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Research Article

Homomorphisms and Derivations in C^*-Algebras

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Using the Hyers-Ulam-Rassias stability method of functional equations, we investigate homomorphisms in C^*-algebras, Lie C^*-algebras, and JC^*-algebras, and derivations on C^*-algebras, Lie C^*-algebras, and JC^*-algebras associated with the following Apollonius-type additive functional equation $f(z-x)+f(z-y)+(1/2)f(x+y)=2f(z-(x+y)/4)$.

1. Introduction and preliminaries

Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $\rho(f(x),h(x))<\epsilon$ for all $x\in G$?

Hyers [2] considered the case of approximately additive mappings $f:E\to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$||f(x+y)-f(x)-f(y)||\leq\epsilon \tag{1.1}$$

for all $x,y\in E$. It was shown that the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n} \tag{1.2}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$||f(x) - L(x)|| \le \epsilon. \tag{1.3}$$

The famous Hyers stability result that appeared in [2] was generalized in the stability involving a sum of powers of norms by Aoki [3]. Th. M. Rassias [4] and J. M. Rassias [5] provided generalizations of Hyers' theorem which allow the *Cauchy difference to be unbounded.*

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$||f(x+y) - f(x) - f(y)|| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.4}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.5}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$||f(x) - L(x)|| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.6}$$

for all $x \in E$. If $p < 0$, then inequality (1.4) holds for $x, y \neq 0$ and (1.6) for $x \neq 0$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

Theorem 1.2 (J. M. Rassias). *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \ge 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies inequality*

$$||f(x+y) - f(x) - f(y)|| \le \theta \cdot ||x||^p \cdot ||y||^q \tag{1.7}$$

for all $x, y \in X$.Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$||f(x) - L(x)|| \le \frac{\theta}{|2^r - 2|}||x||^r \tag{1.8}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Th. M. Rassias [6] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [7], following the same approach as in Th. M. Rassias [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [7], as well as by Th. M. Rassias and Šemrl [8], that one cannot prove Th. M. Rassias' theorem when $p = 1$. The counterexamples of Gajda [7], as well as of Th. M. Rassias and Šemrl [8] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings (cf. Găvruta [9], Jung [10]) who among others studied the stability of functional equations.

In 1982–1994, a generalization of this result was established by J. M. Rassias with a weaker (unbounded) condition controlled by (or involving) a product of different powers of norms. However, there was a singular case. Then for this singularity, a counterxample was given by Găvruta [11]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Găvruta-Rassias stability by Sibaha et al. [12] and Ravi and Arunkumar [13]. This stability is called Hyers-Ulam-Rassias stability involving a product of different powers of norms by Park [14]. Note that both Ulam stabilities specifically called: "Ulam-Găvruta-Rassias stability of mappings" and "Hyers-Ulam-Rassias stability of mappings involving a product of powers of norms" are identical in meaning stability notions. Besides Euler-Lagrange quadratic mappings were introduced by J. M. Rassias [15], motivated from the pertinent algebraic quadratic equation. Thus, he introduced and investigated the relative quadratic functional equation [16, 17]. In addition, he generalized and investigated the general pertinent Euler-Lagrange quadratic mappings [18]. Analogous quadratic mappings were introduced and investigated by the same author [19, 20]. Therefore, these Euler-Lagrange quadratic mappings were named Euler-Lagrange-Rassias mappings and the corresponding Euler-Lagrange quadratic equations were called Euler-Lagrange-Rassias equations by Jun and Kim [21] and Park [22]. Before 1992, these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler-Lagrange partial differential equations was known in calculus of variations. Therefore, this introduction of Euler-Lagrange mappings and equations in functional equations and inequalities provided an interesting cornerstone in analysis, because this kind of Euler-Lagrange-Rassias mappings (resp., Euler-Lagrange-Rassias equations) is of particular interest in probability theory and stochastic analysis by marrying these fields of research results to functional equations and inequalities via the introduction of new Euler-Lagrange-Rassias quadratic weighted means and Euler-Lagrange-Rassias fundamental mean equations [17, 18, 23]. For further research developments in stability of functional equations, the readers are referred to the works of Park [14, 22, 24–29], J. M. Rassias [30, 31, 5, 15–20, 32–40], J. M. Rassias and M. J. Rassias [23, 41–43], Th. M. Rassias [44–47], Skof [48] and the references cited therein.

In an inner product space, the equality

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{x+y}{2}\right\|^2 \tag{1.9}$$

holds and is called the *Apollonius' identity*. The following functional equation, which was motivated by this equation,

$$Q(z-x)+Q(z-y)=\frac{1}{2}Q(x-y)+2Q\left(z-\frac{x+y}{2}\right), \tag{1.10}$$

is quadratic. For this reason, the function equation (1.10) is called a *quadratic functional equation of Apollonius type*, and each solution of the functional equation (1.10) is said to be a *quadratic mapping of Apollonius type*. Jun and Kim [49] investigated the quadratic functional equation of Apollonius type.

In this paper, modifying the above equality (1.10), we introduce a new functional equation, which is called the *Apollonius-type additive functional equation* and whose solution of the functional equation is said to be the *Apollonius-type additive mapping*

$$L(z-x)+L(z-y)=-\frac{1}{2}L(x+y)+2L\left(z-\frac{x+y}{4}\right). \tag{1.11}$$

Gilányi [50] showed that if f has its values in an inner product space and satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.12}$$

then f satisfies the Jordan-von Neumann functional inequality

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}). \tag{1.13}$$

See also [51]. Fechner [52] and Gilányi [53] proved the stability of the functional inequality (1.12). Park et al. [27] proved the stability of functional inequalities associated with Jordan-von-Neumann-type additive functional equations.

In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := (xy+yx)/2$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra*. A Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra*. A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x,y]=(xy-yx)/2$ on $\mathcal{C}$, is called a *Lie C^*-algebra* (see [24, 25, 29]).

In Section 2, we investigate homomorphisms and derivations in C^*-algebras associated with the Apollonius-type additive functional equation.

In Section 3, we investigate homomorphisms and derivations in Lie C^*-algebras associated with the Apollonius-type additive functional equation.

In Section 4, we investigate homomorphisms and derivations in JC^*-algebras associated with the Apollonius-type additive functional equation.

2. Homomorphisms and derivations in C^*-algebras

THEOREM 2.1. *Let A be a uniquely 2-divisible abelian group and B a normed linear space. A mapping $f: A \to B$ satisfies*

$$\left\|f(z-x)+f(z-y)+\frac{1}{2}f(x+y)\right\|_B \leq \left\|2f\left(z-\frac{x+y}{4}\right)\right\|_B \tag{2.1}$$

for all $x,y,z \in A$ if and only if $f: A \to B$ is additive.

Proof. Letting $x=y=z=0$ in (2.1), we get

$$\frac{5}{2}\|f(0)\|_B \leq 2\|f(0)\|_B. \tag{2.2}$$

So $f(0)=0$.

Letting $z=0$ and $y=-x$ in (2.1), we get

$$\|f(-x)+f(x)\|_B \le 2\|f(0)\|_B = 0 \tag{2.3}$$

for all $x \in A$. Hence, $f(-x) = -f(x)$ for all $x \in A$.

Letting $x=y=2z$ in (2.1), we get

$$\left\|2f(-z)+\frac{1}{2}f(4z)\right\|_B \le \|2f(0)\|_B = 0 \tag{2.4}$$

for all $z \in$ A. Hence,

$$f(4z) = -4f(-z) = 4f(z) \tag{2.5}$$

for all $z \in A$.

Letting $z=(x+y)/4$ in (2.1), we get

$$\left\|f\left(\frac{-3x+y}{4}\right)+f\left(\frac{x-3y}{4}\right)+\frac{1}{2}f(x+y)\right\|_B \le \|2f(0)\|_B = 0 \tag{2.6}$$

for all $x,y \in A$. So

$$f\left(\frac{-3x+y}{4}\right)+f\left(\frac{x-3y}{4}\right)+\frac{1}{2}f(x+y) = 0 \tag{2.7}$$

for all $x,y \in$ A. Let $w_1 = (-3x+y)/4$ and $w_2 = (x-3y)/4$ in (2.7). Then

$$f(w_1)+f(w_2) = -\frac{1}{2}f(-2w_1-2w_2) = \frac{1}{2}f(2w_1+2w_2) = 2f\left(\frac{w_1+w_2}{2}\right) \tag{2.8}$$

for all $w_1, w_2 \in A$ and so f is additive.

It is clear that each additive mapping satisfies the inequality (2.1). $\square$

In this section, we investigate C^*-algebra homomorphisms between C^*-algebras and linear derivations on C^*-algebras associated with the Apollonius-type additive functional equation. From now on, assume that A is a C^*-algebra with norm $\|\cdot\|_A$, and that B is a C^*-algebra with norm $\|\cdot\|_B$.

Lemma 2.2 [26]. *Let $f: A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f: A \to B$ be a mapping such that*

$$\left\|f(z-\mu x)+\mu f(z-y)+\frac{1}{2}f(x+y)\right\|_B \le \left\|2f\left(z-\frac{x+y}{4}\right)\right\|_B, \tag{2.9}$$

$$\|f(xy)-f(x)f(y)\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.10}$$

$$\|f(x^*)-f(x)^*\|_B \le 2\theta\|x\|_A^r \tag{2.11}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x,y,z \in A$. Then the mapping $f: A \to B$ is a C^*-algebra homomorphism.

Proof. Let $\mu = 1$ in (2.9). By Theorem 2.1, the mapping $f : A \to B$ is additive.

Letting $y = -x$ and $z = 0$ in (2.9), we get

$$\|f(-\mu x) + \mu f(x)\|_B \leq \|2f(0)\|_B = 0 \tag{2.12}$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So

$$-f(\mu x) + \mu f(x) = f(-\mu x) + \mu f(x) = 0 \tag{2.13}$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence, $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So by Lemma 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.10) that

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right) \right\|_B \\ &\leq \lim_{n\to\infty} \frac{4^n \theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0 \end{aligned} \tag{2.14}$$

for all $x, y \in A$. Thus,

$$f(xy) = f(x)f(y) \tag{2.15}$$

for all $x, y \in$ A.

It follows from (2.11) that

$$\|f(x^*) - f(x)^*\|_B = \lim_{n\to\infty} 2^n \left\| f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^* \right\|_B \leq \lim_{n\to\infty} \frac{2^{n+1}\theta}{2^{nr}} \|x\|_A^r = 0 \tag{2.16}$$

for all $x \in A$. Thus,

$$f(x^*) = f(x)^* \tag{2.17}$$

for all $x \in A$. Hence, the mapping $f : A \to B$ is a C^*-algebra homomorphism. □

THEOREM 2.4. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying (2.9), (2.10), and (2.11). Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.*

Proof. The proof is similar to the proof of Theorem 2.3. □

THEOREM 2.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.9) such that*

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.18}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a linear derivation.

Proof. By the same reasoning as in the proof of Theorem 2.3 and applying Lemma 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (2.18) that

$$\begin{aligned}\|f(xy)-f(x)y-xf(y)\|_A &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right)\frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}\right)\right\|_A \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \cdot \|x\|_A^r \cdot \|y\|_A^r = 0\end{aligned} \tag{2.19}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y) \tag{2.20}$$

for all $x, y \in A$. Thus, the mapping $f : A \to A$ is a linear derivation. □

Theorem 2.6. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (2.9) and (2.18). Then the mapping $f : A \to A$ is a linear derivation.*

Proof. The proof is similar to the proofs of Theorems 2.3 and 2.5. □

3. Homomorphisms and derivations in Lie C^*-algebras

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$, and that B is a Lie C^*-algebra with norm $\|\cdot\|_B$.

Defintion 3.1 [24, 25, 29]. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *Lie C^*-algebra homomorphism* if $H : A \to B$ satisfies

$$H([x,y]) = [H(x), H(y)] \tag{3.1}$$

for all $x, y \in A$.

Defintion 3.2 [24, 25, 29]. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Lie derivation* if $D : A \to A$ satisfies

$$D([x,y]) = [D(x), y] + [x, D(y)] \tag{3.2}$$

for all $x, y \in A$.

In this section, we investigate Lie C^*-algebra homomorphisms between Lie C^*-algebras and Lie derivations on Lie C^*-algebras associated with the Apollonius-type additive functional equation.

Theorem 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.9) such that*

$$\|f([x,y]) - [f(x), f(y)]\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.3}$$

for all $x, y \in A$. Then the mapping $f : A \to B$ is a Lie C^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.3, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (3.3) that

$$\begin{aligned}\|f([x,y])-[f(x),f(y)]\|_B &= \lim_{n\to\infty} 4^n\left\|f\left(\frac{[x,y]}{2^n\cdot 2^n}\right)-\left[f\left(\frac{x}{2^n}\right),f\left(\frac{y}{2^n}\right)\right]\right\|_B \\ &\le \lim_{n\to\infty}\frac{4^n\theta}{4^{nr}}\cdot\|x\|_A^r\cdot\|y\|_A^r = 0\end{aligned} \tag{3.4}$$

for all $x,y\in A$. Thus,

$$f([x,y]) = [f(x),f(y)] \tag{3.5}$$

for all $x,y\in A$. Hence, the mapping $f:A\to B$ is a Lie C^*-algebra homomorphism. □

Theorem 3.4. *Let $r<1$ and θ be positive real numbers, and let $f:A\to B$ be a mapping satisfying (2.9) and (3.3). Then the mapping $f:A\to B$ is a Lie C^*-algebra homomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.3. □

Theorem 3.5. *Let $r>1$ and θ be nonnegative real numbers, and let $f:A\to A$ be a mapping satisfying (2.9) such that*

$$\|f([x,y])-[f(x),y]-[x,f(y)]\|_A \le \theta\cdot\|x\|_A^r\cdot\|y\|_A^r \tag{3.6}$$

for all $x,y\in A$. Then the mapping $f:A\to A$ is a Lie derivation.

Proof. By the same reasoning as in the proof of Theorem 2.3, the mapping $f:A\to A$ is $\mathbb{C}$-linear.

It follows from (3.6) that

$$\begin{aligned}\|f([x,y])-[f(x),y]-[x,f(y)]\|_A &= \lim_{n\to\infty} 4^n\left\|f\left(\frac{[x,y]}{4^n}\right)-\left[f\left(\frac{x}{2^n}\right),\frac{y}{2^n}\right]-\left[\frac{x}{2^n},f\left(\frac{y}{2^n}\right)\right]\right\|_A \\ &\le \lim_{n\to\infty}\frac{4^n\theta}{4^{nr}}\cdot\|x\|_A^r\cdot\|y\|_A^r = 0\end{aligned} \tag{3.7}$$

for all $x,y\in A$. So

$$f([x,y]) = [f(x),y]+[x,f(y)] \tag{3.8}$$

for all $x,y\in A$. Thus, the mapping $f:A\to A$ is a Lie derivation. □

Theorem 3.6. *Let $r<1$ and θ be positive real numbers, and let $f:A\to A$ be a mapping satisfying (2.9) and (3.6). Then the mapping $f:A\to A$ is a Lie derivation.*

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.5. □

4. Homomorphisms and derivations in JC^*-algebras

Throughout this section, assume that A is a JC^*-algebra with norm $\|\cdot\|_A$, and that B is a JC^*-algebra with norm $\|\cdot\|_B$.

Defintion 4.1 [25, 29]. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *JC^*-algebra homomorphism* if $H : A \to B$ satisfies

$$H(x \circ y) = H(x) \circ H(y) \tag{4.1}$$

for all $x, y \in A$.

Defintion 4.2 [25, 29]. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *Jordan derivation* if $D : A \to A$ satisfies

$$D(x \circ y) = D(x) \circ y + x \circ D(y) \tag{4.2}$$

for all $x, y \in A$.

In this section, we investigate JC^*-algebra homomorphisms between JC^*-algebras and Jordan derivations on JC^*-algebras associated with the Apollonius type additive functional equation.

The proofs of the following theorems are similar to the proofs given in Sections 2 and 3.

Theorem 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.9) such that*

$$\big\|f(x \circ y) - f(x) \circ f(y)\big\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.3}$$

for all $x, y \in A$. Then the mapping $f : A \to B$ is a JC^*-algebra homomorphism.

Theorem 4.4. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying (2.9) and (4.3). Then the mapping $f : A \to B$ is a JC^*-algebra homomorphism.*

Theorem 4.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.9) such that*

$$\big\|f(x \circ y) - f(x) \circ y - x \circ f(y)\big\|_A \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.4}$$

for all $x, y \in A$. Then the mapping $f : A \to A$ is a Jordan derivation.

Theorem 4.6. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying (2.9) and (4.4). Then the mapping $f : A \to A$ is a Jordan derivation.*

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