(\tfrac{1}{720}y^{(vi)}(t_n) + \tfrac{1}{2}y''(t_n)\phi_4 + \tfrac{1}{6}y'''(t_n)\phi_3 + \tfrac{1}{24}y^{(iv)}(t_n)\phi_2 + \tfrac{1}{120}y^{(v)}(t_n)\phi_1\right)h^6 \\ & + O(h^7).\end{aligned} \tag{7}$$

Comparing this with (4), we have the following identities:

$$\begin{aligned}
A_0 &= -G + y(t_n),\\
A_1 &= -Hy'(t_n) - G\phi_1 + y(t_n)\phi_1 + y'(t_n),\\
A_2 &= \tfrac{1}{2}y''(t_n) + y(t_n)\phi_2 + y'(t_n)\phi_1 - G\phi_2,\\
A_3 &= \tfrac{1}{6}y'''(t_n) + y(t_n)\phi_3 + y'(t_n)\phi 2 + \tfrac{1}{2}y''(t_n)\phi_1 - G\phi_3,\\
A_4 &= \tfrac{1}{24}y^{(iv)}(t_n) + y(t_n)\phi_4 + y'(t_n)\phi_3 + \tfrac{1}{2}y''(t_n)\phi_2 + \tfrac{1}{6}y'''(t_n)\phi_1 - G\phi_4,\\
A_5 &= \tfrac{1}{120}y^{(v)}(t_n) + y'(t_n)\phi_4 + \tfrac{1}{2}y''(t_n)\phi_3 + \tfrac{1}{6}y'''(t_n)\phi_2 + \tfrac{1}{24}y^{(iv)}(t_n)\phi_1,\\
A_6 &= \tfrac{1}{720}y^{(vi)}(t_n) + \tfrac{1}{2}y''(t_n)\phi_4 + \tfrac{1}{6}y'''(t_n)\phi_3 + \tfrac{1}{24}y^{(iv)}(t_n)\phi_2 + \tfrac{1}{120}y^{(v)}(t_n)\phi_1.
\end{aligned} \tag{8}$$

Based upon the above discussion, we take $A_0 = A_1 = A_2 = A_3 = A_4 = A_5 = 0$ and $A_6 \neq 0$. After solving nonlinear system containing six equations in six unknown parameters, we determine the unknown parameters of the method (5) given as follows:

$$\begin{aligned}
H &= 1,\\
G &= y(t_n),\\
\phi_1 &= -\tfrac{y''(t_n)}{2y'(t_n)},\\
\phi_2 &= \tfrac{3y''(t_n)^2 - 2y'''(t_n)y'(t_n)}{12y'(t_n)^2},\\
\phi_3 &= -\tfrac{3y''(t_n)^3 - 4y'''(t_n)y''(t_n)y' + y^{(iv)}(t_n)(y'(t_n))^2}{24y'(t_n)^3},\\
\phi_4 &= \tfrac{1}{720y'(t_n)^4}\\
&\begin{pmatrix} 45y''(t_n)^4 - 90y''(t_n)^2y'''(t_n)y'(t_n) + 30y^{(iv)}(t_n)y''(t_n)y'(t_n)^2 \\ +20y'''(t_n)^2y'(t_n)^2 - 6y^{(v)}(t_n)y(t_n)^3 \end{pmatrix}.
\end{aligned} \tag{9}$$

Substituting above values into A_6, we obtain the following:

$$\begin{aligned}
A_6 &= \tfrac{1}{1440y'(t_n)^4}\\
&\begin{pmatrix} 2y^{(vi)}(t_n)y'(t_n)^4 + 45y''(t_n)^5 - 120y''(t_n)^3y'(t_n)y'''(t_n) \\ +45y''(t_n)^2y'(t_n)^2y^{(iv)}(t_n) - 12y^{(v)}(t_n)y''(t_n)y'(t_n)^3 \\ +60y''(t_n)y'(t_n)^2y'''(t_n)^2 - 20y'(t_n)^3y'''(t_n)y^{(iv)}(t_n) \end{pmatrix}.
\end{aligned} \tag{10}$$

Under the local assumption, we have $y^{(r)}(t_n) = y_n^{(r)}$, $(r = 0, 1, 2, 3, 4, 5)$ and the new proposed fifth-order rational method suitable for singular and singularly perturbed problems (RMSSP_5) takes the following final structure along with its pseudo-code given in the Algorithm 1 below:

$$^{\text{RMSSP}_5}y_{n+1} = y_n + \frac{720y_n'^5h}{\begin{bmatrix} 720y_n'^4 + \left(-6y_n^{(v)}h^4 - 30h^3y_n^{(iv)} - 120h^2y_n''' - 360y_n''h\right)y_n'^3 + \left(180y_n''^2h^2\right. \\ \left.+\left(30y_n^{(iv)}h^4 + 120y_n'''h^3\right)y_n'' + 20y_n'''^2h^4\right)y_n'^2 - 90h^3y_n''^2\left(hy_n''' + y_n''\right)y_n' + 45y_n''^4h^4 \end{bmatrix}}. \tag{11}$$

Algorithm 1. Pseudo code for the proposed fifth-order rational method RMSSP_5 given in (11) with the fixed stepsize where the

Input:

Define functions $f_1(t, y), f_2(t, y), f_3(t, y), f_4(t, y), f_5(t, y)$.

Initialization

Read values of initial condition (t_0 and y_0), stepsize (h), calculation point (t_n), and calculate number of steps $(n) = \mathtt{ceil}((t_n - t_0)/h)$.

Forward Inclusion

for $i = 1, \ldots, n$ **do**

Use the proposed nonlinear fifth-order method as given in (11)

Update $t_n^{(i)} \leftarrow t_n^{(i-1)}$

Output:

t_n, y_n

symbols $f_1(t,y), f_2(t,y), f_3(t,y)$, $f_4(t,y)$, and $f_5(t,y)$ stand for the first-, second-, third-, fourth-, and fifth-order derivatives of $y(t)$.

3. Theoretical analysis

3.1. Local truncation analysis

To obtain local truncation error of the method RMSSP_5, we follow the usual procedure by considering the functional $\mathscr{L}[w(t),h]$ associated to the method in (11) as $\mathscr{L}[w(t),h] = w(t+h) - \text{RMSSP}_5(t)$, where RMSSP_5 is the proposed method given in (11) and $w(t)$ is an arbitrary function defined along the integration interval $[t_0, t_n]$ and is also differentiable as many times as required. Having expanded it via Taylor's series approach around t and collecting the terms in powers of h, we obtain the following local truncation error of RMSSP_5 which guarantees its fifth order accuracy, and is given as follows:

$$^{\text{RMSSP}_5}L_{n+1} = \frac{h^6}{1440 y_n'^4} \begin{pmatrix} 2y_n^{(vi)}y_n'^4 + 45y_n''^5 - 120y_n''^3 y_n' y_n''' + 45y_n''^2 y_n'^2 y_n^{(iv)} \\ -12y_n^{(v)} y_n'' y_n'^3 + 60y_n'' y_n'^2 y_n'''^2 - 20y_n'^3 y_n''' y_n^{(iv)} \end{pmatrix} + \mathcal{O}(h^7), \quad (12)$$

where $y_n', y_n'', y_n''', y_n^{(iv)}, y_n^{(v)}$, and $y_n^{(vi)}$ stand for the values of first, second, third, fourth, fifth, and sixth derivatives of $y(t)$ respectively at t_n, provided that $y_n' \neq 0$.

3.2. Stability analysis

In this section, we will discuss the stability function and associated regions (2D and 3D) of RMSSP_5 by applying the method to the sample Dahlquist's equation shown as $y'(t) = \omega y(t),\ Re(\omega) < 0$. The following difference equation is obtained:

$$y_{n+1} = y_n \left[\frac{h^4\omega^4 - 60h^2\omega^2 - 360h\omega - 720}{h^4\omega^4 - 60h^2\omega^2 + 360h\omega - 720} \right]. \quad (13)$$

Setting $z = h\omega$ into (13), the rational stability function of RMSSP_5 is obtained as follows:

$$R(z) = \left[\frac{z^4 - 60z^2 - 360z - 720}{z^4 - 60z^2 + 360z - 720} \right]. \quad (14)$$

Taking $z = x + iy$ into (14), we have plotted the absolute stability region and the associated 3D graphic surface of RMSSP_5 as shown in Figs. 1 and 2, respectively which satisfy the condition $|R(z)| \leqslant 1$. However, the plotted stability region of RMSSP_5 does not contain the whole left half complex plane in the Fig. 1 since it is not capable enough to cover a small parabolic region therein whereas that leftover parabolic region has been well covered on right half complex plane. Thus, it can be concluded that the method is not an $\mathscr{A}$-stable method. In addition, it is worth noting that the general structure (2) if considered with $j = 5$ and its higher values then methods obtained with sixth order of accuracy and higher will not be $\mathscr{A}$-stable either. The rational methods obtained for $j = 2, 3$, and 4 in (2) are the only methods that are $\mathscr{A}$-stable.

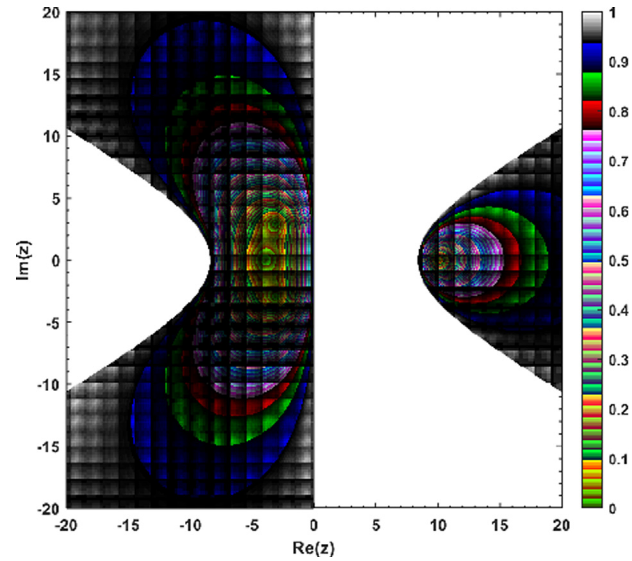

Fig. 1 Stability region of the proposed RMSSP_5 method.

3.3. Consistency and convergence

The method RMSSP_5 given in (11) does satisfy the condition of being consistent owing to having at least fifth order of accuracy as shown in the subSection 3.1. In other words, RMSSP_5 is consistent since the truncation error is

$$L_{n+1} = y(t_{n+1}) - \text{RMSSP}_5(t_n) = \frac{h^6}{1440}\Upsilon(\zeta_n), \quad (15)$$

where

$$\Upsilon(\zeta_n) = \frac{1}{y'(\zeta_n)^4} \begin{pmatrix} 2y^{(vi)}(\zeta_n)y'(\zeta_n)^4 + 45y''(\zeta_n)^5 - 120y''(\zeta_n)^3 y'(\zeta_n)y'''(\zeta_n) \\ +45y''(\zeta_n)^2 y'(\zeta_n)^2 y^{(iv)}(\zeta_n) - 12y^{(v)}(\zeta_n)y''(\zeta_n)y'(\zeta_n)^3 \\ +60y''(\zeta_n)y'(\zeta_n)^2 y'''(\zeta_n)^2 - 20y'(\zeta_n)^3 y'''(\zeta_n)y_n^{(iv)} \end{pmatrix}, \quad (16)$$

and ζ_n lies between t_n and t_{n+1}. For each n, the error is $\mathcal{O}(h^6)$ as $h \to 0$. From consistency perspective, n is fixed so $\Upsilon(\zeta_n)$ is merely a constant which demonstrates that there is no need to have a uniform bound on Υ that is true for all n. Moreover, the convergence of the method RMSSP_5 can easily be claimed since the method's stability in combination with its consistency is sufficient enough to guarantee the convergence as explained by Henrici Peter in [32].

3.4. Applicability to a system of differential equations

We assume a system of s ordinary differential equations in the vector form as follows:

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}),\ \mathbf{y}(t_0) = \mathbf{y}_0,\ t_0 \leqslant t \leqslant t_n,$$

where

$$\begin{aligned} &\mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_s(t))^T, \\ &\mathbf{f}(t, \mathbf{y}(t)) = (f_1(t, y_1(t), y_2(t), \cdots, y_s(t)), \cdots, f_s(t, y_1(t), y_2(t), \cdots, y_s(t))^T, \\ &\mathbf{y}_0 = (y_{1,0}, \cdots y_{s,0})^T. \end{aligned} \quad (17)$$

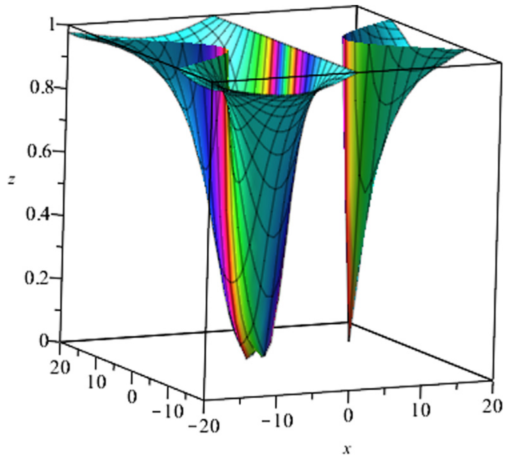

Fig. 2 Graphic surface of the proposed RMSSP_5 method.

The methodRMSSP$_5$ for scalar equation may be written as

$$y_{n+1} = y_n + h\mu_f(t_n, y_n; h),$$

where $\mu_f(t_n, y_n; h)$ is commonly known as the incremental function, and the subscript f on the right hand side shows that the dependence of the μ_f on its variables is through function f (and may be its derivatives). Implementing the proposed method in (11) to each of the scalar equations in the differential system, we have obtained the result as:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mu(t_n, \mathbf{y}_n; h),$$

where

$$\mu(t_n, \mathbf{y}_n; h) = \left(\mu_{f_1}(t_n, y_{1,n}, .., y_{s,n}; h), \cdots, \mu_{f_s}(t_n, y_{1,n}, \ldots, y_{s,n}; h)\right)^T.$$

Finally, this shows that the method RMSSP$_5$ applies to a system of ordinary differential equations.

4. Variable step size approach

The derivation and analysis (local truncation error, stability, consistency) of the proposed RMSSP$_5$ method was performed in the above sections via a constant stepsize approach. Under this approach, we come across a problem to figure out the step size h prior to starting the method. Such a step size will remain the same throughout the entire process. Nonetheless, an iterative method for solving IVPs should also be elegant enough when applied under the variable stepsize approach as recommended by some researchers [33]. When the solution of an IVP has rapid changes over a part of the integration interval and slower changes in remaining intervals, then using a method with constant step size is not effective. Therefore, we will use two approximations such as the fourth-order rational method given in [26] (shown below in (18)) and RMSSP$_5$. Later, the local truncation error will be compared between these two approximations.

$$\begin{aligned} y_{n+1} &= y(t_n) \\ &+ \frac{24y'(t_n)^4 h}{\left[\begin{array}{c} 24y(t_n)^3 - 12hy'(t_n)^2 y''(t_n) + 6h^2 y')t_n)y''(t_n)^2 - 3h^3 y''(t_n)^3 \\ -4h^2 y'(t_n)^2 y'''(t_n) + 4h^3 y'(t_n)y''(t_n)y'''(t_n) - h^3 y'(t_n)^2 y^{(iv)}(t_n) \end{array}\right]}. \end{aligned} \tag{18}$$

There are many researchers who have carried out this sort of approach as discussed in [34,35]. Since the higher order method makes use of values needed in the lower order method therefore, there will not be any additional computational cost. We have followed the process as employed in [36]. In general, surmise that the local error used with a method of order r to get y_{n+1} be identified by

$$M_n = y(t_n + h) - y_{n+1}, \tag{19}$$

where $y(t)$ stands for the analytical solution for the problem (1). At this stage, using a method of order $r+1$ to determine a result $y^\star_{n+1}$ on this step, one obtains the following

$$\begin{aligned} \mathscr{C}_e &= y^\star_{n+1} - y_{n+1} \\ &= \big(y(t_n + h) - y_{n+1}\big) - \big(y(t_n + h) - y^\star_{n+1}\big) \\ &= M_n + \mathcal{O}\big(h^{r+2}\big). \end{aligned} \tag{20}$$

It is said to be a computable estimate of the local error for the lower order numerical method because M_n is $\mathcal{O}(h^{r+1})$ and so dominates in (20) for sufficiently small values of h. It has to be noted that the error can be estimated in y_{n+1} via its comparison with a more accurate solution shown as $y^\star_{n+1}$. However, in embedded kinds of methods, one looks for a pair of numerical methods which share as many function evaluations as possible. For the implementation of such pairs, the lower-order method is merely used to estimate the local error. The higher-order method is used to advance the integration step. Such Advancement in the integration with the more accurate result y_{n+1} is known as the *localextrapolation*. A local error tolerance *tol* must be specified and, if the estimated error is too large relative to this tolerance, the step is rejected, and another attempt is made with a smaller step size. Now, we discuss the approach for changing the stepsize. From (19), we obtain

$$y(t_n + h) - y_{n+1} = h^{r+1}\Lambda(t_n) + \mathcal{O}(h^{r+2}). \tag{21}$$

Now, if we take a step from t_n with a new step size τh, then the following would be the error

$$\begin{aligned} (\tau h)^{r+1}\Lambda(t_n) + \mathcal{O}(h^{r+2}) &= (\tau)^{r+1}(h)^{r+1}\Lambda(t_n) + \mathcal{O}(h^{r+2}) \\ &= \tau^{r+1}\mathscr{C}_e + \mathcal{O}(h^{r+2}). \end{aligned} \tag{22}$$

The predicted largest stepsize passing the error test corresponds to selecting τ such that $|\tau^{r+1}\mathscr{C}_e| \approx tol$. The newly constructed stepsize now becomes as follows

$$h\left[\frac{tol}{|\mathscr{C}_e|}\right]^{1/(r+1)}. \tag{23}$$

On recommendation of various researchers, a safety factor ψ must be included in (23) in order to guarantee the success of next try. For this reason, we additionally comply with the method and achieve the result as $h_{\text{new}} = \psi h\left(\frac{tol}{|\mathscr{C}_e|}\right)^{1/(r+1)}$, where ψ is taken to be a suitable safety factor $\psi \approx 0.9$ [37]. The only reason to choose the safety factor $\psi \in (0, 1)$ is to avoid failed integration steps and r in the above equation stands for the order of the lower order method. In our present scenario, $r = 4$. This approach has been implemented successively to predict the stepsize for the forthcoming step after a successful step is achieved, that is, when $|\mathscr{C}_e| < tol$. Moreover, there is nothing to be worried about the selection of the initial stepsize h_{ini}, we can start with a suitably smaller h_{ini} and later the iter-

Table 1 Maximum Absolute Errors on $[0, 0.5]$ (first row), Absolute Errors at $t = 0.5$ (second row), Absolute Mean Errors on $[0, 0.5]$ (third row), Norm (fourth row) and CPU Time in seconds (fifth row) of each individual method with constant stepsize for Numerical Experiment 1.

Method/h	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
HeM	2.7156e-03	3.1008e-04	1.7975e-05	5.2607e-06	2.1382e-06
	2.7156e-03	3.1008e-04	1.7975e-05	3.9965e-06	2.1382e-06
	6.1836e-04	5.0496e-05	6.3549e-06	2.5496e-06	8.1993e-07
	2.7373e-03	3.2075e-04	3.2395e-05	1.7829e-05	8.4908e-06
	5.5600e-05	1.3520e-04	1.0340e-04	7.5800e-05	4.8200e-05
CoM	1.6092e-03	6.8869e-04	4.5515e-04	2.4303e-04	1.2545e-04
	1.6092e-03	6.8869e-04	4.5515e-04	2.4303e-04	1.2545e-04
	3.6436e-04	1.8871e-04	1.0677e-04	5.4169e-05	2.7273e-05
	1.6144e-03	8.6450e-04	6.8453e-04	4.7955e-04	3.3687e-04
	3.7400e-05	1.1690e-04	7.1500e-05	5.3400e-05	3.5800e-05
Taylor	3.9836e-03	2.2817e-04	9.7309e-06	3.5528e-07	1.1997e-08
	3.9836e-03	2.2817e-04	9.7309e-06	3.5528e-07	1.1997e-08
	9.4453e-04	3.9292e-05	1.3567e-06	4.3903e-08	1.3904e-09
	4.0294e-03	2.4237e-04	1.1730e-05	5.2849e-07	2.3400e-08
	1.3600e-04	3.7910e-04	5.7020e-04	6.2010e-04	7.5670e-04
$RMSSP_5$	5.0742e-08	7.9501e-10	1.2431e-11	1.9496e-13	3.5527e-15
	5.0742e-08	7.9501e-10	1.2431e-11	1.9496e-13	3.5527e-15
	1.6018e-08	2.2352e-10	3.2872e-12	4.9923e-14	6.9859e-16
	5.4778e-08	1.0018e-09	1.9823e-11	4.1417e-13	8.6234e-15
	2.2350e-04	3.0160e-04	3.7970e-04	4.4180e-04	4.2690e-04

ative method will rectify this value if need be, based on the stepsize approach.

5. Numerical dynamics with results and discussion

In this section, we consider different types of IVPs to test the performance of the proposed method $RMSSP_5$ given in (11). Comparison has been carried out with three fifth-order methods chosen from the literature. One of them is known as the rational RK method under Heronian mean [38] abbreviated as HeM, the second one is the rational RK method under contra-harmonic mean [39] abbreviated as CoM, and the third one is the well-known fifth-order Taylor series method denoted as Taylor; in the present computations. Both constant and

Table 2 Maximum Absolute Errors on $[0, 1]$ (first row), Absolute Errors at $t = 1$ (second row), Absolute Mean Errors on $[0, 1]$ (third row), Norm (fourth row) and CPU Time in seconds (fifth row) of each individual method with constant stepsize for Numerical Experiment 1.

Method/h	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
HeM	divergence				
CoM	divergence				
Taylor	divergence				
$RMSSP_5$	4.8168e-06	1.3945e-07	8.9471e-08	1.4017e-09	2.5068e-11
	1.7737e-07	2.7790e-09	4.5752e-11	6.7235e-13	1.0658e-14
	6.8337e-07	1.4837e-08	2.8714e-09	2.7172e-11	4.4271e-13
	4.9038e-06	1.5956e-07	8.9512e-08	1.4157e-09	3.2953e-11
	3.1220e-04	4.9090e-04	2.5860e-04	5.2550e-04	1.1468e-03

Table 3 Maximum Absolute Local Truncation Errors on $[0, 0.5]$ (first row), Absolute Local Truncation Errors at $t = 0.5$ (second row), Absolute Mean Local Truncation Errors on $[0, 0.5]$ (third row), Local Truncation Norm (fourth row) and CPU Time in seconds (fifth row) of each individual method with constant stepsize for Numerical Experiment 1.

Method/h	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
HeM	2.0287e-03	1.9852e-04	1.6471e-05	1.2600e-06	9.5105e-08
	2.0287e-03	1.9852e-04	1.6471e-05	1.2600e-06	9.5105e-08
	4.8163e-04	3.4742e-05	2.4251e-06	1.8662e-07	1.7449e-08
	2.0542e-03	2.1204e-04	1.9882e-05	1.8530e-06	1.9137e-07
	3.0860e-03	3.8998e-03	3.2094e-03	1.8808e-03	2.6432e-03
CoM	1.7611e-03	7.9878e-05	4.6444e-05	1.3290e-05	3.5219e-06
	1.7611e-03	7.9878e-05	4.6444e-05	1.3290e-05	3.5219e-06
	3.8562e-04	3.5130e-05	1.2991e-05	3.4767e-06	8.9626e-07
	1.7642e-03	1.3125e-04	7.4071e-05	2.7510e-05	9.8993e-06
	2.1694e-03	3.1119e-03	2.0536e-03	2.6783e-03	3.0501e-03
Taylor	3.3027e-03	1.7661e-04	7.1702e-06	2.5405e-07	8.4381e-09
	3.3027e-03	1.7661e-04	7.1702e-06	2.5405e-07	8.4381e-09
	7.8857e-04	3.0722e-05	1.0149e-06	3.2025e-08	1.0008e-09
	3.3430e-03	1.8789e-04	8.6731e-06	3.8011e-07	1.6584e-08
	5.8608e-03	6.8722e-03	5.3829e-03	7.7199e-03	6.7755e-03
$RMSSP_5$	1.2685e-08	9.9376e-11	7.7716e-13	6.2172e-15	4.4409e-16
	1.2685e-08	9.9376e-11	7.7716e-13	6.2172e-15	0.0000e+00
	5.1075e-09	3.8740e-11	2.9919e-13	2.3382e-15	8.3694e-17
	1.4929e-08	1.4372e-10	1.4728e-12	1.5693e-14	1.3733e-15
	9.1215e-03	9.9894e-03	8.5818e-03	7.8176e-03	7.9364e-03

Table 4 Number of steps (first row), and Absolute Errors at $t = 0.5$ (second row) of each individual method with variable stepsize for Numerical Experiment 1.

tol	HeM	CoM	Taylor	$RMSSP_5$
10^{-2}	1783	50	5	2
	4.2176e-08	3.5305e-04	2.0898e-03	4.2138e-05
10^{-3}	17964	739	9	3
	3.7030e-10	1.8750e-05	1.2687e-04	1.8593e-06
10^{-4}	179768	7597	15	4
	3.7960e-12	1.7200e-06	1.0851e-05	1.0690e-07
10^{-5}	1797812	76186	28	6
	1.6300e-13	1.7000e-07	5.6135e-07	9.1082e-09

variable stepsize approaches solve IVPs taken under scalar and vector versions.

Problem 1. Consider the following first-order singular type of IVP ([26]):

$$y'(t) = 1 + y^2(t),\ y(0) = 1,\ 0 \leqslant t \leqslant 1 \tag{24}$$

with the exact solution $y(t) = \tan\left(t + \frac{\pi}{4}\right)$, where the singularity occurs at $t = \frac{\pi}{4}$.

As far as numerical experiment 1 is concerned, we have solved it using the fifth-order methods HeM, CoM, Taylor, and $RMSSP_5$ with decreasing constant stepsizes $h = 1/2^i$, $i = 3, \cdots, 7$ in the Table 1 wherein maximum absolute errors, final absolute errors, absolute mean errors, norm and CPU time in seconds have been computed while taking the integration interval $[0, 0.5]$ to avoid the singularity that occurs at $t = \frac{\pi}{4}$. The numerical results show that the proposed method $RMSSP_5$ yields the smallest errors in every case with comparable CPU times. In Table 2, we have carried out the

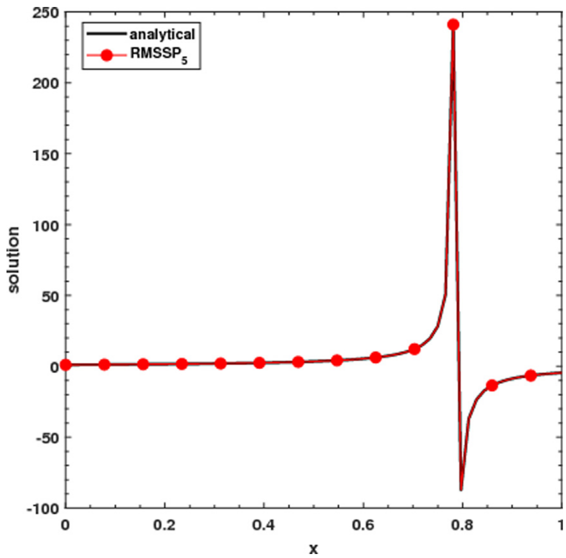

Fig. 3 Comparison between analytical and approximate solution via $RMSSP_5$ with $h = 1/2^6$ for Numerical Experiment 1.

same computations but considered the interval [0, 1] to include the singularity wherein it can easily be observed that the only method giving highly satisfactory results is the proposed method $RMSSP_5$. Moreover, local truncation errors are also computed in Table 3 which, once again, shows better performance of the proposed method. Also, Table 4 demonstrates the usefulness of the variable stepsize approach for each method with $RMSSP_5$ being the method utilizing a minimum number of steps while maintaining an acceptable magnitude of errors in contrast to the other three methods. Finally, Fig. 3 demonstrates the usefulness of $RMSSP_5$ wherein the approximate solution smoothly crosses singularity $(t = \pi/4)$ without being failed thereat. In contrast, the other three methods were unsuccessful in capturing this behavior and, thus, not plotted.

Problem 2. Consider the following singularly perturbed IVP ([31]):

$$y'(t) = -2py(t)^2,\ y(-1) = 4,\ -1 \leqslant t \leqslant 1, \quad p = 100, \tag{25}$$

with the exact solution $y(t) = 4/(1 + 8(t + 1)p)$.

For the numerical experiment 2, we have computed the maximum absolute errors, final absolute errors, absolute mean errors, norm, and CPU time in seconds while using decreasing constant stepsizes $h = 1/2^i$, $i = 2, 4, 6, 8, 10$ over the integration interval $[-1, 1]$. Among all methods, the method $RMSSP_5$ produced the acceptable results when $i = 2, 4, 6, 8$ in the stepsize, whereas the remaining methods ultimately failed. However, in the case of $i = 10$, we start to get suitable

Table 5 Maximum Absolute Errors on $[-1, 1]$ (first row), Absolute Errors at $t = 1$ (second row), Absolute Mean Errors on $[-1, 1]$ (third row), Norm (fourth row) and CPU Time in seconds (fifth row) of each individual method with constant stepsize for Numerical Experiment 2.

Method/h	$1/2^2$	$1/2^4$	$1/2^6$	$1/2^8$	$1/2^{10}$
HeM	divergence				1.0675e+00 3.3501e-03 2.1869e-02 3.4000e+00 1.0382e-03
CoM	divergence				2.1506e-01 2.9491e-07 3.1757e-04 2.5962e-01 7.2160e-04
Taylor	divergence				5.1059e-01 8.2751e-07 7.8936e-04 6.2322e-01 1.8027e-02
$RMSSP_5$	5.6324e-09 8.8778e-11 9.5798e-10 5.8607e-09 3.3570e-04	1.6653e-16 4.3368e-19 8.4633e-18 1.7148e-16 5.6280e-04	1.3878e-17 4.3368e-19 1.7919e-18 4.1441e-17 1.0351e-03	1.1102e-16 4.3368e-19 2.0424e-18 1.4945e-16 4.2445e-03	2.2204e-16 2.6021e-18 5.8178e-18 4.6437e-16 1.3429e-02

Table 6 Maximum Absolute Errors on $[0, 10]$ (first row), Absolute Errors at $t = 10$ (second row), Absolute Mean Errors on $[0, 10]$ (third row), Norm (fourth row) and CPU Time in seconds (fifth row) of each individual method with constant stepsize for Numerical Experiment 3.

Method/h	$10/2^3$	$10/2^4$	$10/2^5$	$10/2^6$	$10/2^7$
HeM	3.8497e-01	3.5414e-01	3.3148e-01	3.1702e-01	3.0994e-01
	4.1284e-02	2.6806e-02	1.9585e-02	1.6462e-02	1.5041e-02
	1.7310e-01	1.5790e-01	1.4470e-01	1.3751e-01	1.3383e-01
	6.5781e-01	8.1195e-01	1.0368e+00	1.3856e+00	1.9029e+00
	2.7605e-03	3.3608e-03	2.7648e-03	4.2512e-03	3.7735e-03
CoM	1.5012e-01	1.1083e-03	9.5614e-05	4.7965e-05	2.2179e-05
	4.5280e-05	1.3861e-06	1.1838e-07	5.9350e-08	2.7382e-08
	2.7067e-02	2.8150e-04	2.5043e-05	1.2833e-05	5.9772e-06
	1.6447e-01	1.9476e-03	2.3289e-04	1.6526e-04	1.0786e-04
	3.8250e-03	4.6010e-03	3.4558e-03	2.7610e-03	4.6417e-03
Taylor	4.4817e-03	8.1282e-05	1.9879e-06	5.4304e-08	1.5899e-09
	5.3799e-06	1.0294e-07	2.4582e-09	6.7153e-11	1.9623e-12
	9.7179e-04	2.0669e-05	5.2057e-07	1.4528e-08	4.2846e-10
	5.2869e-03	1.4289e-04	4.8414e-06	1.8710e-07	7.7320e-09
	7.6193e-03	8.5629e-03	7.2074e-03	6.6841e-03	7.3259e-03
$RMSSP_5$	4.9448e-05	7.2212e-07	1.1371e-08	1.7692e-10	2.7667e-12
	6.2646e-08	9.1542e-10	1.4062e-11	2.1878e-13	3.4148e-15
	1.0788e-05	1.8365e-07	2.9779e-09	4.7330e-11	7.4559e-13
	5.8487e-05	1.2695e-06	2.7695e-08	6.0955e-10	1.3455e-11
	3.4620e-03	3.6687e-03	4.8939e-03	3.5712e-03	3.7160e-03

error values in each method even though the proposed $RMSSP_5$ outperforms as shown by the last column of the Table 5.

Problem 3. Consider the following stiff IVP ([39, p. 113]):

$$y'(t) = -y(t),\ y(0) = 1,\ 0 \leqslant t \leqslant 10, \tag{26}$$

with the exact solution $y(t) = \exp(-t)$.

For the numerical experiment 3, we have performed computations to determine the maximum absolute errors, final absolute errors, absolute mean errors, norm, and CPU time in seconds while using decreasing constant stepsize $h = 10/2^i,\ i = 3, \ldots, 7$ over the integration interval $[0, 10]$ while using the methods HeM, CoM, Taylor and $RMSSP_5$. Table 6 shows that errors produced by $RMSSP_5$ are smallest in each case while consuming the comparable amount of CPU times (seconds). Variable stepsize approach is used within the Table 7

Table 7 Number of steps (first row), and Absolute Errors at $t = 10$ (second row) of each individual method with variable stepsize for Numerical Experiment 3.

tol	CoM	Taylor	$RMSSP_5$
10^{-2}	21	8	6
	2.2201e-03	4.2843e-05	3.9894e-05
10^{-3}	94	12	8
	1.9386e-04	4.2837e-05	5.0264e-06
10^{-4}	754	17	11
	1.1435e-06	2.2518e-06	4.2008e-07
10^{-5}	7226	27	17
	2.6872e-08	2.5300e-07	6.4187e-08

Table 8 No. of Steps (first row), Maximum Absolute Error in $y_1(t)$ (second row), Maximum Absolute Error in $y_2(t)$ (third row), Absolute Error in $y_1(t)$ at $t = 1$ (fourth row), Absolute Error in $y_2(t)$ at $t = 1$ (fifth row) via Taylor's and Proposed methods with variable stepsize for Numerical Experiment 4.

Method/tol	10^{-3}	10^{-6}	10^{-12}	10^{-15}
Taylor	12	21	194	742
	3.1980e-04	3.1980e-04	3.1980e-04	4.2383e-04
	4.2627e-04	4.2627e-04	4.2627e-04	4.2627e-04
	5.6468e-05	2.9994e-05	3.2461e-06	1.4199e-06
	9.7585e-05	4.0039e-05	4.3272e-06	1.5884e-06
$RMSSP_5$	12	16	96	347
	1.4047e-05	3.1261e-08	5.4845e-14	8.8818e-16
	3.6277e-05	5.7431e-08	9.0317e-14	8.8818e-16
	1.1706e-06	4.6055e-09	1.5506e-14	2.7132e-16
	5.2416e-06	1.5302e-08	4.6816e-14	3.5930e-16

while avoiding the HeM method because the number of steps was substantially large enough to be included for this particular method. However, a comparison of $RMSSP_5$ with the remaining two methods reveals the importance variable stepsize approach considered within the proposed method since it yields better errors in the smallest possible number of steps.

Problem 4. Consider the following singular system ([31]):

$$\begin{aligned} &\tfrac{dy_1(t)}{dt} = y_2(t), \quad y_1(0) = 1, \\ &\tfrac{dy_2(t)}{dt} = \tfrac{-4y_2(t)}{2t+3}, \quad y_2(0) = 1, \end{aligned} \tag{27}$$

with the rational solution $y_1(t) = \frac{5}{2} - \frac{9}{4(t+3/2)}$, $y_2(t) = \frac{9}{4(t+3/2)^2}$, $t \in [0, 1]$.

For the last numerical experiment 4, a singular system has been considered where the singularity occurs at $t = -3/2$. Here, the methods called HeM and CoM failed to produce error values when considered with the variable stepsize approach. On the other hand, Taylor and the proposed $RMSSP_5$ were successful, as shown in Table 4. The number of steps, maximum and final absolute errors in both state variables $y_1(t)$ and $y_2(t)$ are computed wherein the method $RMSSP_5$ reaches the size of error as small as 10^{-16} while consuming a fewer number of steps in comparison with the Taylor method. Therefore, it can be said that the variable stepsize approach in $RMSSP_5$ is highly efficient. Table 8.

6. Concluding remarks

A new rational method $RMSSP_5$ has been developed with fifth-order accuracy via multi-variable Taylor series expansion. The method is self-starting, stable, and consistent. Its principal term in the local truncation error contains the term $\mathcal{O}(h^6)$, which confirms the fifth-order convergence of the proposed method. The variable stepsize approach employed improved its performance when different kinds of absolute errors were computed. The proposed method was also compared with some existing rational fifth-order methods. This confirmation depends upon various initial value problems taken as single and systems of differential equations. Generally, the method performs better for problems with singularities, having stiff nature, and are singularly perturbed. The future plans consist of a proposal with a new nonlinear method having $\mathscr{A}$-stability features and a higher-order convergence.

Availability of data and material

Not applicable.

Code availability

Not applicable.

Authors' contributions

Each author equally contributed towards writing and finalizing the article.

Ethics approval

We affirm that the contents of this article are original. Furthermore, it has been neither published elsewhere fully or partially in any language nor submitted for publication (fully or partially) elsewhere simultaneously. It contains no matter that is scandalous, obscene, fraud, plagiarism, libelous, or otherwise contrary to law.

Consent to participate

Each author has approved of and agreed to submit the article.

Consent for publication

Each author agreed to publish the article.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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