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Research article

A study of fixed point sets based on Z-soft rough covering models

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Abstract: Z-soft rough covering models are important generalizations of classical rough set theory to deal with uncertain, inexact and more complex real world problems. So far, the existing study describes various forms of approximation operators and their properties by means of soft neighborhoods. In this paper, we propose the notion of Z-soft rough covering fixed point set (briefly, $\mathcal{Z}$-$\mathcal{SRCFP}$-set) induced by covering soft set. We study the conditions that the family of $\mathcal{Z}$-$\mathcal{SRCFP}$-sets become lattice structure. For any covering soft set, the $\mathcal{Z}$-$\mathcal{SRCFP}$-set is a complete and distributive lattice, and at the same time, it is also a double p-algebra. Furthermore, when soft neighborhood forms a partition of the universe, then $\mathcal{Z}$-$\mathcal{SRCFP}$-set is both a boolean lattice and a double stone algebra. Some main theoretical results are obtained and investigated with the help of examples.

Keywords: Z-soft rough covering set; soft nieghborhood; fixed point; lattice; double stone algebra
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The classical rough set theory introduced by Pawlak in 1982 [31] is an excellent mathematical tool to handle uncertain, vague or inexact knowledge and has been successfully applied to different fields like pattern recognition, data mining, machine learning and many others [17, 23, 32]. Rough set theory is based on equivalence relations which partitions the universe and every block of partition is an equivalence class. The key concept in rough set theory is the lower and upper approximations and equivalence relations is used to find such approximations. However, the notion of equivalence relation and partition is too restricted for many practical applications of real world. Therefore, generalizations of rough set theory have been introduced by many authors such that tolerance based rough sets, similarity relation based rough sets, arbitrary binary relation based rough sets, covering based rough sets and many others [34, 41].

Covering is an approach to extend any partition and is a more general concept used to deal with the attribute subset. Covering-based rough sets are more reasonable than classical rough sets for dealing with the problems of uncertainty, and this theory has obtained a lot of attention and many meaningful research fruits. In order to establish an applicable mathematical systems for covering-based rough set and promote its applications in various fields of life, it has been linked with some other theories like fuzzy set theory, soft set theory, neutrosophic set theory, graph theory and blend of theories [7, 9, 15, 21, 22, 24, 25, 44–49, 51]. Some important work on fuzzy β-covering and Noise-tolerant fuzzy covering based multigranulation rough sets, along with applications can be seen it [18–20].

Lattice theory and partial order play an important role in many fields of engineering and computer science, e.g., they have many applications in distributed computing, that is, vector clocks and global predicate detection, concurrency theory, occurrence nets and pomsets, programming language semantics (fixed-point semantics), and data mining. They have also useful in other disciplines of mathematics such as combinatorics, group theory and number theory. Many authors have combined the rough set theory and lattice theory, and some useful results have been obtained. Based on the existing works about the connection of rough sets and lattice theory, Chen et al. [8] used the notion of covering to define the approximation operators on a completely distributive lattice and set up a unified framework for generalizations of rough sets.

Molodtsov [29] introduced the notion of soft sets to overthrow the problem of handling multi-attributes. This theory has been amended in some appearances to tackle many problems [1–3, 30]. A number of utilizations and applications has been established and used regarding multi-attributes modeling and decision making problems. Shah et al. [36] discussed another approach to roughness of soft graphs with applications in decision making [14–16, 27, 28, 53]. In [35], Praba et al. defined a novel rough set called minimal soft rough set by using minimal soft description of the objects. They also analyzed the relation between modified soft rough set and minimal soft rough set. They proposed a lattice structure on minimal soft rough sets. Uncertainty measures associated with neighborhood based soft covering rough graphs such as roughness measure, entropy measure and granularity is proposed in [40]. Li and Zhu [26] introduced the lattice structures of fixed points of the lower approximations of two types of covering-based rough sets in which they discussed that under what conditions two partially ordered sets are some lattice structures. They defined two types of sets called the fixed point set of neighborhoods and the fixed point set of covering, respectively. Fixed points of covering upper and lower approximation operators are introduced in [13], in which by using some results about the Feynman paths, they have shown that the family of all fixed points of covering upper and lower approximation operators is an atomic frame and a complete lattice, respectively.

Z-soft rough covering models introduced by Zhan et al. [49] are important generalizations of classical rough set theory to deal with data structure and more complex problems of real world. Different kinds of uncertainty measures related to Z-soft rough covering sets and their limitations are presented in [39]. So far, the existing study describes various forms of approximation operators and their related properties both by means of soft graphs and soft neighborhoods [36, 39, 40, 50, 51].

In this paper, we introduce the notion of Z-soft rough covering fixed point set (briefly, $\mathcal{Z}$-$\mathcal{SRCFP}$-set) induced by covering soft set $\mathfrak{C}_{\mathcal{U}}$ over universe $\mathcal{U}$. The $\mathcal{Z}$-$\mathcal{SRCFP}$-set is equal to the one induced by the reduction of covering soft set. We discuss the conditions that the family of $\mathcal{Z}$-$\mathcal{SRCFP}$-sets become lattice structure, which is an important algebraic system and can be used in investigating some generalized rough sets. For any two elements of $\mathcal{Z}$-$\mathcal{SRCFP}$-set , the least upper bound is the join of such two elements and the greatest lower bound is the intersection of these two elements. Further, the soft neighborhood of any element of the universe set $\mathcal{U}$ is the is a join-irreducible element of $\mathcal{Z}$-$\mathcal{SRCFP}$-set. It is shown that for any covering soft set $\mathfrak{C}_{\mathcal{U}}$, the $\mathcal{Z}$-$\mathcal{SRCFP}$-set is a complete and distributive lattice, and at the same time, it is also a double p- algebra. Furthermore, when soft neighborhood forms a partition of the universe, then $\mathcal{Z}$-$\mathcal{SRCFP}$-set is both a boolean lattice and a double Stone algebra. In this study, some main theoretical results are obtained and investigated with the help of examples.

This paper is organized as follows. In Section 2, we review some fundamental and basic knowledge about rough sets, covering soft sets, Z-soft rough covering models and lattices. In Section 3, we study under what conditions that the Z-soft rough covering fixed point set of soft neighborhood becomes some special lattice structures. In Section 4, we study some algebra related Z-soft rough covering fixed point sets. Finally, we conclude our paper in Section 5.

2. Preliminaries

In this section, some basic ideas related to fixed points, soft sets, rough sets, soft rough sets and Z-soft rough covering fixed point sets are being recalled which will help us in understanding rest of the sections. Throughout this paper, U will represent a universe of discourse and E will represent the set of parameters.

Definition 1. [27] Let E be the set of all parameters and $Q \subseteq E$. A pair (g, Q) is called a soft set over the set $\mathcal{U}$ of universe, where $g : Q \to P(\mathcal{U})$ is a set valued mapping and $P(\mathcal{U})$ is the power set of $\mathcal{U}$.

Definition 2. Let $\mathcal{U}$ be a non-void set (universe of discourse). A non-empty sub-family $\mathcal{F}$ ($\mathcal{F} \subseteq P(\mathcal{U})$) is called a soft covering of $\mathcal{U}$ if
(i) each element of $\mathcal{F}$ is non-empty, that is, for $\lambda(\tau) \in \mathcal{F}$, $\lambda(\tau) \neq \emptyset$ for all $\tau \in \mathcal{R}$.
(ii) $\bigcup\limits_{\tau \in \mathcal{R}} \{\lambda(\tau) : \lambda(\tau) \in \mathcal{F}\} = \mathcal{U}$, where $P(\mathcal{U})$ is the power set of $\mathcal{U}$.

In this case, the pair $(\mathcal{U}, \mathcal{F})$ is called soft covering approximation space (briefly, SCAS).

Definition 3. [15] Let $S = (\mu, Q)$ be a soft set over $\mathcal{U}$. Then, the pair $\mathcal{P} = (\mathcal{U}, S)$ is called soft approximation space. Based on the soft approximation space $\mathcal{P} = (\mathcal{U}, S)$, we define the following two sets

$$\begin{aligned} \underline{appr}_{\mathcal{P}}(X) &= \{u \in \mathcal{U} : \exists \rho \in Q, [u \in \mu(\rho) \subseteq X]\} \text{ and} \\ \overline{appr}_{\mathcal{P}}(X) &= \{u \in \mathcal{U} : \exists \rho \in Q, [u \in \mu(\rho), \mu(\rho) \cap X \neq \emptyset]\}, \text{ where } X \subseteq \mathcal{U}, \end{aligned}$$

called soft $\mathcal{P}$-lower approximation and soft $\mathcal{P}$-upper approximation of X, respectively.

If $\overline{appr}_{\mathcal{P}}(X) = \underline{appr}_{\mathcal{P}}(X)$, X is said to be soft $\mathcal{P}$-definable; otherwise X is called a soft $\mathcal{P}$-rough set.

Definition 4. Let C be a covering of $\mathcal{U}$. For any $X \subseteq \mathcal{U}$, the sets $XL_C(X) = \{x \in \mathcal{U} | N(x) \subseteq X\}$ and $XH_C(X) = \{x \in \mathcal{U} | N(x) \cap X \neq \emptyset\}$ are called respectively the sixth type of covering lower and upper approximations of X.

Proposition 1. *[55,56] Let C be a covering of $\mathcal{U}$ and $\emptyset$ be the empty set. For any $X \subseteq \mathcal{U}$, the following properties hold:*

(1) $XL_C(\emptyset) = \emptyset$;
(2) $XL_C(\mathcal{U}) = \mathcal{U}$;
(3) $XL_C(X) \subseteq X$;
(4) $XL_C(XL_C(X)) = XL_C(X)$;
(5)$X \subseteq Y \Rightarrow, XL_C(X) \subseteq XL_C(Y)$;
(6)$\forall K \in C, FL(K) = K, XL_C(K) = K$.

Definition 5. Let (μ, Q) be a full soft set. Then (μ, Q) is called a covering soft set if $\mu(\rho) \neq \emptyset$ for all $\rho \in Q$. In this case (μ, Q) is called a covering soft set over $\mathcal{U}$, denoted by $\mathfrak{C}_{\mathcal{U}}$. Denote by $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ and call it a soft covering approximation space.

Definition 6. [39] Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space and $x \in \mathcal{U}$.
(i) $C_{\mathcal{B}}(x) = \{\mu(\rho) \in \mathfrak{C}_{\mathcal{U}} : x \in \mu(\rho)\}$ is called a soft association of x.
(ii) $N_{\mathcal{B}}(x) = \cap\{\mu(\rho) \in \mathfrak{C}_{\mathcal{U}} : x \in \mu(\rho)\} = \cap C_{\mathcal{B}}(x)$ is called a soft neighborhood of x.

Definition 7. [51] Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space. For any $X \subseteq \mathcal{U}$, the soft covering lower and soft covering upper approximation operators are, respectively, defined as:

$$\begin{aligned} \underline{Z_{\mathcal{W}}}(X) &= \{x \in \mathcal{U} : N_{\mathcal{W}}(x) \subseteq X\}, \\ \overline{Z_{\mathcal{B}}}(X) &= \{x \in \mathcal{U} : N_{\mathcal{W}}(x) \cap X \neq \emptyset\}. \end{aligned}$$

If $\underline{Z_{\mathcal{B}}}(X) = \overline{Z_{\mathcal{B}}}(X)$, then X is called a Z-soft covering definable set. In opposite case, if $\underline{Z_{\mathcal{B}}}(X) = \overline{Z_{\mathcal{B}}}(X)$, then X is called a Z-soft rough covering set.

Definition 8. [39] Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ and $\mathcal{W}^* = \left(\mathcal{U}, \mathfrak{C}^*_{\mathcal{U}}\right)$ be two soft covering approximation spaces such that
(i) for all $\beta(\theta) \in C_{\mathcal{W}}(x)$, there exists $\beta^*(\theta) \in C_{\mathcal{W}^*}(x)$ such that $\beta(\theta) \subseteq \beta^*(\theta)$;
(ii) for all $\beta^*(\theta) \in C_{\mathcal{W}^*}(x)$, there exists $\beta(\theta) \in C_{\mathcal{W}}(x)$ such that $\beta(\theta) \subseteq \beta^*(\theta)$.
Then we say $\mathfrak{C}_{\mathcal{U}}$ is finer than $\mathfrak{C}^*_{\mathcal{U}}$ and denote it by $\mathfrak{C}_{\mathcal{U}} \leq \mathfrak{C}^*_{\mathcal{U}}$.

One can see that if $\mathcal{U}$ is a finite universe and if $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ is a soft covering approximation space. Then $(\mathfrak{C}_{\mathcal{U}}, \leq)$ is partially ordered.

Definition 9. [4] A relation ρ on a non-empty set X is called a partial order if (i) ρ is reflexive, that is, $x\rho x$, for all $x \in X$, (ii) antisymmetric, that is, $x\rho y$ and $y\rho x$ imply $x = y$, for all $x, y \in X$ and (iii) transitive, that is, $x\rho y$ and $y\rho z$ imply $x\rho z$, for all $x, y, z \in X$.

Usually, the partially ordered relation is denoted by " $\leq$ ".

Definition 10. [4] A set X with the partial order relation " $\leq$ " is called a partially ordered set or simply a poset. It is denoted by $(X, \leq)$.

Definition 11. [4] A lattice is a poset $(\mathcal{L}, \leq)$ in which any subset $\{m, n\}$ consisting of two members has a least upper bound and a greatest lower bound. A lattice $\mathcal{L}$ is a complete lattice if $\wedge S$ and $\vee S$ are both in $\mathcal{L}$ for all $S \subseteq \mathcal{L}$.

Definition 12. [4] A lattice $\mathcal{L}$ is a distributive lattice if for all, $m, n \in \mathcal{L}$, we have $l \vee (m \wedge n) = (l \vee m) \wedge (l \vee n)$ or $l \wedge (m \vee n) = (l \wedge m) \vee (l \wedge n)$.

Definition 13. [4] Let $\mathcal{L}$ be a bounded lattice having a least element 0 and a greatest element 1. For a point $m \in \mathcal{L}$, we say that an element $n \in \mathcal{L}$ is a complement of l if $l \vee m = 1$ and $l \wedge m = 0$. If the element a has a unique complement, we denote it by l^c. A lattice $\mathcal{L}$ is a complemented lattice if each element has a complement and $\mathcal{L}$ is a boolean lattice if it is both complemented and distributive lattice.

Definition 14. [4] An element l of a lattice $\mathcal{L}$ is called a join-irreducible if $l = m \vee n$ implies $l = m$ or $l = n$ for every $m, n \in \mathcal{L}$. And $J(\mathcal{L})$ denotes all join-irreducible elements in $\mathcal{L}$.

Definition 15. [4] Let $\mathcal{L}$ be a lattice having a least element. An element m^* is a pseudocomplement of $m \in \mathcal{L}$, if $m \wedge m^* = 0$ and for all $n \in \mathcal{L}$, $m \wedge n = 0$ implies $n \leq m^*$. A lattice is pseudocomplemented if every member has a pseudocomplement. If $\mathcal{L}$ is a distributive pseudocomplemented, and it satisfies the Stone identity $m^* \vee m^{**} = 1$ for all $m \in \mathcal{L}$, then $\mathcal{L}$ is called a Stone algebra.

Definition 16. [4] Let $\mathcal{L}$ be a lattice with a greatest element. An element m^+ is a dual pseudocomplement of $m \in \mathcal{L}$, if $m \vee m^+ = 1$ and for all $n \in \mathcal{L}$, $m \vee n = 1$ implies $m^+ \leq n$. A lattice is dual pseudocomplemented if each element has a dual pseudocomplement. If $\mathcal{L}$ is a distributive dual pseudocomplemented, and it satisfies the dual Stone identity $m^+ \wedge m^{++} = 0$ for every $m \in \mathcal{L}$, then $\mathcal{L}$ is called a dual Stone algebra.

Definition 17. [4] A lattice $\mathcal{L}$ is called a double p-algebra if it is pseudocomplemented and dual pseudocomplemented. A lattice $\mathcal{L}$ is called a double Stone algebra if it is a Stone and a dual Stone algebra.

3. Z-Soft rough covering fixed point sets

The concept of soft neighborhood of an element and Z-soft rough covering sets is introduced by Zhan et al. [51]. In this section, we propose the concept of soft union reducible elements, soft union irreducible elements and Z-soft rough covering fixed point sets. Some basic properties are established and related examples are constructed.

Definition 18. Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space, where $\mathfrak{C}_{\mathcal{U}} = (\sigma, \mathcal{P})$ is a covering soft set over $\mathcal{U}$. An element $\sigma(\rho) \in \mathfrak{C}_{\mathcal{U}}$ is called soft union reducible if $\sigma(\rho_i)$ is the union of some $\sigma\left(\rho_j\right) \in \mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}$ for $i \neq j$. Any other element which is not soft union reducible, is called soft union irreducible element.

Definition 19. Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space, then the set of all soft union irreducible elements of $\mathfrak{C}_{\mathcal{U}}$ is called soft reduct of $\mathfrak{C}_{\mathcal{U}}$.

Example 1. Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$ be a finite universe and let $\mathcal{P} = \left\{\rho_g, \rho_y, \rho_v, \sigma_w\right\}$ be the set of parameters such that $(\sigma, \mathcal{P})$ is covering soft set over $\mathcal{U}$, see Table 1 such that $\sigma\left(\rho_g\right) = \{x_1\}$, $\sigma\left(\rho_y\right) = \{x_2, x_3\}$, $\sigma(\rho_v) = \{x_1, x_2, x_3\}$, $\sigma(\rho_w) = \{x_2, x_4\}$ and $\mathfrak{C}_{\mathcal{U}} = \{\sigma\left(\rho_g\right), \sigma\left(\rho_y\right), \sigma(\rho_v), \sigma(\rho_w)\} = \{\{x_1\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_2, x_4\}\}$.

Table 1. Tabular representation of soft set $(\sigma, \mathcal{P})$.

	x_1	x_2	x_3	x_4
ρ_g	1	0	0	0
ρ_y	0	1	1	0
ρ_v	1	1	1	0
ρ_w	0	1	0	1

Clearly, $\sigma(\rho_v) = \{x_1, x_2, x_3\} = \sigma(\rho_g) \cup \sigma(\rho_y)$, where $\sigma(\rho_g), \sigma(\rho_y) \in \mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_3)\}$ showing that $\sigma(\rho_v)$ is soft union reducible element in $\mathfrak{C}_{\mathcal{U}}$. The elements $\sigma(\rho_g), \sigma(\rho_y)$ and $\sigma(\rho_w)$ are soft union irreducible elements in $\mathfrak{C}_{\mathcal{U}}$.

Proposition 2. *Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space. If $\sigma(\rho)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}}$. Then, $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho)\}$ is still a covering soft set over $\mathcal{U}$.*

Proposition 3. *Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space and $\sigma(\rho_i) \in \mathfrak{C}_{\mathcal{U}}$. Let $\sigma(\rho_i)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}}$ and $\sigma(\rho_j) \in \mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}$. Then, $\sigma(\rho_j)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}}$ if and only if it is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}$ for parameters ρ_i, ρ_j.*

Proof. Suppose $\sigma(\rho_j)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)$. Then, by definition, $\sigma(\rho_j)$ can be expressed as a union of some sets in $(\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}) - \{\sigma(\rho_j)\} = \mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i), \sigma(\rho_j)\}$. It can further be written as union of sets in $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_j)$. This implies that $\sigma(\rho_j)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}}$. Conversely suppose that $\sigma(\rho_j)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}}$. Then, $\sigma(\rho_j)$ can be expressed as a union of some sets in $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_j)$, say $\sigma(\rho_1), \sigma(\rho_2), ..., \sigma(\rho_n)$. It is can be seen that $\sigma(\rho_r) \subset \sigma(\rho_j)$, $r = 1, 2, ..., n$. If all the sets among $\sigma(\rho_1), \sigma(\rho_2), ..., \sigma(\rho_n)$ are not equal to $\sigma(\rho_i)$, then $\sigma(\rho_j)$ can be expressed as the union of sets $\sigma(\rho_1), \sigma(\rho_2), ..., \sigma(\rho_n)$ in $(\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}) - \{\sigma(\rho_j)\}$ which shows that $\sigma(\rho_j)$ is a soft union reducible element of $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}$. If some one among $\sigma(\rho_1)$, $\sigma(\rho_2), ..., \sigma(\rho_n)$ is equal to $\sigma(\rho_i)$, say $\sigma(\rho_1) = \sigma(\rho_i)$ from that $\sigma(\rho_i)$ is soft union reducible element in $\mathfrak{C}_{\mathcal{U}}$, there are sets $\sigma(\theta_1), \sigma(\theta_2), ..., \sigma(\theta_m) \in \mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}$ such that $\sigma(\rho_1)$ is the union of $\sigma(\theta_1)$, $\sigma(\theta_2), ..., \sigma(\theta_m)$. Because $\sigma(\rho_1) \subset \sigma(\rho_j)$, $\sigma(\theta_1), \sigma(\theta_2), ..., \sigma(\theta_m)$ cannot be equal to $\sigma(\rho_j)$, so we have $\sigma(\rho_j) = \sigma(\rho_1) \cup \sigma(\rho_2) \cup, ..., \cup\sigma(\rho_n) = \sigma(\theta_1) \cup \sigma(\theta_2) \cup, ..., \cup\sigma(\theta_m) \cup \sigma(\rho_2) \cup, ..., \cup\sigma(\rho_n)$ and $\sigma(\theta_1), \sigma(\theta_2), ..., \sigma(\theta_m), \sigma(\rho_2), ..., \sigma(\rho_n)$ are not equal to either $\sigma(\rho_i)$ or $\sigma(\rho_j)$. It follows that $\sigma(\rho_j)$ is a soft union reducible element in $\mathfrak{C}_{\mathcal{U}} - \{\sigma(\rho_i)\}$. □

Definition 20. Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space. Then, the set $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}) = \{X \in P(\mathcal{U}) : \underline{Z_{\mathcal{B}}}(X) = X\}$ is called Z-soft rough covering fixed point set (briefly $\mathcal{Z}$-$\mathcal{SRCFP}$-set) induced by covering $\mathfrak{C}_{\mathcal{U}}$.

Example 2. Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$ be a finite universe and let $\mathcal{P} = \{\rho_g, \rho_y, \rho_v, \rho_w\}$ be the set of parameters such that $(\sigma, \mathcal{P})$ is covering soft set over $\mathcal{U}$, see Table 2 and $\sigma(\rho_g) = \{x_1, x_2\}$, $\sigma(\rho_y) = \{x_1, x_2, x_3\}$, $\sigma(\rho_v) = \{x_3\}$, $\sigma(\rho_w) = \mathcal{U}$.

Table 2. Tabular representation of soft set $(\sigma, \mathcal{P})$.

	x_1	x_2	x_3	x_4
θ_g	1	1	0	0
θ_y	1	1	1	0
θ_v	0	0	1	0
θ_w	1	1	1	1

Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space with $\mathfrak{C}_{\mathcal{U}} = \{\{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_3\}, \{x_1, x_2, x_3, x_4\}\}$. Then, for $x \in \mathcal{U}$ we have $N_{\mathcal{B}}(x) = \cap\{\sigma(\rho) \in \mathfrak{C}_{\mathcal{U}} : x \in \sigma(\rho)\}$. That is, shows $N_{\mathcal{B}}(x_1) = \{x_1, x_2\} = N_{w}(x_2)$, $N_{\mathcal{B}}(x_3) = \{x_3\}$, $N_{\mathcal{B}}(x_4) = \mathcal{U}$. Let $X = \{x_1, x_2\} \subseteq \mathcal{U}$, then $\underline{Z_{\mathcal{B}}}(X) = \{x \in \mathcal{U} : N_{w}(x) \subseteq X\} = \{x_1, x_2\} = X$. In this case, X is a member of Z-soft covering rough fixed point set induced by covering $\mathfrak{C}_{\mathcal{U}}$. Similarly, when $X = \{x_1, x_2, x_3\}$ then $\underline{Z_{\mathcal{B}}}(X) = X$, showing that $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ and Soft Reduct= $\{\{x_1, x_2\}, \{x_3\}, \{x_1, x_2, x_3, x_4\}\}$. One can easily verify that $\underline{Z_{Red\mathcal{U}ct\mathcal{B}}}(X) = X$. Thus, $\underline{Z_{Red\mathcal{U}ct\mathcal{B}}}(X) = \underline{Z_{\mathcal{B}}}(X)$

The following proposition shows that the Z-soft rough covering fixed point set induced by any covering soft set over $\mathcal{U}$ is equal to the one induced by the reduction of the covering soft set.

Proposition 4. *Suppose $\mathfrak{C}_{\mathcal{U}}$ is a covering soft set over $\mathcal{U}$, then $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}) = \mathcal{S}_{(Reduct\ fix)}(\mathfrak{C}_{\mathcal{U}})$.*

Proof. By definition, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}) = \{X \in P(\mathcal{U}) : \underline{Z_{\mathcal{B}}}(X) = X\}$ and $\mathcal{S}_{(Red\mathcal{U}ct\ fix)}(\mathfrak{C}_{\mathcal{U}}) = \{X \in P(\mathcal{U}) : \underline{Z_{Reduct\mathcal{B}}}(X) = X\}$. But according to Proposition 1, $\underline{Z_{Reduct\mathcal{B}}}(X) = \underline{Z_{\mathcal{B}}}(X)$ for any $X \in P(\mathcal{U})$. Thus, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}) = \mathcal{S}_{(Reduct\ fix)}(\mathfrak{C}_{\mathcal{U}})$. □

For any covering $\mathfrak{C}_{\mathcal{U}}$ of $\mathcal{U}$, the Z-soft rough covering fixed point set induced by any covering soft set over $\mathcal{U}$ together with the set inclusion, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$, is a partially ordered set.

The following proposition presents an equivalent characterization of the element of the Z-soft rough covering fixed point set.

Proposition 5. *For any subset X of $\mathcal{U}$, $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ if and only if $\bigcup_{x\in X} N_{\mathcal{B}}(x) = X$.*

Proof. Suppose $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}) = \{X \in P(\mathcal{U}) : \underline{Z_{\mathcal{B}}}(X) = X\}$. Then $\underline{Z_{\mathcal{B}}}(X) = X$. That is, $\underline{Z_{\mathcal{B}}}(X) = \{x \in \mathcal{U} : N_{\mathcal{B}}(x) \subseteq X\} = X$. As $N_{\mathcal{B}}(x) \subseteq X$ for all $x \in X$. So $\bigcup_{x\in X} N_{\mathcal{B}}(x) \subseteq X$. Also, $x \in N_{\mathcal{B}}(x)$ then $X \subseteq \bigcup_{x\in X} N_{\mathcal{B}}$. Therefore, $\bigcup_{x\in X} N_{\mathcal{B}}(x) = X$. Conversely, suppose that $\bigcup_{x\in X} N_{\mathcal{B}}(x) = X$. Then clearly $\bigcup_{x\in X} N_{\mathcal{B}}(x) \subseteq X$ for all $x \in X$. For any element $y \notin X$, $y \in N_{\mathcal{B}}(y) \nsubseteq X$. Hence $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}) = \{x \in \mathcal{U} | N_{\mathcal{B}}(x) \subseteq X\} = X$, That is, $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. □

Since, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$, is a partially ordered set, so we consider that whether this partially ordered set is a lattice or not. In the following, we will investigate lattice structures of the partially ordered set $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$.

Proposition 6. *Suppose, $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Then, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is a lattice, where $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$.*

Proof. Suppose, $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, if $X \cup Y \notin \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, then there exists $x \in X \cup Y$ such that $N_{\mathcal{B}}(x) \nsubseteq X \cup Y$. Since $x \in X \cup Y$, so $x \in X$ or $x \in Y$. Hence $N_{\mathcal{B}}(x) \nsubseteq X$ or $N_{\mathcal{B}}(x) \nsubseteq Y$, which is contradictory with the fact $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Therefore, $X \cup Y \in X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Now for any $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, if $X \cap Y \notin \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, then there exists $y \in X \cap Y$ such that $N_{\mathcal{B}}(x) \nsubseteq X \cap Y$. Since $x \in X \cap Y$ so $x \in X$ and $x \in Y$. Hence there exist three cases as follows:
(1) $N_{\mathcal{B}}(y) \nsubseteq X$ and $N_{\mathcal{B}}(y) \nsubseteq Y$,
(2) $N_{\mathcal{B}}(y) \nsubseteq X$ *and* $N_{\mathcal{B}}(y) \subseteq Y$,
(3) $N_{\mathcal{B}}(y) \subseteq X$ *and* $N_{\mathcal{B}}(y) \nsubseteq Y$. But all these three cases are contradictory with the fact that $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Therefore, $X \cap Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ .Thus, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is lattice. □

Remark 1. In above Proposition, $\emptyset$ and $\mathcal{U}$ are the least and greatest elements of $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$, respectively. Therefore, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is a bounded lattice.

The following proposition shows that the neighborhood of any element of the universe belongs to the Z-soft rough covering fixed point set.

Proposition 7. *Let $\mathfrak{C}_{\mathcal{U}}$ is a covering soft set over $\mathcal{U}$. Then, for all $x \in \mathcal{U}$, $N_{\mathcal{B}}(x) \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$.*

Proof. For any $y \in N_{\mathcal{B}}(x)$, we have $N_{\mathcal{B}}(y) \subseteq N_{\mathcal{B}}(x)$, which implies $y \in \{z \in \mathcal{U} : N_{\mathcal{B}}(z) \subseteq N_{\mathcal{B}}(x)\} = \underline{Z_{\mathcal{B}}}(X)$. Hence $N_{\mathcal{B}}(x) \subseteq \underline{Z_{\mathcal{B}}}(X)$. But by Proposition 1, $\underline{Z_{\mathcal{B}}}(X) \subseteq N_{\mathcal{B}}(x)$. Thus, $\underline{Z_{\mathcal{B}}}(X) = N_{\mathcal{B}}(x)$, i.e., $N_{\mathcal{B}}(x) \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. □

The following proposition points out that the neighborhood of any element of the universe is a join-irreducible element of the Z-soft covering rough fixed point set.

Proposition 8. *Let $\mathfrak{C}_{\mathcal{U}}$ is a covering soft set over $\mathcal{U}$. Then, for any $x \in \mathcal{U}$, $N_{\mathcal{B}}(x)$ is a join-irreducible element of the lattice $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$.*

Proof. Suppose there exist $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ such that $N_{\mathcal{B}}(x) = X \cup Y$. Since $x \in N(x)$, $x \in X \cup Y$. Therefore, $x \in X$ or $x \in Y$. Furthermore, as $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, then $N_{\mathcal{B}}(x) \subseteq X \subseteq X \cup Y = N_{\mathcal{B}}(x)$ or $N_{\mathcal{B}}(x) \subseteq Y \subseteq X \cup Y = N_{\mathcal{B}}(x)$ Therefore, $N_{\mathcal{B}}(x) = X$ or $N_{\mathcal{B}}(x) = Y$. Thus $N_{\mathcal{B}}(x)$ is a join-irreducible element of the lattice $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, for every $x \in \mathcal{U}$. □

We have already shown that the set $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is a lattice, where $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ with $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$. Now in the following we show that $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$, is a complete lattice.

Proposition 9. *Suppose $\mathfrak{C}_{\mathcal{U}}$ is a covering soft set over $\mathcal{U}$ and $X, Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Then, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is a complete lattice.*

Proof. Let $G \subseteq \cup G$, we need to prove that $\cap G \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ and $\cup G \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. If $\cap G \notin \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, then there exists $t \in \cap G$ such that $N_{\mathcal{B}}(t) \subsetneq \cap G$, that is, there are two index sets I and J with I, $J \subseteq \{1, 2, \cdots, |G|\}$ with $I \cap J = \emptyset$ and $|I \cup J| = |G|$ such that $N_{\mathcal{B}}(t) \subsetneq X_i$ and $N_{\mathcal{B}}(t) \subseteq X_j$ for any $i \in I, j \in J$, where X_i, $X_j \in G$. This is contradictory with the fact that the sets $X_i (i \in I)$, $X_j (j \in J) \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Hence $\cap G \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$.

If $\cup G \notin \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, then there exists $p \in \cup G$ such that $N_{\mathcal{B}}(p) \subsetneq \cup G$, that is, there exists $X \in G$ such that $p \in X$ and $N(p) * X$, which is contradictory with the fact that $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Hence $\cup G \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. □

The following proposition shows that the Z-soft rough covering fixed point set induced by any covering soft set over $\mathcal{U}$ is a distributive lattice.

Proposition 10. *Let $\mathfrak{C}_{\mathcal{U}}$ is a covering soft set over $\mathcal{U}$. Then, $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is a distributive lattice.*

Proof. Suppose X, Y and $Z \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ where $X, Y, Z \subseteq \mathcal{U}$. It can easily be seen that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$, and $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ showing that $\left(\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}}), \subseteq\right)$ is a distributive lattice. □

4. Some algebra related to Z-soft rough covering fixed point sets

In the following, we show that Z-soft rough covering fixed point set ($\mathcal{Z}$-$\mathcal{SRCFP}$-*set*) induced by any covering soft set $\mathfrak{C}_{\mathcal{U}}$ over $\mathcal{U}$ is both a pseudocomplemented and a dual pseudocomplemented lattice. That is to say any element of Z-soft rough covering fixed point of neighborhoods has a pseudocomplement and a dual pseudocomplement. For any element, its pseudocomplement is the lower approximation of its complement and dual pseudocomplement is the union of all the neighborhood of its complement. We also discuss some algebras connected with ($\mathcal{Z}$-$\mathcal{SRCFP}$-*set*).

Proposition 11. *Let $\mathfrak{C}_{\mathcal{U}}$ is a covering soft set over $\mathcal{U}$. Then,*
(i) $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a pseudocomplemented lattice, and $X^* = \underline{Z_{\mathcal{B}}}(\sim X)$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$;
(ii) $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a dual pseudocomplemented lattice, and $X^+ = \cup_{x\in\sim X} N_{\mathcal{B}}(x)$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$,where $\sim X$ is the complement of X in $\mathcal{U}$.

Proof. (i) Suppose $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, then by Proposition 1, we have $\underline{Z_{\mathcal{B}}}\left(\underline{Z_{\mathcal{B}}}(\sim X)\right) = \underline{Z_{\mathcal{B}}}(\sim X)$, showing that $\underline{Z_{\mathcal{B}}}(\sim X) \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. According to Proposition 1, $\underline{Z_{\mathcal{B}}}\left(\underline{Z_{\mathcal{B}}}(\sim X)\right) \subseteq \underline{Z_{\mathcal{B}}}(\sim X)$. Hence $X \cap \left(\underline{Z_{\mathcal{B}}}(\sim X)\right) = \emptyset$. Now suppose for any $Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, if $X \cap Y = \emptyset$, then $Y \subseteq\sim X$. According to Proposition 1, $Y = \underline{Z_{\mathcal{B}}}(Y) \subseteq \underline{Z_{\mathcal{B}}}(\sim X)$. Therefore, we have $X^* = \underline{Z_{\mathcal{B}}}(\sim X)$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$; that is, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a pseudocomplemented lattice.
(ii) First, we prove $\cup_{x\in\sim X} N_{\mathcal{B}}(x) \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. For any $y \in \cup_{x\in\sim X} N_{\mathcal{B}}(x)$, there exists $z \in\sim X$ such that $y \in \underline{Z_{\mathcal{B}}}(z)$. Thus $N_{\mathcal{B}}(y) \subseteq N_{\mathcal{B}}(z)$, i.e.,$N_{\mathcal{B}}(y) \subseteq \cup_{x\in\sim X} N_{\mathcal{B}}(x)$. Therefore, $y \in \underline{Z_{\mathcal{B}}}(\cup_{x\in\sim X} N_{\mathcal{B}}(x))$, that is, $\cup_{x\in\sim X} N_{\mathcal{B}}(x) \subseteq \underline{Z_{\mathcal{B}}}(\cup_{x\in\sim X} N_{\mathcal{B}}(x))$. According to Proposition 1, $\underline{Z_{\mathcal{B}}}(\cup_{x\in\sim X} N_{\mathcal{B}}(x)) \subseteq (\cup_{x\in\sim X} N_{\mathcal{B}}(x))$. Consequently, $\underline{Z_{\mathcal{B}}}(\cup_{x\in\sim X} N_{\mathcal{B}}(x)) = \cup_{x\in\sim X} N_{\mathcal{B}}(x)$, that is, $\cup_{x\in\sim X} N_{\mathcal{B}}(x) \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. It is straightforward that $X \cup (\cup_{x\in\sim X} N_{\mathcal{B}}(x)) = \mathcal{U}$. Secondly, we need to prove that for any $Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, if $X \cup Y = \mathcal{U}$, then $\cup_{x\in\sim X} N_{\mathcal{B}}(x) \subseteq Y$. Then we have the following two cases to prove it.
Case 1. If $\cup_{x\in\sim X} N_{\mathcal{B}}(x) = \sim X$, then $\cup_{x\in\sim X} N_{\mathcal{B}}(x) \subseteq Y$.
Case 2. If $\sim X \subset \cup_{x\in\sim X} N_{\mathcal{B}}(x)$, then $\sim X \subset Y$. Suppose $Y \subset \cup_{x\in\sim X} N_{\mathcal{B}}(x)$, then there exists $y \in \cup_{x\in\sim X} N_{\mathcal{B}}(x)$ such that $y \notin Y$, so $y \notin\sim X$, which implies there exists $z \in\sim X$, such that $y \in N_{\mathcal{B}}(z)$. Since $\sim X \subset Y$, $z \in Y$. So $N_{\mathcal{B}}(z) \subsetneq Y$, that is, $z \notin \underline{Z_{\mathcal{B}}}(Y)$. In other words, $\underline{Z_{\mathcal{B}}}(Y) \neq Y$, which is contradictory with $Y \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Hence $\cup_{x\in\sim X} N_{\mathcal{B}}(x) \subseteq Y$. Consequently, $X^+ = \cup_{x\in\sim X} N_{\mathcal{B}}(x)$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, that is, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a dual pseudocomplemented lattice. □

Thus, we have seen that Z-soft rough covering fixed point set of neighborhoods induced by any covering soft set over $\mathcal{U}$ is both a pseudocomplemented and a dual pseudocomplemented lattice. Moreover, according to Definition17, it is obvious that $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a double p-algebra.

Remark 2. Generally, the Z-soft rough covering fixed point set of neighborhoods neither a Stone algebra nor a dual Stone algebra.

Example 3. Let $\mathcal{U} = \{1,2,3,4\}$ be a finite universe and let $\mathcal{P} = \left\{\rho_g, \rho_y, \rho_v, \rho_w\right\}$ be the set of parameters such that $(\sigma, \mathcal{P})$ is covering soft set over $\mathcal{U}$, see Table 3 and $\sigma\left(\rho_g\right) = \{x_1, x_2, x_3\}$, $\sigma\left(\rho_y\right) = \{x_1\}$, $\sigma(\rho_v) = \{x_1, x_3, x_4\}$, $\sigma(\rho_w) = \{x_2, x_3\}$.

Table 3. Tabular representation of soft set $(\sigma, \mathcal{P})$.

	x_1	x_2	x_3	x_4
θ_g	1	1	1	0
θ_y	1	0	0	0
θ_v	1	0	1	1
θ_w	0	1	1	0

Let $\mathcal{B} = (\mathcal{U}, \mathfrak{C}_{\mathcal{U}})$ be a soft covering approximation space with $\mathfrak{C}_{\mathcal{U}} = \{\{x_1, x_2, x_3\}, \{x_1\}, \{x_1, x_3, x_4\}, \{x_2, x_3\}\}$. Then, for $x \in \mathcal{U}$ we have $N_{\mathcal{B}}(x) = \cap\{\sigma(\rho) \in \mathfrak{C}_{\mathcal{U}} : x \in \sigma(\rho)\}$. Then $N_{\mathcal{B}}(x_1) = \{x_1\}$, $N_{\mathcal{B}}(x_2) = \{x_2, x_3\}$, $N_{\mathcal{B}}(x_3) = \{x_3\}$, $N_{\mathcal{B}}(x_4) = \{x_1, x_3, x_4\}$. If $X = \{x_2, x_3\}$, then $X^* = \underline{Z_{\mathcal{B}}}(\sim X) = \{x_1\}$, $X^{**} = \underline{Z_{\mathcal{B}}}((\sim X^*)) = \{x_2, x_3\}$, that is., $X^* \cup X^{**} \neq \mathcal{U}$. Therefore, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is not a Stone algebra. Similarly, $X^+ = \cup_{x \in \sim X} N_{\mathcal{B}}(x) = \{x_1, x_3, x_4\}$, $X^{++} = \cup_{x \in \sim X^+} N_{\mathcal{B}}(y) = \{x_2, x_3\}$, that is, $X^+ \cap X^{++} \neq \emptyset$. Thus,$\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is not a dual Stone algebra.

According to Example 3, the fixed point set of neighborhoods induced by any covering is not always a double Stone algebra. In the following, we study under what conditions that the fixed point set of neighborhoods induced by a covering is a boolean lattice and a double Stone algebra, respectively.

Proposition 12. *If $\{N_{\mathcal{B}}(x) : x \in \mathcal{U}\}$ is a partition of $\mathcal{U}$, then $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a boolean lattice.*

Proof. According to Proposition 10, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a distributive lattice. Furthermore, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a bounded lattice. In the following,we need to prove only that $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a complemented lattice. In other words, we need to prove that $\sim X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. If $\sim X \notin \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, that is, $\cup_{x \in \sim X} N_{\mathcal{B}}(x) \neq \sim X$, then there exists $y \in \cup_{x \in \sim X} N_{\mathcal{B}}(x)$ such that $y \notin \sim X$. Since $y \in \cup_{x \in \sim X} N_{\mathcal{B}}(x)$, then there exists $z \in \sim X$ such that $y \in N_{\mathcal{B}}(z)$. Since $\{N(x) | x \in \mathcal{U}\}$ is a partition of $\mathcal{U}$, $z \in N_{\mathcal{B}}(y)$. Therefore, $N_{\mathcal{B}}(y) \subsetneq X$, that is., $\cup_{x \in \sim X} N_{\mathcal{B}}(x) \neq X$, which is contradictory with $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Hence, $\sim X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ for any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, showing that $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a complemented lattice. Consequently, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a boolean lattice. □

Proposition 13. *If $\{N_{\mathcal{B}}(x) | x \in \mathcal{U}\}$ is a partition of $\mathcal{U}$, then $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a double Stone algebra.*

Proof. For any $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$, we prove $X^* = \sim X = X^+$. Suppose for any $y \in \sim X$ there exists $z \in X$ such that $z \in N_{\mathcal{B}}(y)$, that is, $N_{\mathcal{B}}(y) \subsetneq \sim X$. Since $\{N_{\mathcal{B}}(x) | x \in \mathcal{U}\}$ is a partition of $\mathcal{U}$, then $y \in N_{\mathcal{B}}(z)$, that is, $N_{\mathcal{B}}(z) \subsetneq X$. So $z \notin \underline{Z_{\mathcal{B}}}(X)$, which is contradictory with $X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Hence $N_{\mathcal{B}}(y) \subseteq \sim X$. Then $y \in \underline{Z_{\mathcal{B}}}(\sim X)$ and $\cup_{x \in \sim X} \underline{N_{\mathcal{B}}}(x) \subseteq \sim X$, that is, $\sim X \subseteq \underline{Z_{\mathcal{B}}}(Xc)$. According to Proposition 3, $\underline{Z_{\mathcal{B}}}(\sim X) \subseteq \sim X$. It is straightforward that $\sim X \subseteq \cup_{x \in \sim X} N_{\mathcal{B}}(x)$. Consequently, $X^* = \underline{Z_{\mathcal{B}}}(\sim X) = \sim X = \cup_{x \in \sim X} N_{\mathcal{B}}(x) = X^+$. Since $\underline{Z_{\mathcal{B}}}(\sim X) = \sim X$, then $\sim X \in \mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$. Similarly, we can prove that $X^{**} = \sim\sim X = X = X^{++}$. Therefore, $X^* \cup X^{**} = \mathcal{U}$, $X^+ \cap X^{++} = \emptyset$, showing that $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is both a Stone and a dual Stone algebra. Consequently, $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a double Stone algebra. □

According to Propositions 12 and 13, the the Z-soft rough covering fixed point set of soft neighborhoods is both a boolean lattice and a double Stone algebra when the soft neighborhoods forms a partition of the universe $\mathcal{U}$.

5. Conclusions

In our present paper, we introduce a new kind of partial order with the help of Z-soft rough covering fixed point sets (briefly, $\mathcal{Z}$-$\mathcal{SRCFP}$- set) denoted by $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ which is based on soft covering lower approximation operators $\underline{Z_{\mathcal{B}}}(X)$. Z-soft rough covering fixed point set is also Z-soft rough covering fixed point set of soft neighborhoods. It is shown that $\mathcal{S}_{fix}(\mathfrak{C}_{\mathcal{U}})$ is a lattice where the least upper bound of any two elements of the Z-soft rough covering fixed point set of soft neighborhoods is the join of these two elements and the greatest lower bound is the intersection of these two elements. For any covering soft set $\mathfrak{C}_{\mathcal{U}}$ over $\mathcal{U}$, we prove that the Z-Soft rough covering fixed point set of soft neighborhoods is both a complete and a distributive lattice. It is also a double p-algebra. Especially, when the soft neighborhoods form a partition of the universe, the Z-soft rough covering fixed point set of soft neighborhoods is both a boolean lattice and a double Stone algebra.

Conflict of interest

The authors declare that they have no competing interests.

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