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Research Article

Isomorphisms and Derivations in Lie C^*-Algebras

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Recommended by John Michael Rassias

We investigate isomorphisms between C^*-algebras, Lie C^*-algebras, and JC^*-algebras, and derivations on C^*-algebras, Lie C^*-algebras, and JC^*-algebras associated with the Cauchy–Jensen functional equation $2f((x+y/2)+z) = f(x)+f(y)+2f(z)$.

1. Introduction and preliminaries

Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, containing the stability problem of homomorphisms. Hyers [2] proved the stability problem of additive mappings in Banach spaces. Rassias [3] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded:* Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. The inequality (1.1) that was introduced by Rassias [3] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations. Găvruta [4] provided a further generalization of Th. M. Rassias' theorem. Several mathematicians have contributed works on these subjects (see [4–14]).

Rassias [15] provided an alternative generalization of Hyers' stability theorem which allows the *Cauchy difference to be unbounded*, as follows.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$||f(x+y) - f(x) - f(y)|| \le \epsilon \|x\|^p \|y\|^p \tag{1.2}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < 1/2$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.3}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$||f(x) - L(x)|| \le \frac{\epsilon}{2 - 4^p} \|x\|^{2p} \tag{1.4}$$

for all $x \in E$. If $p < 0$, then inequality (1.2) holds for $x, y \ne 0$, and (1.4) for $x \ne 0$. If $p > 1/2$, then inequality (1.2) holds for all $x, y \in E$, and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{1.5}$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$||f(x) - A(x)|| \le \frac{\epsilon}{4^p - 2} \|x\|^{2p} \tag{1.6}$$

for all $x \in E$.

In 1982–1994, a generalization of this result was established by J. M. Rassias with a weaker (unbounded) condition controlled by (or involving) a product of different powers of norms. However, there was a singular case. Then for this singularity, a counterexample was given by Găvruta [16]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Găvruta-Rassias stability by Sibaha et al. [17] and Ravi and Arunkumar[18]. This stability is called Hyers-Ulam-Rassias stability involving a product of different powers of norms by Park [10]. Note that both Ulam stabilities specifically called: “Ulam-Găvruta-Rassias stability of mappings” and “Hyers-Ulam-Rassias stability of mappings involving a product of powers of norms are identical in meaning stability notions. Besides Euler-Lagrange quadratic mappings were introduced by Rassias [19], motivated from the pertinent algebraic quadratic equation. Thus he introduced and investigated the relative quadratic functional equation [20, 21]. In addition, he generalized and investigated the general pertinent Euler-Lagrange quadratic mappings [22]. Analogous quadratic mappings were introduced and investigated by the same author [23, 24]. Therefore, this introduction of Euler-Lagrange mappings and equations in functional equations and inequalities provided an interesting cornerstone in analysis, because this kind of Euler-Lagrange-Rassias mappings (resp., Euler-Lagrange-Rassias equations) is of particular interest in probability theory and stochastic analysis by marrying these fields of research results to functional equations and inequalities via the introduction of new Euler-Lagrange-Rassias quadratic weighted means and Euler-Lagrange-Rassias fundamental mean equations [21, 22, 25]. For further research developments in stability of functional equations, the readers are referred to the works of Park [6–13], Rassias [15, 19–24, 26–36], J. M. Rassias and M. J. Rassias [25, 37–39], Rassias [40–43], Skof [44], and the references cited therein.

Gilányi [45] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\|, \tag{1.7}$$

then f satisfies the Jordan-von Neumann functional inequality

$$2f(x)+2f(y)=f(x+y)+f(x-y) \tag{1.8}$$

(see also [46]). Fechner [47] and Gilányi [48] proved the Hyers-Ulam-Rassias stability of the functional inequality (1.7). Park et al.[11] proved the Hyers-Ulam-Rassias stability of functional inequalities associated with Jordan-von Neumann-type additive functional equations.

Jordan observed that $\mathscr{L}(\mathscr{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := (xy+yx)/2$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra*. A Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra*. A C^*-algebra $\mathscr{C}$, endowed with the Lie product $[x,y]=(xy-yx)/2$ on $\mathscr{C}$, is called a *Lie C^*-algebra* (see [6, 7, 13]).

This paper is organized as follows. In Section 2, we investigate isomorphisms and derivations in C^*-algebras associated with the Cauchy-Jensen functional equation. In Section 3, we investigate isomorphisms and derivations in Lie C^*-algebras associated with the Cauchy-Jensen functional equation. In Section 4, we investigate isomorphisms and derivations in JC^*-algebras associated with the Cauchy-Jensen functional equation.

2. Isomorphisms and derivations in C^*-algebras

Throughout this section, assume that A is a C^*-algebra with norm $\|\cdot\|_A$, and that B is a C^*-algebra with norm $\|\cdot\|_B$.

Lemma 2.1 [11]. *Let $f: A \to B$ be a mapping such that*

$$\|f(x)+f(y)+2f(z)\|_B \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_B \tag{2.1}$$

for all $x,y,z \in A$. Then f is Cauchy additive, that is, $f(x+y)=f(x)+f(y)$.

In this section, we investigate C^*-algebra isomorphisms between C^*-algebras and linear derivations on C^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 2.2. *Let $r>1$ and θ be nonnegative real numbers, and let $f: A \to B$ be a bijective mapping such that*

$$\|\mu f(x)+f(y)+2f(z)\|_B \le \left\|2f\left(\frac{\mu x+y}{2}+z\right)\right\|_B, \tag{2.2}$$

$$\|f(xy)-f(x)f(y)\|_B \le \theta(\|x\|_A^{2r}+\|y\|_A^{2r}), \tag{2.3}$$

$$\|f(x^*)-f(x)^*\|_B \le \theta(\|x\|_A^{r}+\|x\|_A^{r}) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ *and all* $x, y, z \in A$. *Then the mapping* $f : A \to B$ *is a* C^**-algebra isomorphism.*

Proof. Let $\mu = 1$ in (2.2). By Lemma 2.1, the mapping $f : A \to B$ is Cauchy additive. So $f(0) = 0$ and $f(x) = \lim_{n\to\infty} 2^n f(x/2^n)$ for all $x \in A$.

Letting $y = -\mu x$ and $z = 0$, we get

$$||\mu f(x) + f(-\mu x)||_B \leq ||2f(0)||_B = 0 \tag{2.5}$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$.So

$$\mu f(x) - f(\mu x) = \mu f(x) + f(-\mu x) = 0 \tag{2.6}$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [8, Theorem 2.1], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.3) that

$$\begin{aligned} ||f(xy) - f(x)f(y)||_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right) \right\|_B \\ &\leq \lim_{n\to\infty} \frac{4^n \theta}{4^{nr}} \left(\|x\|_A^{2r} + \|y\|_A^{2r}\right) = 0 \end{aligned} \tag{2.7}$$

for all $x, y \in A$. Thus

$$f(xy) = f(x)f(y) \tag{2.8}$$

for all $x, y \in A$.

It follows from (2.4) that

$$\begin{aligned} ||f(x^*) - f(x)^*||_B &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^* \right\|_B \\ &\leq \lim_{n\to\infty} \frac{2^n \theta}{2^{nr}} \left(\|x\|_A^r + \|x\|_A^r\right) = 0 \end{aligned} \tag{2.9}$$

for all $x \in A$. Thus

$$f(x^*) = f(x)^* \tag{2.10}$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

Theorem 2.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2), (2.3), and (2.4). Then the mapping $f : A \to B$ is a C^*-algebra isomorphism.*

Proof. The proof is similar to the proof of Theorem 2.2. □

Theorem 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that*

$$||f(xy) - f(x)y - xf(y)||_A \le \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{2.11}$$

for all $x, y \in A$.Then the mapping $f : A \to A$ is a linear derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (2.11) that

$$\begin{aligned} ||f(xy) - f(x)y - xf(y)||_A &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right)\frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}\right)\right\|_A \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}}(\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned} \tag{2.12}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y) \tag{2.13}$$

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a linear derivation. □

Theorem 2.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (2.11). Then the mapping $f : A \to A$ is a linear derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 2.4. □

Theorem 2.6. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that*

$$||f(xy) - f(x)f(y)||_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.14}$$

$$||f(x^*) - f(x)^*||_B \le \theta \cdot \|x\|_A^{r/2} \cdot \|x\|_A^{r/2} \tag{2.15}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then the mapping $f : A \to B$ is a C^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.14) that

$$\begin{aligned} ||f(xy) - f(x)f(y)||_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0 \end{aligned} \tag{2.16}$$

for all $x, y \in A$. Thus

$$f(xy) = f(x)f(y) \tag{2.17}$$

for all $x, y \in A$.

It follows from (2.15) that

$$\begin{aligned} \|f(x^*) - f(x)^*\|_B &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^* \right\|_B \\ &\leq \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}} \cdot \|x\|_A^{r/2} \cdot \|x\|_A^{r/2} = 0 \end{aligned} \tag{2.18}$$

for all $x \in A$. Thus

$$f(x^*) = f(x)^* \tag{2.19}$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a C^*-algebra isomorphism. □

Theorem 2.7. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2), (2.14), and (2.15). Then the mapping $f : A \to B$ is a C^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 2.6. □

Theorem 2.8. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that*

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.20}$$

for all $x, y \in A$.Then the mapping $f : A \to A$ is a linear derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (2.20) that

$$\begin{aligned} \|f(xy) - f(x)y - xf(y)\|_A &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right)\frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}\right) \right\|_A \\ &\leq \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0 \end{aligned} \tag{2.21}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y) \tag{2.22}$$

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a linear derivation. □

Theorem 2.9. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (2.20). Then the mapping $f : A \to A$ is a linear derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 2.8. □

3. Isomorphisms and derivations in Lie C^*-algebras

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$, and that B is a Lie C^*-algebra with norm $\|\cdot\|_B$.

Definition 3.1 [6, 7, 13]. A bijective $\mathbb{C}$-linear mapping $H : A \to B$ is called a *Lie C^*-algebra isomorphism* if $H : A \to B$ satisfies

$$H([x,y]) = [H(x), H(y)] \tag{3.1}$$

for all $x, y \in A$.

Definition 3.2 [6, 7, 13]. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Lie derivation* if $D : A \to A$ satisfies

$$D([x,y]) = [Dx, y] + [x, Dy] \tag{3.2}$$

for all $x, y \in A$.

In this section, we investigate Lie C^*-algebra isomorphisms between Lie C^*-algebras and Lie derivations on Lie C^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that*

$$\|f([x,y]) - [f(x), f(y)]\|_B \le \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{3.3}$$

for all $x, y \in A$. Then the mapping $f : A \to B$ is a Lie C^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (3.3) that

$$\begin{aligned} \|f([x,y]) - [f(x), f(y)]\|_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x,y]}{2^n \cdot 2^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right]\right\|_B \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} (\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned} \tag{3.4}$$

for all $x, y \in A$. Thus

$$f([x,y]) = [f(x), f(y)] \tag{3.5}$$

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a Lie C^*-algebra isomorphism, as desired. □

Theorem 3.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (3.3). Then the mapping $f : A \to B$ is a Lie C^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.3. □

Theorem 3.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \le \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{3.6}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Lie derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.6) that

$$\begin{aligned}\|f([x,y]) &- [f(x),y] - [x,f(y)]\|_A \\ &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x,y]}{4^n}\right) - \left[f\left(\frac{x}{2^n}\right), \frac{y}{2^n}\right] - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right)\right]\right\|_A \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}}(\|x\|_A^{2r} + \|y\|_A^{2r}) = 0\end{aligned} \tag{3.7}$$

for all $x, y \in A$. So

$$f([x,y]) = [f(x),y] + [x,f(y)] \tag{3.8}$$

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a Lie derivation. □

Theorem 3.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (3.6). Then the mapping $f : A \to A$ is a Lie derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.5. □

Theorem 3.7. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that*

$$\|f([x,y]) - [f(x),f(y)]\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.9}$$

for all $x, y \in A$. Then the mapping $f : A \to B$ is a Lie C^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (3.9) that

$$\begin{aligned}\|f([x,y]) - [f(x),f(y)]\|_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x,y]}{2^n \cdot 2^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right]\right\|_B \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0\end{aligned} \tag{3.10}$$

for all $x, y \in A$. Thus

$$f([x,y]) = [f(x),f(y)] \tag{3.11}$$

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a Lie C^*-algebra isomorphism, as desired. □

Theorem 3.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (3.9). Then the mapping $f : A \to B$ is a Lie C^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.2, 2.6, and 3.7. □

Theorem 3.9. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.12}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Lie derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.12) that

$$\begin{aligned}
&\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \\
&\quad = \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x,y]}{4^n}\right) - \left[f\left(\frac{x}{2^n}\right), \frac{y}{2^n}\right] - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right)\right]\right\|_A \\
&\quad \leq \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0
\end{aligned} \tag{3.13}$$

for all $x, y \in A$. So

$$f([x,y]) = [f(x),y] + [x,f(y)] \tag{3.14}$$

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a Lie derivation. □

Theorem 3.10. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2)and (3.12). Then the mapping $f : A \to A$ is a Lie derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2, 2.8, and 3.9. □

4. Isomorphisms and derivations in JC^*-algebras

Throughout this section, assume that A is a JC^*-algebra with norm $\|\cdot\|_A$, and that B is a JC^*-algebra with norm $\|\cdot\|_B$.

Definition 4.1 [7, 13]. A bijective $\mathbb{C}$-linear mapping $H : A \to B$ is called a *JC^*-algebra isomorphism* if $H : A \to B$ satisfies

$$H(x \circ y) = H(x) \circ H(y) \tag{4.1}$$

for all $x, y \in A$.

Definition 4.2 [7, 13]. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Jordan derivation* if $D : A \to A$ satisfies

$$D(x \circ y) = Dx \circ y + x \circ Dy \tag{4.2}$$

for all $x, y \in A$.

In this section, we investigate JC^*-algebra isomorphisms between JC^*-algebras and Jordan derivations on JC^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that*

$$\|f(x \circ y) - f(x) \circ f(y)\|_B \le \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{4.3}$$

for all $x, y \in A$. Then the mapping $f : A \to B$ is a JC^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (4.3) that

$$\begin{aligned} \|f(x \circ y) - f(x) \circ f(y)\|_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x \circ y}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) \circ f\left(\frac{y}{2^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} (\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned} \tag{4.4}$$

for all $x, y \in A$. Thus

$$f(x \circ y) = f(x) \circ f(y) \tag{4.5}$$

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism, as desired. □

Theorem 4.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (4.3). Then the mapping $f : A \to B$ is a JC^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 4.3. □

Theorem 4.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that*

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_A \le \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) \tag{4.6}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Jordan derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (4.6) that

$$\begin{aligned} \|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_A &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x \circ y}{4^n}\right) - f\left(\frac{x}{2^n}\right) \circ \frac{y}{2^n} - \frac{x}{2^n} \circ f\left(\frac{y}{2^n}\right)\right\|_A \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} (\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned} \tag{4.7}$$

for all $x, y \in A$. So

$$f(x \circ y) = f(x) \circ y + x \circ f(y) \tag{4.8}$$

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a Jordan derivation. □

THEOREM 4.6. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (4.6). Then the mapping $f : A \to A$ is a Jordan derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 4.5. □

THEOREM 4.7. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) such that*

$$\|f(x \circ y) - f(x) \circ f(y)\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.9}$$

for all $x, y \in A$. Then the mapping $f : A \to B$ is a JC^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (4.9) that

$$\begin{aligned}\|f(x \circ y) - f(x) \circ f(y)\|_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x \circ y}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) \circ f\left(\frac{y}{2^n}\right) \right\|_B \\ &\leq \lim_{n\to\infty} \frac{4^n \theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0\end{aligned} \tag{4.10}$$

for all $x, y \in A$. Thus

$$f(x \circ y) = f(x) \circ f(y) \tag{4.11}$$

for all $x, y \in A$. Hence the bijective mapping $f : A \to B$ is a JC^*-algebra isomorphism, as desired. □

THEOREM 4.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.2) and (4.9). Then the mapping $f : A \to B$ is a JC^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.2, 2.6, and 4.7. □

THEOREM 4.9. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) such that*

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.12}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Jordan derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (4.6) that

$$\begin{aligned}\|f(x\circ y)-f(x)\circ y-x\circ f(y)\|_A &= \lim_{n\to\infty}4^n\left\|f\left(\frac{x\circ y}{4^n}\right)-f\left(\frac{x}{2^n}\right)\circ\frac{y}{2^n}-\frac{x}{2^n}\circ f\left(\frac{y}{2^n}\right)\right\|_A\\ &\le \lim_{n\to\infty}\frac{4^n\theta}{4^{nr}}\cdot\|x\|_A^r\cdot\|y\|_A^2=0\end{aligned} \tag{4.13}$$

for all $x, y \in A$. So

$$f(x\circ y)=f(x)\circ y+x\circ f(y) \tag{4.14}$$

for all $x, y \in A$. Thus the mapping $f : A \to A$ is a Jordan derivation. □

Theorem 4.10. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (2.2) and (4.12). Then the mapping $f : A \to A$ is a Jordan derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2, 2.8, and 4.9. □

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