

Research Article

Generalized Stability of C^* -Ternary Quadratic Mappings

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We prove the generalized stability of C^* -ternary quadratic mappings in C^* -ternary rings for the quadratic functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.

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1. Introduction and preliminaries

A C^* -ternary ring is a complex Banach space A , equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A , which is \mathbb{C} -linear in the outer variables, conjugate \mathbb{C} -linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [1]).

If a C^* -ternary ring $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A , endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^* -algebra. Conversely, if (A, \circ) is a unital C^* -algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^* -ternary ring (see [2]).

Ulam [3] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, containing the stability problem of homomorphisms. Hyers [4] proved the stability problem of additive mappings in Banach spaces. Rassias [5] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded*: let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \quad (1.1)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Inequality (1.1) provided a lot of influence in the development of a generalization of the Hyers-Ulam stability

concept. Găvruta [6] provided a further generalization of Hyers-Ulam theorem (see [7, 8]).

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.2}$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.3}$$

is called the *quadratic functional equation* whose solution is said to be a *quadratic mapping*. A generalized stability problem for the quadratic functional equation was proved by Skof [9] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [11] proved the generalized stability of the quadratic functional equation, and Park [12] proved the generalized stability of the quadratic functional equation in Banach modules over a C^* -algebra. Jun and Lee [13] proved the further generalized stability of a Pexiderized quadratic functional equation

$$f(x + y) + g(x - y) = 2h(x) + 2k(y). \tag{1.4}$$

Recently, a fixed point approach to the stability of Pexiderized quadratic equation was established by Mirzavaziri and Moslehian [14].

Throughout this paper, assume that A is a C^* -ternary ring with norm $\| \cdot \|_A$ and that B is a C^* -ternary ring with norm $\| \cdot \|_B$.

A quadratic mapping $Q : A \rightarrow B$ is called a *C^* -ternary quadratic mapping* if

$$Q([x, y, z]) = [Q(x), Q(y), Q(z)] \tag{1.5}$$

for all $x, y, z \in A$.

Example 1.1. Let $(A, [\cdot, \cdot, \cdot])$ be a C^* -ternary ring derived from a unital commutative C^* -algebra A , and let $Q : A \rightarrow A$ satisfy $Q(x) = x^2$ for all $x \in A$. It is easy to show that the mapping $Q : A \rightarrow A$ is a C^* -ternary quadratic mapping.

In this paper, we prove the further generalized stability of C^* -ternary quadratic mappings in C^* -ternary rings.

2. Stability of C^* -ternary quadratic mappings

We prove the further generalized stability of C^* -ternary quadratic mappings in C^* -ternary rings for the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y). \tag{2.1}$$

THEOREM 2.1. Let $f : A \rightarrow B$ be a mapping for which there exists a function $\varphi : A^3 \rightarrow [0, \infty)$ such that

$$\sum_{j=0}^{\infty} 4^{3j} \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \quad (2.2)$$

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_B \leq \varphi(x, y, 0), \quad (2.3)$$

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \varphi(x, y, z) \quad (2.4)$$

for all $x, y, z \in A$. Then there exists a unique C^* -ternary quadratic mapping $Q : A \rightarrow B$ such that

$$\|f(x) - Q(x)\|_B \leq \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}, 0\right) \quad (2.5)$$

for all $x \in A$. Here,

$$\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \quad (2.6)$$

for all $x, y, z \in A$.

Proof. It follows from (2.3) that $f(0) = 0$. Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\|_B \leq \varphi(x, x, 0) \quad (2.7)$$

for all $x \in A$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\|_B \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \quad (2.8)$$

for all $x \in A$. Hence,

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|_B \leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \quad (2.9)$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.9) that the sequence $\{4^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{4^n f(x/2^n)\}$ converges. So one can define the mapping $Q : A \rightarrow B$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right) \quad (2.10)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.9), we get (2.5).

4 Abstract and Applied Analysis

It follows from (2.3) that

$$\begin{aligned}
 & \|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\|_B \\
 &= \lim_{n \rightarrow \infty} 4^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right\|_B \\
 &\leq \lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right) = 0
 \end{aligned} \tag{2.11}$$

for all $x, y \in A$. So

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{2.12}$$

for all $x, y \in A$.

It follows from (2.4) and the continuity of the ternary product that

$$\begin{aligned}
 & \|Q([x, y, z]) - [Q(x), Q(y), Q(z)]\|_B \\
 &= \lim_{n \rightarrow \infty} 4^{3n} \left\| f\left(\frac{[x, y, z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right) \right] \right\|_B \\
 &\leq \lim_{n \rightarrow \infty} 4^{3n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
 \end{aligned} \tag{2.13}$$

for all $x, y, z \in A$. So

$$Q([x, y, z]) = [Q(x), Q(y), Q(z)] \tag{2.14}$$

for all $x, y, z \in A$.

Now, let $T : A \rightarrow B$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned}
 \|Q(x) - T(x)\|_B &= 4^n \left\| Q\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|_B \\
 &\leq 4^n \left(\left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_B + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_B \right) \\
 &\leq 2 \cdot 4^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, 0\right),
 \end{aligned} \tag{2.15}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in A$. So we can conclude that $Q(x) = T(x)$ for all $x \in A$. This proves the uniqueness of Q . Thus, the mapping $Q : A \rightarrow B$ is a unique C^* -ternary quadratic mapping satisfying (2.5). \square

THEOREM 2.2. *Let $f : A \rightarrow B$ be a mapping for which there exists a function $\varphi : A^3 \rightarrow [0, \infty)$ satisfying (2.3) and (2.4) such that*

$$\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{2.16}$$

for all $x, y, z \in A$. Then there exists a unique C^* -ternary quadratic mapping $Q : A \rightarrow B$ such that

$$\|f(x) - Q(x)\|_B \leq \frac{1}{4} \tilde{\varphi}(x, x, 0) \tag{2.17}$$

for all $x \in A$.

Proof. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\|_B \leq \frac{1}{4} \varphi(x, x, 0) \tag{2.18}$$

for all $x \in A$. So

$$\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\|_B \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\|_B \leq \sum_{j=l}^{m-1} \frac{1}{4^{j+1}} \varphi(2^j x, 2^j x, 0) \tag{2.19}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.19) that the sequence $\{(1/4^n) f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{(1/4^n) f(2^n x)\}$ converges. So one can define the mapping $Q : A \rightarrow B$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x) \tag{2.20}$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.19), we get (2.17).

It follows from (2.4) and the continuity of the ternary product that

$$\begin{aligned} & \|Q([x, y, z]) - [Q(x), Q(y), Q(z)]\|_B \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^{3n}} \|f(2^{3n}[x, y, z]) - [f(2^n x), f(2^n y), f(2^n z)]\|_B \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{4^{3n}} \varphi(2^n x, 2^n y, 2^n z) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z) = 0 \end{aligned} \tag{2.21}$$

for all $x, y, z \in A$. So

$$Q([x, y, z]) = [Q(x), Q(y), Q(z)] \tag{2.22}$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 2.3. For a Pexiderized quadratic functional equation

$$f(x + y) + g(x - y) = 2h(x) + 2k(y), \tag{2.23}$$

one can obtain similar results to Theorems 2.1 and 2.2.

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