Bull. Sci. math. 132 (2008) 87–96

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras

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Received 5 February 2006

Available online 22 August 2006

Abstract

In this paper, we prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras. This is applied to investigate isomorphisms between quasi-Banach algebras.

MSC: 39B72; 46B03; 47Jxx

Keywords: Cauchy functional equation; Jensen functional equation; Homomorphism in quasi-Banach algebra; Hyers–Ulam–Rassias stability; p-Banach algebra

1. Introduction and preliminaries

In 1940, S.M. Ulam [23] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

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[1] Supported by grant No. F01-2006-000-10111-0 from the Korea Science & Engineering Foundation.

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2006.07.004

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f: G \to G'$ an *approximate homomorphism*.

In 1941, D.H. Hyers [5] considered the case of approximately additive mappings $f: E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\| f(x+y) - f(x) - f(y) \| \leqslant \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L: E \to E'$ is the unique additive mapping satisfying

$$\| f(x) - L(x) \| \leqslant \epsilon.$$

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed $x \in E$, then L is linear, and if f is continuous at a single point of E then $L: E \to E'$ is also continuous. A generalization of this result was proved by J.M. Rassias [14–16,18]. J.M. Rassias assumed *the following weaker inequality*

$$\| f(x+y) - f(x) - f(y) \| \leqslant \theta \|x\|^p \|y\|^q, \quad \forall x, y \in E,$$

involving a product of different powers of norms, where $\theta > 0$ and real p, q such that $r = p + q \neq 1$, and retained the condition of continuity $f(tx)$ in t for fixed x. J.M. Rassias [17] investigated that it is possible to replace ϵ in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. The stability phenomenon that was introduced and proved by J.M. Rassias is called the *Hyers–Ulam–Rassias stability*. In all the cases investigated in this article, the approach to the existence question was to prove *asymptotic type formulas* of the form $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, or $L(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. However, in 2002, J.M. Rassias and M.J. Rassias [19] considered and investigated quadratic equations involving a product of powers of norms in which an approximate quadratic mapping *degenerates* to a *genuine* quadratic mapping. Analogous results could be investigated with additive type equations involving a product of powers of norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2,4,6–13,21]).

Theorem 1.1. [14,15,18] *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f: X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geqslant 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies inequality*

$$\| f(x+y) - f(x) - f(y) \| \leqslant \theta \|x\|^{p/2} \|y\|^{p/2}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L: X \to Y$ satisfying

$$\| f(x) - L(x) \| \leqslant \frac{\theta}{|2^p - 2|} \|x\|^p$$

for all $x \in X$. If, in addition, $f: X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Theorem 1.2. [16] *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geqslant 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\left\| f(x) - L(x) \right\| \leqslant \frac{\theta}{|2^r - 2|} \|x\|^r$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Theorem 1.3. [17] *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exists a constant $\theta \geqslant 0$ such that f satisfies inequality*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \leqslant \theta K(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in X^n$ and $K : X^n \to \mathbb{R}^+ \cup \{0\}$ is a non-negative real-valued function such that

$$R_n(x) = \sum_{j=0}^{\infty} \frac{1}{n^j} K\left(n^j x, \ldots, n^j x\right) < \infty$$

is a non-negative function of x, and the condition

$$\lim_{m \to \infty} \frac{1}{n^m} K\left(n^m x_1, \ldots, n^m x_n\right) = 0$$

holds. Then there exists a unique additive mapping $L_n : X \to Y$ satisfying

$$\left\| f(x) - L_n(x) \right\| \leqslant \frac{\theta}{n} R_n(x)$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L_n is an $\mathbb{R}$-linear mapping.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.4. [3,22] Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $K \geqslant 1$ such that $\|x + y\| \leqslant K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$.

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* ($0 < p \leqslant 1$) if

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [22] (see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.5. [1] Let $(A, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(A, \|\cdot\|)$ is called a *quasi-normed algebra* if A is an algebra and there is a constant $C > 0$ such that $\|xy\| \leqslant C\|x\|\|y\|$ for all $x, y \in A$.

A *quasi-Banach algebra* is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a *p-norm* then the quasi-Banach algebra is called a *p-Banach algebra*.

In Section 2, we prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation and the Jensen functional equation.

In Section 3, we investigate isomorphisms between quasi-Banach algebras.

2. Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras

Throughout this section, assume that A is a quasi-normed algebra with quasi-norm $\|\cdot\|_A$ and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$. Let K be the modulus of concavity of $\|\cdot\|_B$.

We prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation.

Theorem 2.1. *Let $r > 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that*

$$\big\|f(x+y) - f(x) - f(y)\big\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r, \tag{2.1}$$

$$\big\|f(xy) - f(x)f(y)\big\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r \tag{2.2}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}}\|x\|_A^{2r} \tag{2.3}$$

for all $x \in A$.

Proof. Letting $y = x$ in (2.1), we get

$$\big\|f(2x) - 2f(x)\big\|_B \leqslant \theta\|x\|_A^{2r} \tag{2.4}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \leqslant \frac{\theta}{4^r}\|x\|_A^{2r}$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B^p \leqslant \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B^p$$
$$\leqslant \frac{\theta^p}{4^{pr}} \sum_{j=l}^{m-1} \frac{2^{pj}}{4^{prj}} \|x\|_A^{2pr} \tag{2.5}$$

for all non-negative integers m and l with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

It follows from (2.1) that

$$\left\|H(x+y) - H(x) - H(y)\right\|_B = \lim_{n\to\infty} 2^n \left\|f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\|_B$$
$$\leqslant \lim_{n\to\infty} \frac{2^n\theta}{4^{nr}} \|x\|_A^r \|y\|_A^r = 0$$

for all $x, y \in A$. So

$$H(x+y) = H(x) + H(y)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

By the same reasoning as in the proof of Theorem of [20], the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2.2) that

$$\left\|H(xy) - H(x)H(y)\right\|_B = \lim_{n\to\infty} 4^n \left\|f\left(\frac{xy}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right)\right\|_B$$
$$\leqslant \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \|x\|_A^r \|y\|_A^r = 0$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Now, let $T : A \to B$ be another Cauchy additive mapping satisfying (2.3). Then we have

$$\left\|H(x) - T(x)\right\|_B = 2^n \left\|H\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_B$$
$$\leqslant 2^n K\left(\left\|H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B\right)$$
$$\leqslant \frac{2^{n+1}K\theta}{(4^{pr} - 2^p)^{1/p} 4^{nr}} \|x\|_A^{2r},$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique homomorphism satisfying (2.3). □

Theorem 2.2. *Let $r < \frac{1}{2}$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying* (2.1) *and* (2.2). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\| f(x) - H(x) \|_B \leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}} \|x\|_A^{2r} \tag{2.6}$$

for all $x \in A$.

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \leqslant \frac{\theta}{2} \|x\|_A^{2r}$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned} \left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\|_B^p &\leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\|_B^p \\ &\leqslant \frac{\theta^p}{2^p} \sum_{j=l}^{m-1} \frac{4^{prj}}{2^{pj}} \|x\|_A^{2pr} \end{aligned} \tag{2.7}$$

for all non-negative integers m and l with $m > l$ and all $x \in A$. It follows from (2.7) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

We prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Jensen functional equation.

Theorem 2.3. *Let $r < \frac{1}{2}$ and θ be positive real numbers, and let $f : A \to B$ be a mapping with $f(0) = 0$ satisfying* (2.2) *such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\|_B \leqslant \theta \|x\|_A^r \|y\|_A^r \tag{2.8}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\| f(x) - H(x) \|_B \leqslant \frac{K(1 + 3^r)\theta}{(3^p - 9^{pr})^{1/p}} \|x\|_A^{2r} \tag{2.9}$$

for all $x \in A$.

Proof. Letting $y = -x$ in (2.8), we get

$$\| -f(x) - f(-x) \|_B \leqslant \theta \|x\|_A^{2r}$$

for all $x \in A$. Letting $y = 3x$ and replacing x by $-x$ in (2.8), we get

$$\left\|2f(x) - f(-x) - f(3x)\right\|_B \leqslant 3^r\theta\|x\|_A^{2r}$$

for all $x \in A$. Thus

$$\left\|3f(x) - f(3x)\right\|_B \leqslant K\left(3^r + 1\right)\theta\|x\|_A^{2r} \tag{2.10}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{3}f(3x)\right\|_B \leqslant \frac{K(3^r+1)\theta}{3}\|x\|_A^{2r}$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned}\left\|\frac{1}{3^l}f\left(3^l x\right) - \frac{1}{3^m}f\left(3^m x\right)\right\|_B^p &\leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}f\left(3^j x\right) - \frac{1}{3^{j+1}}f\left(3^{j+1}x\right)\right\|_B^p \\ &\leqslant \frac{K^p(3^r+1)^p\theta^p}{3^p}\sum_{j=l}^{m-1}\frac{9^{prj}}{3^{pj}}\|x\|_A^{2pr} \end{aligned} \tag{2.11}$$

for all non-negative integers m and l with $m > l$ and all $x \in A$. It follows from (2.11) that the sequence $\{\frac{1}{3^n}f(3^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{3^n}f(3^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{3^n}f\left(3^n x\right)$$

for all $x \in A$.

By (2.8),

$$\begin{aligned}\left\|2H\left(\frac{x+y}{2}\right) - H(x) - H(y)\right\|_B &= \lim_{n\to\infty}\frac{1}{3^n}\left\|2f\left(3^n\frac{x+y}{2}\right) - f\left(3^n x\right) - f\left(3^n y\right)\right\|_B \\ &\leqslant \lim_{n\to\infty}\frac{9^{rn}}{3^n}\theta\|x\|_A^r\|y\|_A^r = 0\end{aligned}$$

for all $x, y \in A$. So

$$2H\left(\frac{x+y}{2}\right) = H(x) + H(y)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.9).

It follows from (2.2) that

$$\begin{aligned}\left\|H(xy) - H(x)H(y)\right\|_B &= \lim_{n\to\infty}\frac{1}{9^n}\left\|f\left(9^n xy\right) - f\left(3^n x\right)f\left(3^n y\right)\right\|_B \\ &\leqslant \lim_{n\to\infty}\frac{9^{nr}\theta}{9^n}\|x\|_A^r\|y\|_A^r = 0\end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Now, let $T : A \to B$ be another Jensen additive mapping satisfying (2.9). Then we have

$$\begin{aligned}\|H(x)-T(x)\|_B^p &= \frac{1}{3^{pn}}\|H(3^n x)-T(3^n x)\|_B^p \\ &\leqslant \frac{1}{3^{pn}}\left(\|H(3^n x)-f(3^n x)\|_B^p+\|T(3^n x)-f(3^n x)\|_B^p\right) \\ &\leqslant 2\frac{9^{prn}}{3^{pn}}\frac{K^p(1+3^r)^p\theta^p}{3^p-9^{pr}}\|x\|_A^{2pr},\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x\in A$. So we can conclude that $H(x)=T(x)$ for all $x\in A$. This proves the uniqueness of H.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let $r>1$ and θ be positive real numbers, and let $f:A\to B$ be a mapping with $f(0)=0$ satisfying* (2.2) *and* (2.8). *If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, then there exists a unique homomorphism $H:A\to B$ such that*

$$\|f(x)-H(x)\|_B \leqslant \frac{K(1+3^r)\theta}{(9^{pr}-3^p)^{1/p}}\|x\|_A^{2r} \tag{2.12}$$

for all $x\in A$.

Proof. It follows from (2.10) that

$$\left\|f(x)-3f\left(\frac{x}{3}\right)\right\|_B \leqslant \frac{K(3^r+1)\theta}{9^r}\|x\|_A^{2r}$$

for all $x\in A$. Since B is a p-Banach algebra,

$$\begin{aligned}\left\|3^l f\left(\frac{x}{3^l}\right)-3^m f\left(\frac{x}{3^m}\right)\right\|_B^p &\leqslant \sum_{j=l}^{m-1}\left\|3^j f\left(\frac{x}{3^j}\right)-3^{j+1}f\left(\frac{x}{3^{j+1}}\right)\right\|_B^p \\ &\leqslant \frac{K^p(3^r+1)^p\theta^p}{9^{pr}}\sum_{j=l}^{m-1}\frac{3^{pj}}{9^{prj}}\|x\|_A^{2pr}\end{aligned} \tag{2.13}$$

for all non-negative integers m and l with $m>l$ and all $x\in A$. It follows from (2.13) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x\in A$. Since B is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $H:A\to B$ by

$$H(x):=\lim_{n\to\infty}3^n f\left(\frac{x}{3^n}\right)$$

for all $x\in A$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3. □

3. Isomorphisms between quasi-Banach algebras

Throughout this section, assume that A is a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and unit e and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$ and unit e'. Let K be the modulus of concavity of $\|\cdot\|_B$.

We investigate isomorphisms between quasi-Banach algebras, associated to the Cauchy functional equation.

Theorem 3.1. *Let $r > 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *such that*

$$f(xy) = f(x)f(y) \tag{3.1}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} 2^n f(\frac{e}{2^n}) = e'$, then the mapping $f : A \to B$ is an isomorphism.

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.1, there exists a homomorphism $H : A \to B$ satisfying (2.3). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

It follows from (3.1) that

$$\begin{aligned} H(x) = H(ex) &= \lim_{n\to\infty} 2^n f\left(\frac{ex}{2^n}\right) = \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\cdot x\right) = \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) f(x) \\ &= e' f(x) = f(x) \end{aligned}$$

for all $x \in A$. So the bijective mapping $f : A \to B$ is an isomorphism. □

Theorem 3.2. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *and* (3.1). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is an isomorphism.*

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.2, there exists a homomorphism $H : A \to B$ satisfying (2.6). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

We investigate isomorphisms between quasi-Banach algebras, associated to the Jensen functional equation.

Theorem 3.3. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping with $f(0) = 0$ satisfying* (2.8) *and* (3.1). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{3^n} f(3^n e) = e'$, then the mapping $f : A \to B$ is an isomorphism.*

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.3, there exists a homomorphism $H : A \to B$ satisfying (2.9). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} f\left(3^n x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.4. *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping with $f(0) = 0$ satisfying* (2.8) *and* (3.1). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} 3^n f(\frac{e}{3^n}) = e'$, then the mapping $f : A \to B$ is an isomorphism.*

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.4, there exists a homomorphism $H : A \to B$ satisfying (2.12). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

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