Bull. Korean Math. Soc. **45** (2008), No. 1, pp. 111–118

ISOMORPHISMS IN QUASI-BANACH ALGEBRAS

CHOONKIL PARK AND JONG SU AN

ABSTRACT. Using the Hyers–Ulam–Rassias stability method, we investigate isomorphisms in quasi-Banach algebras and derivations on quasi-Banach algebras associated with the Cauchy–Jensen functional equation

$$2f(\frac{x+y}{2}+z) = f(x) + f(y) + 2f(z),$$

which was introduced and investigated in [2, 17]. The concept of Hyers–Ulam–Rassias stability originated from the Th. M. Rassias' stability theorem that appeared in the paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300. Furthermore, isometries and isometric isomorphisms in quasi-Banach algebras are studied.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Received July 10, 2007.

2000 *Mathematics Subject Classification.* Primary 46B03, 47Jxx, 47B48.

Key words and phrases. Cauchy–Jensen functional equation, isomorphism, isometry, derivation, quasi-Banach algebra.

Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th. M. Rassias [19] introduced the following inequality: Assume that there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th. M. Rassias [19] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$. The above inequality has provided a lot of influence in the development of what is now known as *Hyers–Ulam–Rassias stability* of functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [13]-[16], [20], [22]-[25]).

Gilányi [8] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \le \|f(x+y)\|, \tag{1.1}$$

then f satisfies the Jordan–von Neumann functional inequality

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [27]. Fechner [7] and Gilányi [9] proved the Hyers–Ulam–Rassias stability of the functional inequality (1.1). Park, Cho, and Han [18] proved the Hyers–Ulam–Rassias stability of functional inequalities associated with Jordan–von Neumann type additive functional equations.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 ([5], [28]). Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [28] (see also [5]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.2 ([1]). Let $(A, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(A, \|\cdot\|)$ is called a *quasi-normed algebra* if A is an algebra and there is a constant $K > 0$ such that $\|xy\| \leq K\|x\| \cdot \|y\|$ for all $x, y \in A$.

A *quasi-Banach algebra* is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a *p-norm* then the quasi-Banach algebra is called a *p-Banach algebra.*

Throughout this paper, assume that A is a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and unit e and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$ and unit e'.

This paper is organized as follows: In Section 2, we investigate isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen additive mapping.

In Section 3, we investigate derivations on quasi-Banach algebras associated with the Cauchy–Jensen additive mapping.

In Section 4, we investigate isometries and isometric isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen additive mapping.

2. Isomorphisms in quasi-Banach algebras

Definition 2.1. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism in quasi-Banach algebras* if $H(xy) = H(x)H(y)$ for all $x, y \in A$. If, in addition, the mapping $H : A \to B$ is bijective, then the mapping $H : A \to B$ is called an *isomorphism in quasi-Banach algebras.*

In this section, we investigate isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen functional equation.

Theorem 2.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a bijective mapping such that*

$$\|\mu f(x) + f(y) + 2f(z)\|_B \leq \|2f(\frac{\mu x + y}{2} + z)\|_B, \tag{2.1}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{2.2}$$

for $\mu = 1, i$ and all $x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the bijective mapping $f : A \to B$ is an isomorphism in quasi-Banach algebras.

Proof. Let $\mu = 1$ in (2.1). By Proposition 2.3 of [18], the mapping $f : A \to B$ is Cauchy additive. By Theorem of [19], the mapping $f : A \to B$ is $\mathbb{R}$-linear.

Letting $\mu = i$, $z = 0$ and $y = -ix$ in (2.1), we get

$$if(x) - f(ix) = if(x) + f(-ix) = 0$$

for all $x \in A$. So $f(ix) = if(x)$ for all $x \in A$. For each $\lambda \in \mathbb{C}$, $\lambda = a + ib$ $(a, b \in \mathbb{R})$. Hence

$$f(\lambda x) = f(ax + ibx) = af(x) + bf(ix) = af(x) + ibf(x) = \lambda f(x)$$

for all $x \in A$. Thus $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.2),

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)f(2^n y)\|_B \\ &\le \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

Therefore, the bijective mapping $f : A \to B$ is an isomorphism in quasi-Banach algebras, as desired. □

Theorem 2.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a bijective mapping satisfying* (2.1) *such that*

$$\|f(xy) - f(x)f(y)\|_B \le \theta \cdot \|w\|_A^r \cdot \|x\|_A^r \tag{2.3}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the bijective mapping $f : A \to B$ is an isomorphism in quasi-Banach algebras.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.3),

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)f(2^n y)\|_B \\ &\le \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0 \end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

Therefore, the bijective mapping $f : A \to B$ is an isomorphism in quasi-Banach algebras, as desired. □

3. Derivations on quasi-Banach algebras

Definition 3.1. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We investigate derivations on quasi-Banach algebras associated with the Cauchy–Jensen functional equation.

Theorem 3.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping such that*

$$\|\mu f(x) + f(y) + 2f(z)\|_A \leq \|2f(\frac{\mu x + y}{2} + z)\|_A, \tag{3.1}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{3.2}$$

for $\mu = 1, i$ and all $x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to A$ is a derivation on A.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.2),

$$\begin{aligned}\|f(xy) - f(x)y - xf(y)\|_A &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)\cdot 2^n y - 2^n x f(2^n y)\|_A \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|x\|_A^{2r} + \|y\|_A^{2r}) = 0\end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

Therefore, the mapping $f : A \to A$ is a derivation on A, as desired. □

Theorem 3.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying (3.1) such that*

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.3}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to A$ is a derivation on A.

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.2. □

4. Isometries and isometric isomorphisms in quasi-Banach algebras

Surjective isometries between normed vector spaces have been investigated by several authors ([4], [6], [11], [12], [21], [26]).

Definition 4.1. A mapping $I : A \to B$ is called an *isometry in quasi-Banach algebras* if

$$\|I(x) - I(y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.

We investigate isometries in quasi-Banach algebras associated with the Cauchy-Jensen functional equation.

Theorem 4.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.1) *such that*

$$| \ \|f(x)\|_B - \|x\|_A \ | \le \theta \|x\|_A^r \tag{4.1}$$

for all $x \in A$. Then the mapping $f : A \to B$ is an isometry in quasi-Banach algebras.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is additive.

(i) Assume that $r < 1$. It follows from (4.1) that

$$| \ \|f(x)\|_B - \|x\|_A \ | = \lim_{n\to\infty} \frac{1}{2^n} | \ \|f(2^n x)\|_B - \|2^n x\|_A \ | \le \lim_{n\to\infty} \frac{2^{nr}}{2^n} \theta \|x\|_A^r = 0$$

for all $x \in A$. So $\|f(x)\|_B = \|x\|_A$ for all $x \in A$. Since $f : A \to B$ is additive,

$$\|f(x) - f(y)\|_B = \|f(x - y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$\|f(x) - f(y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.

Therefore, the mapping $f : A \to B$ is an isometry in quasi-Banach algebras. □

Definition 4.3. A quasi-Banach algebra isomorphism $H : A \to B$ is called an *isometric isomorphism in quasi-Banach algebras* if H is an isometry in quasi-Banach algebras.

We investigate isometric isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen functional equation.

Theorem 4.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a bijective mapping satisfying (2.1), (2.2) and (4.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is an isometric isomorphism in quasi-Banach algebras.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 4.2. □

References

[1] J. M. Almira and U. Luther, *Inverse closedness of approximation algebras*,J. Math. Anal. Appl. **314** (2006), no. 1, 30–44.

[2] C. Baak, *Cauchy-Rassias stability of Cauchy-Jensen additive mappings in Banach spaces*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 6, 1789–1796.

[3] C. Baak and M. S. Moslehian, *On the stability of θ-derivations on* JB*-*triples*, Bull. Braz. Math. Soc. (N.S.) **38** (2007), no. 1, 115–127.

[4] J. Baker, *Isometries in normed spaces*, Amer. Math. Monthly **78** (1971), 655–658.

[5] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.

[6] J. Bourgain, *Real isomorphic complex Banach spaces need not be complex isomorphic*, Proc. Amer. Math. Soc. **96** (1986), no. 2, 221–226.

[7] W. Fechner, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. 71 (2006), no. 1-2, 149–161.

[8] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), no. 3, 303–309.

[9] ———, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), no. 4, 707–710.

[10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.

[11] N. Kalton, *An elementary example of a Banach space not isomorphic to its complex conjugate*, Canad. Math. Bull. **38** (1995), no. 2, 218–222.

[12] S. Mazur and S. Ulam, *Sur les transformation d'espaces vectoriels normé*, C. R. Acad. Sci. Paris **194** (1932), 946–948.

[13] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. (N.S.) **37** (2006), no. 3, 361–376.

[14] C. Park, *On an approximate automorphism on a C^*-algebra*, Proc. Amer. Math. Soc. **132** (2004), no. 6, 1739–1745.

[15] ———, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. (N.S.) **36** (2005), no. 1, 79–97.

[16] ———, *A generalized Jensen's mapping and linear mappings between Banach modules*, Bull. Braz. Math. Soc. (N.S.) **36** (2005), no. 3, 333–362.

[17] ———, *Isomorphisms between C^*-ternary algebras*, J. Math. Phys. **47**, no. 10, Article ID 103512 (2006), 12 pages.

[18] C. Park, Y. Cho, and M. Han, *Functional inequalities associated with Jordan–von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Article ID 41820 (2007), 13 pages.

[19] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.

[20] ———, *Problem 16; 2*, Report of the 27^{th} Internat. Symp. on Functional Equations, Aequationes Math. **39** (1990), 292–293; 309.

[21] ———, *Properties of isometic mappings*, J. Math. Anal. Appl. **235** (1997), 108–121.

[22] ———, *The problem of S. M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), no. 2, 352–378.

[23] ———, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), no. 1, 264–284.

[24] ———, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), no. 1, 23–130.

[25] ———, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 2003.

[26] Th. M. Rassias and P. Šemrl, *On the Mazur-Ulam theorem and the Aleksandrov problem for unit distance preserving mappings*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 919–925.

[27] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), no. 1-2, 191–200.

[28] S. Rolewicz, *Metric Linear Spaces*, Second edition. PWN—Polish Scientific Publishers, Warsaw; D. Reidel Publishing Co., Dordrecht, 1984.

[29] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

Jong Su An
Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail address: jsan63@pusan.ac.kr