Research Article

Fixed Points and Stability of an Additive Functional Equation of n-Apollonius Type in C^*-Algebras

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Using the fixed point method, we prove the generalized Hyers-Ulam stability of C^*-algebra homomorphisms and of generalized derivations on C^*-algebras for the following functional equation of Apollonius type $\sum_{i=1}^{n} f(z - x_i) = -(1/n)\sum_{1 \le i < j \le n} f(x_i + x_j) + nf(z - (1/n^2)\sum_{i=1}^{n} x_i)$.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "when is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$, must be close to an exact solution of $\mathcal{E}$?" If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the stability theory for functional equations. The result of Hyers was extended by Aoki [3] in 1950 by considering the unbounded Cauchy differences. In 1978, Rassias [4] proved that the additive mapping T, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$, the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$. Găvruța [5] generalized the Rassias' result. Following the techniques of the proof of the corollary of Hyers [2], we observed that Hyers introduced (in 1941) the following Hyers continuity condition about the continuity of the mapping for each fixed point and then he proved homogeneity of degree one and, therefore, the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms (see [6]). Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7–21]).

Rassias [22], following the spirit of the innovative approach of Hyers [2], Aoki [3], and Rassias [4] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [23, 24] for a number of other new results).

In 2003, Cădariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation [25] (see also [8, 26–30]). They could present a short and a simple proof (different of the "*direct method*," initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation [25], for Cauchy functional equation [8], and for quadratic functional equation [26].

The following functional equation:

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \tag{1.1}$$

is called a *quadratic functional equation,* and every solution of (1.1) is said to be a *quadratic mapping*. Skof [31] proved the Hyers-Ulam stability of the quadratic functional equation (1.1) for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [32], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation (1.1). Borelli and Forti [33] generalized the stability result of the quadratic functional equation (1.1). Jun and Lee [34] proved the Hyers-Ulam stability of the Pexiderized quadratic equation

$$f(x + y) + g(x - y) = 2h(x) + 2k(y) \tag{1.2}$$

for mappings f, g, h, and k. The stability problem of the quadratic equation has been extensively investigated by some mathematicians [35].

In an inner product space, the equality

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{x + y}{2}\right\|^2 \tag{1.3}$$

holds, then it is called the *Apollonius' identity.* The following functional equation, which was motivated by this equation,

$$Q(z - x) + Q(z - y) = \frac{1}{2}Q(x - y) + 2Q\left(z - \frac{x + y}{2}\right), \tag{1.4}$$

holds, then it is called *quadratic* (see [36]). For this reason, the functional equation (1.4) is called a *quadratic functional equation of Apollonius type,* and each solution of the functional equation (1.4) is said to be a *quadratic mapping of Apollonius type.* The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [37].

In [36], Park and Rassias introduced and investigated a functional equation, which is called a *generalized Apollonius type quadratic functional equation.* In [38], Najati introduced and investigated a functional equation, which is called a *quadratic functional equation of n-Apollonius type.* Recently in [39], Park and Rassias introduced and investigated the following functional equation:

$$f(z - x) + f(z - y) = -\frac{1}{2}f(x + y) + 2f\left(z - \frac{x + y}{4}\right) \tag{1.5}$$

which is called an *Apollonius type additive functional equation,* and whose solution is called an *Apollonius type additive mapping.* In [40], Park introduced and investigated a functional equation, which is called a *generalized Apollonius-Jensen type additive functional equation* and whose solution is said to be a *generalized Apollonius-Jensen type additive mapping.*

In this paper, employing the above equality (1.5), for a fixed positive integer $n \geq 2$, we introduce a new functional equation, which is called an *additive functional equation of n-Apollonius type* and whose solution is said to be an *additive mapping of n-Apollonius type;*

$$\sum_{i=1}^{n} f(z - x_i) = -\frac{1}{n} \sum_{1 \leq i < j \leq n} f(x_i + x_j) + nf\left(z - \frac{1}{n^2}\sum_{i=1}^{n} x_i\right).$$

(1.6)

We will adopt the idea of Cădariu and Radu [8, 25, 28] to prove the generalized Hyers-Ulam stability results of C^*-algebra homomorphisms as well as to prove the generalized Ulam-Hyers stability of generalized derivations on C^*-algebra for additive functional equation of n-Apollonius type.

We recall two fundamental results in fixed-point theory.

Theorem 1.1 (see [25]). *Let (X, d) be a complete metric space and let $J : X \rightarrow X$ be strictly contractive, that is,*

$$d(Jx, Jy) \leq Lf(x, y), \quad \forall x, y \in X$$

(1.7)

for some Lipschitz constant $L < 1$. Then, the following hold:

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point x^* is globally attractive, that is,*

$$\lim_{n \to \infty} J^n x = x^*$$

(1.8)

for any starting point $x \in X$;

(3) *one has the following estimation inequalities:*

$$d(J^n x, x^*) \leq L^n d(x, x^*),$$

$$d(J^n x, x^*) \leq \frac{1}{1 - L} d(J^n x, J^{n+1} x),$$

$$d(x, x^*) \leq \frac{1}{1 - L} d(x, Jx)$$

(1.9)

for all nonnegative integers n and all $x \in X$.

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies the following:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.2 (see [41]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.10}$$

for all nonnegative integers n or there exists a positive integer n_0 such that the following hold:

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1 - L)) d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, assume that A is a C^*-algebra with norm $\|\cdot\|_A$ and that B is a C^*-algebra with norm $\|\cdot\|_B$.

2. Stability of C^*-algebra homomorphisms

Lemma 2.1. *Let X and Y be real-vector spaces. A mapping $f : X \to Y$ satisfies (1.6) for all $x_1, \dots, x_n, z$ if and only if the mapping f is additive.*

Proof. Letting $x_1 = \cdots = x_n = z = 0$ in (1.6), we get that $f(0) = 0$. Let j and k be fixed integers with $1 \leq j < k \leq n$. Setting $x_i = 0$ for all $1 \leq i \leq n$, $i \neq j, k$ in (1.6), we have

$$f(z - x_j) + f(z - x_k) + (n - 2)f(z) = -\frac{1}{n} f(x_j + x_k) - \frac{n - 2}{n} (f(x_j) + f(x_k)) + nf\left(z - \frac{1}{n^2}(x_j + x_k)\right) \tag{2.1}$$

for all $x_j, x_k, z \in X$. Replacing x_j by $-x_j$ and x_k by x_j in (2.1), respectively, we get

$$f(z + x_j) + f(z - x_j) = -\frac{n - 2}{n}(f(-x_j) + f(x_j)) + 2f(z) \tag{2.2}$$

for all $x_j, z \in X$. Putting $z = 0$ in (2.2), we conclude that $f(-x_j) = -f(x_j)$ for all $x_j \in X$. This means that f is an odd function. Letting $x_k = z = 0$ in (2.1) and using the oddness of f, we obtain that

$$f\left(\frac{1}{n^2} x_j\right) = \frac{1}{n^2} f(x_j), \qquad f(n^2 x_j) = n^2 f(x_j) \tag{2.3}$$

for all $x_j \in X$. Letting $z = 0$ in (2.1), using the oddness of f and (2.3), we have

$$f(x_j + x_k) = f(x_j) + f(x_k) \tag{2.4}$$

for all $x_j, x_k \in X$. Therefore, $f : X \to Y$ is an additive mapping.

The converse is obviously true. $\qquad\square$

For a given mapping $f : A \to B$ and for a fixed positive integer $n \geq 2$, we define

$$C_\mu f(z, x_1, \dots, x_n) := \sum_{i=1}^{n} \mu f(z - x_i) + \frac{1}{n} \sum_{1 \leq i < j \leq n} f(\mu x_i + \mu x_j) - nf\left(\mu z - \frac{1}{n^2} \sum_{i=1}^{n} \mu x_i\right) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $z, x_1, \dots, x_n \in A$.

We prove the generalized Hyers-Ulam stability of C^*-algebra homomorphisms for the functional equation $C_\mu f(z, x_1, \dots, x_n) = 0$.

Theorem 2.2. *Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} \left(\frac{n^2}{n^2-1}\right)^{2j} \varphi\left(\left(\frac{n^2-1}{n^2}\right)^j z, \left(\frac{n^2-1}{n^2}\right)^j x_1, \ldots, \left(\frac{n^2-1}{n^2}\right)^j x_n\right) < \infty, \tag{2.6}$$

$$\|C_\mu f(z, x_1, \ldots, x_n)\|_B \leq \varphi(z, x_1, \ldots, x_n), \tag{2.7}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \varphi(x, y, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}), \tag{2.8}$$

$$\|f(x^*) - f(x)^*\|_B \leq \varphi(\underbrace{x, \ldots, x}_{n+1 \text{ times}}) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_n \in A$. If for some $1 \leq j \leq n$ there exists a Lipschitz constant $L < 1$ such that

$$\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \leq \frac{n^2-1}{n^2} L\varphi\left(\frac{n^2}{n^2-1}x, 0, \ldots, 0, \underbrace{\frac{n^2}{n^2-1}x}_{j\text{th}}, 0, \ldots, 0\right) \tag{2.10}$$

for all $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{n}{(n^2-1) \times (1-L)} \varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \tag{2.11}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \longrightarrow B, \ g(0) = 0\} \tag{2.12}$$

and introduce the *generalized metric* on X:

$$d(g, h) = \inf\left\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \ \forall x \in A\right\}. \tag{2.13}$$

It is easy to show that (X, d) is complete.

For convenience, set

$$\varphi_j(x, y) := \varphi(x, 0, \ldots, 0, \underbrace{y}_{j\text{th}}, 0, \ldots, 0) \tag{2.14}$$

for all $x, y \in A$ and all $1 \leq j \leq n$.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{n}{\alpha} g\left(\frac{\alpha}{n} x\right) \tag{2.15}$$

for all $x \in A$, where $\alpha = (n^2 - 1)/n$.

For any $g, h \in X$, we have

$$
\begin{aligned}
d(g, h) < C &\Longrightarrow \|g(x) - h(x)\|_B \leq C\varphi_j(x, x) \quad \forall x \in A \\
&\Longrightarrow \left\| \frac{n}{\alpha} g\left(\frac{\alpha}{n}x\right) - \frac{n}{\alpha} h\left(\frac{\alpha}{n}x\right) \right\|_B \leq \frac{n}{\alpha} C\varphi_j\left(\frac{\alpha}{n}x, \frac{\alpha}{n}x\right) \\
&\Longrightarrow \left\| \frac{n}{\alpha} g\left(\frac{\alpha}{n}x\right) - \frac{n}{\alpha} h\left(\frac{\alpha}{n}x\right) \right\|_B \leq LC\varphi_j(x, x) \\
&\Longrightarrow d(Jg, Jh) \leq LC.
\end{aligned}
\tag{2.16}
$$

Therefore, we see that

$$
d(Jg, Jh) \leq Ld(g, h), \quad \forall g, h \in A.
\tag{2.17}
$$

This means J is a strictly contractive self-mapping of X, with the Lipschitz constant L.

Letting $\mu = 1$, $z = x_j = x$, and for each $1 \leq k \leq n$ with $k \neq j$, $x_k = 0$ in (2.7), we get

$$
\left\| \alpha f(x) - nf\left(\frac{\alpha}{n}x\right) \right\|_B \leq \varphi_j(x, x)
\tag{2.18}
$$

for all $x \in A$. So

$$
\left\| f(x) - \frac{n}{\alpha} f\left(\frac{\alpha}{n}x\right) \right\|_B \leq \frac{1}{\alpha} \varphi_j(x, x)
\tag{2.19}
$$

for all $x \in A$. Hence $d(f, Jf) \leq 1/\alpha$.

By Theorem 1.2, there exists a mapping $H : A \to B$ such that the following hold:

(1) H is a fixed point of J, that is,

$$
H\left(\frac{\alpha}{n}x\right) = \frac{\alpha}{n}H(x)
\tag{2.20}
$$

for all $x \in A$; the mapping H is a unique fixed point of J in the set

$$
Y = \{g \in X : d(f, g) < \infty\};
\tag{2.21}
$$

and this implies that H is a unique mapping satisfying (2.20) such that there exists $C \in (0, \infty)$ satisfying

$$
\|H(x) - f(x)\|_B \leq C\varphi_j(x, x)
\tag{2.22}
$$

for all $x \in A$.

(2) $d(J^m f, H) \to 0$ as $m \to \infty$; and this implies the equality

$$
\lim_{m \to \infty} \left(\frac{n}{\alpha}\right)^m f\left(\left(\frac{\alpha}{n}\right)^m x\right) = H(x)
\tag{2.23}
$$

for all $x \in A$;

(3) $d(f, H) \leq (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{\alpha - \alpha L};$$
(2.24)

and this implies that the inequality (2.11) holds.

It follows from (2.6), (2.7), and (2.23) that

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} H(z - x_i) + \frac{1}{n} \sum_{1 \leq i < j \leq n} H(x_i + x_j) - nH\left(z - \frac{1}{n^2} \sum_{i=1}^{n} x_i \right) \right\|_B \\
= \lim_{m \to \infty} \left(\frac{n}{\alpha} \right)^m \left\| \sum_{i=1}^{n} f\left(\left(\frac{\alpha}{n} \right)^m (z - x_i) \right) + \frac{1}{n} \sum_{1 \leq i < j \leq n} f\left(\left(\frac{\alpha}{n} \right)^m (x_i + x_j) \right) \right. \\
\left. - nf\left(\left(\frac{\alpha}{n} \right)^m z - \left(\frac{\alpha}{n} \right)^m \times \frac{1}{n^2} \sum_{i=1}^{n} x_i \right) \right\|_B \\
\leq \lim_{m \to \infty} \left(\frac{n}{\alpha} \right)^m \varphi\left(\left(\frac{\alpha}{n} \right)^m z, \left(\frac{\alpha}{n} \right)^m x_1, \ldots, \left(\frac{\alpha}{n} \right)^m x_n \right) \\
\leq \lim_{m \to \infty} \left(\frac{n}{\alpha} \right)^{2m} \varphi\left(\left(\frac{\alpha}{n} \right)^m z, \left(\frac{\alpha}{n} \right)^m x_1, \ldots, \left(\frac{\alpha}{n} \right)^m x_n \right) = 0
\end{aligned}
$$
(2.25)

for all $x_1, \ldots, x_n, z \in A$. So

$$\sum_{i=1}^{n} H(z - x_i) = -\frac{1}{n} \sum_{1 \leq i < j \leq n} H(x_i + x_j) + nH\left(z - \frac{1}{n^2} \sum_{i=1}^{n} x_i \right)$$
(2.26)

for all $x_1, \ldots, x_n, z \in A$. By Lemma 2.1, the mapping $H : A \to B$ is Cauchy additive, that is, $H(x + y) = H(x) + H(y)$ for all $x, y \in A$.

By a similar method to the proof of [14], one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.8) that

$$
\begin{aligned}
\|H(xy) - H(x)H(y)\|_B &= \lim_{m \to \infty} \left(\frac{n}{\alpha} \right)^{2m} \left\| f\left(\left(\frac{\alpha}{n} \right)^{2m} xy \right) - f\left(\left(\frac{\alpha}{n} \right)^m x \right) f\left(\left(\frac{\alpha}{n} \right)^m y \right) \right\|_B \\
&\leq \lim_{m \to \infty} \left(\frac{n}{\alpha} \right)^{2m} \varphi\left(\left(\frac{\alpha}{n} \right)^m x, \left(\frac{\alpha}{n} \right)^m y, \underbrace{0, \ldots, 0}_{n-1 \text{ times}} \right) = 0
\end{aligned}
$$
(2.27)

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$
(2.28)

for all $x, y \in A$.

It follows from (2.9) that

$$
\begin{aligned}
\|H(x^*) - H(x)^*\|_B &= \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^m\left\|f\left(\left(\frac{\alpha}{n}\right)^m x^*\right) - f\left(\left(\frac{\alpha}{n}\right)^m x\right)^*\right\|_B \\
&\leq \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^m \varphi\Big(\underbrace{\left(\frac{\alpha}{n}\right)^m x, \ldots, \left(\frac{\alpha}{n}\right)^m x}_{n+1 \text{ times}}\Big) \\
&\leq \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{2m} \varphi\Big(\underbrace{\left(\frac{\alpha}{n}\right)^m x, \ldots, \left(\frac{\alpha}{n}\right)^m x}_{n+1 \text{ times}}\Big) = 0
\end{aligned}
\tag{2.29}
$$

for all $x \in A$. So

$$
H(x^*) = H(x)^* \tag{2.30}
$$

for all $x \in A$.

Thus $H : A \to B$ is a C^*-algebra homomorphism satisfying (2.11) as desired. □

Corollary 2.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$
\|C_\mu f(z, x_1, \ldots, x_n)\|_B \leq \theta\left(\|z\|_A^r + \sum_{i=1}^n \|x_i\|_A^r\right), \tag{2.31}
$$

$$
\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r), \tag{2.32}
$$

$$
\|f(x^*) - f(x)^*\|_B \leq (n+1)\theta\|x\|_A^r \tag{2.33}
$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$
\|f(x) - H(x)\|_B \leq \frac{2n(n^2-1)^{-r}\theta}{(n^2-1)^{1-r} - n^{2(1-r)}}\|x\|_A^r \tag{2.34}
$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 by taking

$$
\varphi(z, x_1, \ldots, x_n) := \theta\left(\|z\|_A^r + \sum_{i=1}^n \|x_i\|_A^r\right) \tag{2.35}
$$

for all $x, y, z \in A$. It follows from (2.31) that $f(0) = 0$. We can choose $L = (n^2/(n^2-1))^{1-r}$ to get the desired result. □

Theorem 2.4. *Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0, \infty)$ satisfying (2.7), (2.8), and (2.9) such that*

$$
\sum_{j=0}^\infty \left(\frac{n^2-1}{n^2}\right)^j \varphi\left(\left(\frac{n^2}{n^2-1}\right)^j z, \left(\frac{n^2}{n^2-1}\right)^j x_1, \ldots, \left(\frac{n^2}{n^2-1}\right)^j x_n\right) < \infty \tag{2.36}
$$

for all $z, x_1, \ldots, x_n \in A$. If for some $1 \leq j \leq n$ there exists a Lipschitz constant $L < 1$ such that

$$\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \leq \frac{n^2}{n^2 - 1} L \varphi\left(\frac{n^2 - 1}{n^2} x, 0, \ldots, 0, \underbrace{\frac{n^2 - 1}{n^2} x}_{j\text{th}}, 0, \ldots, 0\right) \tag{2.37}$$

for all $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{nL}{(n^2 - 1) \times (1 - L)} \varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \tag{2.38}$$

for all $x \in A$.

Proof. Similar to proof of Theorem (2.2), we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{\alpha}{n} g\left(\frac{n}{\alpha} x\right) \tag{2.39}$$

for all $x \in A$, where $\alpha = (n^2 - 1)/n$. We can conclude that J is a strictly contractive self mapping of X with the Lipschitz constant L.

It follows from (2.18) that

$$\left\| f(x) - \frac{\alpha}{n} f\left(\frac{n}{\alpha} x\right) \right\|_B \leq \frac{1}{n} \varphi_j\left(\frac{n}{\alpha} x, \frac{n}{\alpha} x\right) \leq \frac{L}{\alpha} \varphi_j(x, x) \tag{2.40}$$

for all $x \in A$. Hence, $d(f, Jf) \leq (L/\alpha)$.

By Theorem 1.2, there exists a mapping $H : A \to B$ such that the following hold:
(1) H is a fixed point of J, that is,

$$H\left(\frac{n}{\alpha} x\right) = \frac{n}{\alpha} H(x) \tag{2.41}$$

for all $x \in A$; the mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}; \tag{2.42}$$

and this implies that H is a unique mapping satisfying (2.41) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi_j(x, x) \tag{2.43}$$

for all $x \in A$;
(2) $d(J^m f, H) \to 0$ as $m \to \infty$; and this implies the equality

$$\lim_{m \to \infty} \left(\frac{\alpha}{n}\right)^m f\left(\left(\frac{\alpha}{n}\right)^m x\right) = H(x) \tag{2.44}$$

for all $x \in A$;
(3) $d(f, H) \leq (1/(1 - L))d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{L}{\alpha - \alpha L}, \tag{2.45}$$

which implies that the inequality (2.38) holds.

The rest of the proof is similar to the proof of Theorem 2.2. $\qquad \square$

Corollary 2.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.31), (2.32), *and* (2.33). *Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{2n(n^2 - 1)^{r-2}L\theta}{(n^2 - 1)^{r-1} - n^{2(r-1)}}\|x\|_A^r \tag{2.46}$$

for all $x \in A$ and $L = (n^2/(n^2 - 1))^{r-1}$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(z, x_1, \ldots, x_n) := \theta\left(\|z\|_A^r + \sum_{i=1}^n \|x_i\|_A^r\right) \tag{2.47}$$

for all $z, x_1, \ldots, x_n \in A$. It follows from (2.31) *that $f(0) = 0$. We can choose $L = (n^2/(n^2-1))^{r-1}$ to get the desired result.* □

3. Stability of generalized derivations on C^*-algebras

For a given mapping $f : A \to A$ and for a fixed positive integer $n \geq 2$, we define

$$C_\mu f(z, x_1, \ldots, x_n) := \sum_{i=1}^n \mu f(z - x_i) + \frac{1}{n}\sum_{1 \leq i < j \leq n} f(\mu x_i + \mu x_j) - nf\left(\mu z - \frac{1}{n^2}\sum_{i=1}^n \mu x_i\right) \tag{3.1}$$

for all $\mu \in \mathbb{T}^1$ and all $z, x_1, \ldots, x_n \in A$.

Definition 3.1 (see [42]). A generalized derivation $\delta : A \to A$ is involutive $\mathbb{C}$-linear and fulfills

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz)$$

for all $x, y, z \in A$.

   We prove the generalized Hyers-Ulam stability of derivations on C^*-algebras for the functional equation $C_\mu f(z, x_1, \ldots, x_n) = 0$.

Theorem 3.2. *Let $f : A \to A$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^\infty \left(\frac{n^2}{n^2 - 1}\right)^{3j} \varphi\left(\left(\frac{n^2 - 1}{n^2}\right)^j z, \left(\frac{n^2 - 1}{n^2}\right)^j x_1, \ldots, \left(\frac{n^2 - 1}{n^2}\right)^j x_n\right) < \infty, \tag{3.2}$$

$$\|C_\mu f(x_1, \ldots, x_n, z)\|_A \leq \varphi(z, x_1, \ldots, x_n), \tag{3.3}$$

$$\|f(xyz) - f(xy)z + xf(y)z - xf(yz)\|_A \leq \varphi(x, y, z, \underbrace{0, \ldots, 0}_{n-2 \text{ times}}), \tag{3.4}$$

$$\|f(x^*) - f(x)^*\|_A \leq \varphi(\underbrace{x, \ldots, x}_{n+1 \text{ times}}) \tag{3.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_n \in A$. If for some $1 \le j \le n$ there exists a Lipschitz constant $L < 1$ such that

$$\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \le \frac{n^2 - 1}{n^2} L\varphi\left(\frac{n^2}{n^2 - 1}x, 0, \ldots, 0, \underbrace{\frac{n^2}{n^2 - 1}x}_{j\text{th}}, 0, \ldots, 0\right) \tag{3.6}$$

for all $x \in A$, then there exists a unique generalized derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{n}{(n^2 - 1) \times (1 - L)} \varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \tag{3.7}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.7). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \left(\frac{n}{\alpha}\right)^m f\left(\left(\frac{n}{\alpha}\right)^m x\right) \tag{3.8}$$

for all $x \in A$.

It follows from (3.4) that

$$\|\delta(xyz) - \delta(xy)z + x\delta(y)z - x\delta(yz)\|_A$$

$$= \lim_{m \to \infty} \left(\frac{n}{\alpha}\right)^{3m} \left\| f\left(\left(\frac{\alpha}{n}\right)^{3m} xyz\right) - f\left(\left(\frac{\alpha}{n}\right)^{2m} xy\right) \cdot \left(\frac{\alpha}{n}\right)^m z \right.$$

$$\left. + \left(\frac{\alpha}{n}\right)^m xf\left(\left(\frac{\alpha}{n}\right)^m y\right) \cdot \left(\frac{\alpha}{n}\right)^m z - \left(\frac{\alpha}{n}\right)^m xf\left(\left(\frac{\alpha}{n}\right)^{2m} yz\right)\right\|_A \tag{3.9}$$

$$\le \lim_{m \to \infty} \left(\frac{n}{\alpha}\right)^{3m} \varphi\left(\left(\frac{\alpha}{n}\right)^m x, \left(\frac{\alpha}{n}\right)^m y, \left(\frac{\alpha}{n}\right)^m z, \underbrace{0, \ldots, 0}_{n-2\text{ times}}\right) = 0$$

for all $x, y, z \in A$. So

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz) \tag{3.10}$$

for all $x, y, z \in A$. Thus $\delta : A \to A$ is a generalized derivation satisfying (3.7). $\qquad\square$

Theorem 3.3. *Let $f : A \to A$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0, \infty)$ satisfying (2.36),(3.3), (3.4) and (3.5) for all $x, y, z, x_1, \ldots, x_n \in A$. If for some $1 \le j \le n$ there exists a Lipschitz constant $L < 1$ such that*

$$\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \le \frac{n^2}{n^2 - 1} L\varphi\left(\frac{n^2 - 1}{n^2}x, 0, \ldots, 0, \underbrace{\frac{n^2 - 1}{n^2}x}_{j\text{th}}, 0, \ldots, 0\right) \tag{3.11}$$

for all $x \in A$, then there exists a unique generalized derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_B \le \frac{nL}{(n^2 - 1) \times (1 - L)} \varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \tag{3.12}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. $\qquad\square$

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