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Research article

Fixed points of non-linear multivalued graphic contractions with applications

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Abstract: In this paper, a novel and more general type of sequence of non-linear multivalued mappings as well as the corresponding contractions on a metric space equipped with a graph is initiated. Fixed point results of a single-valued mapping and the new sequence of multivalued mappings are examined under suitable conditions. A non-trivial comparative illustration is provided to support the assumptions of our main theorem. A few important results in ϵ-chainable metric space and cyclic contractions are deduced as some consequences of the concepts obtained herein. As a result of our findings, new criteria for solving a broader form of Fredholm integral equation are established. An open problem concerning discretized population balance model whose solution may be investigated using any of the ideas proposed in this note is highlighted as a future assignment.

Keywords: fixed point; non-linear multivalued graphic contraction; coincidence point; sequence of multivalued maps
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1. Introduction and preliminaries

Fixed point theory is one of the main tools in modern functional analysis. Its primary role is in the existence criteria for solutions of different types of equations arising in science and engineering.

One of the first most celebrated results in this context is the Banach contraction principle (BCP). The prototypical idea of the BCP has been fine-tuned by many examiners in different domains.

Definition 1.1. A metric space (MS) $(\widetilde{\bigwedge}, \rho)$ is called ϵ-chainable, for some $\epsilon > 0$, if for any $u, v \in \widetilde{\bigwedge}$, we can find $\alpha \in \mathbb{N}$ and a sequence $\{\jmath_i\}_{i=0}^{\alpha}$ in $\widetilde{\bigwedge}$ such that $\jmath_0 = u$, $\jmath_\alpha = v$ and $\rho(\jmath_{i-1}, \jmath_i) < \epsilon$ for $i = \overline{1, \alpha}$.

Definition 1.2. Let $(\widetilde{\bigwedge}, \rho)$ be an MS, $\epsilon > 0, 0 \leq l < 1$ and $u, v \in \widetilde{\bigwedge}$. A mapping $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ is called (ϵ, l)-uniformly locally contractive, if $0 < \rho(u, v) < \epsilon$ implies $\rho(gu, gv) < ld(u, v)$.

As one of the improvements of the BCP, Edelstein [6] proved that every (ϵ, l)-uniformly locally contractive mapping on a complete ϵ-chainable MS has a unique fixed point

Let $(\widetilde{\bigwedge}, \rho)$ be an MS. Consistent with Nadler [18] and Hu [7], denote by $C\widehat{B}(\widetilde{\bigwedge}), \mathcal{K}(\widetilde{\bigwedge})$ and $2^{\widetilde{\bigwedge}}$, the collection of all non-empty closed and bounded, compact and non-empty subsets of $\widetilde{\bigwedge}$, respectively. Let $\widehat{A}, \widehat{B} \in C\widehat{B}(\widetilde{\bigwedge})$. The Pompeiu-Hausdorff distance $\aleph$ on $C\widehat{B}(\widetilde{\bigwedge})$ induced by the metric ρ is given as:

$$\aleph(\widehat{A}, \widehat{B}) = \inf\{\eta > 0 : \widehat{A} \subseteq N_\eta(\eta, \widehat{B}), \widehat{B} \subseteq N_\eta(\eta, \widehat{A})\},$$

where

$$N_\eta(\eta, \Theta) = \{\jmath \in \widetilde{\bigwedge} : \rho(\jmath, r) < \eta, \text{ for some } r \in \Theta\}.$$

In 1969, Nadler [18] brought up a multivalued version of the BCP by availing the Hausdorff distance function. Along this line, Reich [26] presented a fixed point theorem for multivalued mappings (MVM) on compact subsets of an MS and noted the puzzle: "can $\mathcal{K}(\widetilde{\bigwedge})$ be replaced with $C\widehat{B}(\widetilde{\bigwedge})$?". Mizoguchi and Takahashi [16, Theorem 5] gave an affirmative response to this puzzle. In similar development, the multivalued fixed point theorem given by Nadler was extended to an ϵ-chainable MS by Hu [7]. Azam and Arshad [3] improved [18, Theorem 6] by investigating fixed point results of a sequence of locally contractive MVMs in an ϵ-chainable MS. Muhammad et al. [17] refined the ideas in [3, 18] to the setting of an MS with a directed graph. On similar line, Phikul and Suthep [24] improved the ideas of Berinde [5], Jachysmki [8] and Nadler [18] by examining common fixed point results of a pair of two MVMs on an MS endowed with a graph. On the other side, Jachysmki [8] studied the fixed point notions putforward by Nieto and Rodriguez-Lopez [23] and Ran and Reuring [25] by launching the concept of a graphic contraction (also named a G_r-contraction) on an MS.

Following the above trend of investigations (in particular, the ideas in [3, 5, 8, 16, 24] and the references therein), we noticed that fixed point results of set-valued maps connecting the notions of M-function, graphic contractions and Berinde-type weak contractions have not been sufficiently examined. Hence, this paper introduces a more general notion of a graphic contraction, viz. a non-linear multivalued G_{rg}-contraction. Sufficient conditions for the existence of coincidence points (CoPs) of the sequence of MVMs and a single-valued map given on an MS endowed with a graph are examined. Comparative examples which dwell on the preeminence of our obtained results are constructed. A significant number of results in an ϵ-chainable MS and cyclic contractions are derived as some special cases of our results. From an application point, one of our findings is employed to investigate new criteria for solution to a more general form of Fredholm integral equation. As a future assignment of the ideas presented here, we note down an open problem regarding a discretized population balance model whose solution may be analyzed using any of the concepts proposed in this work.

We now present some concepts and results that will be needed hereafter. Let $(\widetilde{\bigwedge}, \rho)$ be an MS and ∇ denote the diagonal of the Cartesian product $\widetilde{\bigwedge} \times \widetilde{\bigwedge}$. Given a directed graph G_r, let $\widetilde{\bigwedge} = V_{et}(G_r)$, where $V_{et}(G_r)$ depicts the set of vertices of the graph G_r and $E_{ed}(G_r)$ be the set of all edges of G_r. Assume that G_r has no parallel edges so that $G_r = (V_{et}(G_r), E_{ed}(G_r))$.

If G_r is a directed graph, then ${G_r}^{-1}$ depicts the graph derived from G_r by inverting the direction of edges. And, if we overlook the direction of the edges in G_r, then, we obtain an undirected graph denoted by $\widetilde{G_r}$. The pair (V_{et}', E_{ed}') is a subgraph of G_r, if $V_{et}' \subseteq V_{et}(G_r)$ and $E_{ed}' \subseteq E_{ed}(G_r)$, and for each $(p, q) \in E_{ed}'$ for all $p, q \in V_{et}'$. Moreover, we record the needed concepts of connectivity of graph from [9] as follows.

Definition 1.3. A path in a graph G_r from the vertex q to p of length $\alpha \in \mathbb{N} \cup \{0\}$, is a sequence $\{\varsigma_i\}_{i=0}^{\alpha}$ of $\alpha + 1$ vertices such that $\varsigma_0 = q, \varsigma_\alpha = p$ and $(\varsigma_{i-1}, \varsigma_i) \in E_{ed}(G_r)$ for $i = \overline{i, \alpha}$.

Definition 1.4. A graph G_r is said to be connected if we can find a path between any two of its vertices. G_r is weakly connected if $\widetilde{G_r}$ is connected.

Definition 1.5. For $\jmath, \ell, z \in V_{et}(G_r)$, $[\jmath]_{G_r}$ depicts the equivalence class of the relation $\sim$ given on $V_{et}(G_r)$ by the rule $\ell \sim z$ if we can find a path in G_r from ℓ to z.

For $\eta \in V_{et}(G_r)$ and $\alpha \in \mathbb{N} \cup \{0\}$, define the set $[\eta]_{G_r}^{\alpha}$, as follows:

$$[\eta]_{G_r}^{\alpha} = \{\omega \in V_{et}(G_r) : \text{there is a path of length } \alpha \text{ from } \eta \text{ to } \omega\}.$$

Jachymski [8] brought up the idea of a G_r-contraction in the following manner.

Definition 1.6. Let $(\widetilde{\bigwedge}, \rho)$ be an MS equipped with a graph G_r. The mapping $\Upsilon : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ is called a G_r-contraction if it preserves the edges of G_r; that is:

$$\forall_{\jmath, \ell \in \widetilde{\bigwedge}} (\jmath, \ell) \in E_{ed}(G_r) \Rightarrow (\Upsilon \jmath, \Upsilon \ell) \in E_{ed}(G_r),$$

and we can find $\beta \in [0, 1)$ such that

$$\forall_{\jmath, \ell \in \widetilde{\bigwedge}} (\jmath, \ell) \in E_{ed}(G_r) \Rightarrow \rho(\Upsilon \jmath, \Upsilon \ell) \leq \beta \rho(\jmath, \ell).$$

Mizoguchi and Takahashi [16] introduced an auxiliary function, named an MT-function, as follows.

Definition 1.7. [16] A function $\psi : \mathbb{R}_+ = [0, \infty) \longrightarrow [0, 1)$ is known as an MT-function if it satisfies the Mizoguchi and Takahashi condition, that is, if $\lim_{r \longrightarrow t^+} \sup \psi(t) < 1$ for all $t \in \mathbb{R}_+$.

Now, we present specific results from the theory of MVMs.

Lemma 1.8. *[2] Let $\{\widehat{A}_n\}$ be a sequence in $C\widehat{B}(\widetilde{\bigwedge})$ and we can find $\widehat{A} \in C\widehat{B}(\widetilde{\bigwedge})$ such that $\lim_{n \longrightarrow \infty} \aleph(\widehat{A}_n, \widehat{A}) = 0$. If $\jmath_n \in \widehat{A}_n (n \in \mathbb{N})$ and we can find $\jmath \in \widetilde{\bigwedge}$ such that $\lim_{n \longrightarrow \infty} \rho(\jmath_n, \jmath) = 0$, then $\jmath \in \widehat{A}$.*

Lemma 1.9. *[7] Let $(\widetilde{\bigwedge}, \rho)$ be an MS and $\widehat{A}, \widehat{\widehat{B}} \in C\widehat{B}(\widetilde{\bigwedge})$ with $\aleph(\widehat{A}, \widehat{\widehat{B}}) < \epsilon$ for every $\epsilon > 0$. Then for each $\jmath \in \widehat{A}$, we can find $\ell \in \widehat{\widehat{B}}$ such that $\rho(\jmath, \ell) < \epsilon$.*

2. Main results

We begin this section by introducing a new type of sequence of MVM in an MS with a directed graph.

Definition 2.1. Let $(\widetilde{\bigwedge}, \rho)$ be an MS, $G_r = (V_{et}(G_r), E_{ed}(G_r))$ be a directed graph such that $V_{et}(G_r) = \widetilde{\bigwedge}$ and $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be a surjection. A sequence of MVM $\{\Upsilon_p\}_{p\in\mathbb{N}}$ from $\widetilde{\bigwedge}$ into $CB(\widetilde{\bigwedge})$ is called a non-linear multivalued G_{rg}-contraction, if we can find an MT-function $\psi : \mathbb{R}_+ \longrightarrow [0, 1)$ and some constant $K \geq 0$ such that for $(gu, gv) \in E_{ed}(G_r)$,

(a)

$$\begin{aligned}\aleph(\Upsilon_p(u), \Upsilon_r(v)) &\leq \psi(\rho(gu, gv))\rho(gu, gv)\\ &\quad +Kd(gv, \Upsilon_p(u));\end{aligned}$$

(b) if $\jmath \in \Upsilon_p(u), \ell \in \Upsilon_r(v)$ and $\rho(g\jmath, g\ell) \leq \rho(gu, gv)$, then $(g\jmath, g\ell) \in E_{ed}(G_r)$.

If $\Upsilon_p : \widetilde{\bigwedge} \longrightarrow CB(\widetilde{\bigwedge})$, then the graph of Υ_p is given by

$$G_r(\Upsilon_p) = \{(\jmath, \ell) : \jmath \in \widetilde{\bigwedge}, \ell \in \Upsilon_p(\jmath), p \in \mathbb{N}\}.$$

We now study conditions for the existence of CoPs of a single-valued mapping and a sequence of MVM.

Theorem 2.2. *Let $(\widetilde{\bigwedge}, \rho)$ be a complete MS, $\{\Upsilon_p : p \in \mathbb{N}\}$ be a sequence of multivalued G_{rg}-contraction from $\widetilde{\bigwedge}$ into $CB(\widetilde{\bigwedge})$ and $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be a surjection. If we can find $\alpha \in \mathbb{N}$ and $v_0 \in \widetilde{\bigwedge}$ such that*

(i) $\Upsilon_1(v_0) \cap [g(v_0)]^\alpha_{G_r} \neq \emptyset$;

(ii) for any sequence $\{v_n\}$ in $\widetilde{\bigwedge}$, if $v_n \longrightarrow v$ and $v_n \in \Upsilon_n(v_{n-1}) \cap [v_{n-1}]^\alpha_{G_r}$ for all $n \in \mathbb{N}$, then we can find a subsequence $\{v_{n_\gamma}\}$ of $\{v_n\}$ such that $(v_{n_\gamma}, v) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$.

Then we can find $u^* \in \widetilde{\bigwedge}$ such that $gu^* \in \bigcap_{p\in\mathbb{N}} \Upsilon_p(u^*)$.

Proof. Choose $v_1 \in \widetilde{\bigwedge}$ such that $gv_1 \in \Upsilon_1(v_0) \cap [gv_0]^\alpha_{G_r}$, then, we can find a path from gv_0 to gv_1, that is,

$$gv_0 = gu^{(1)}_0, \cdots, gu^{(1)}_\alpha = gv_1 \in \Upsilon_1(v_0),$$

and $(gu^{(1)}_i, gu^{(1)}_{i+1}) \in E_{ed}(G_r)$ for all $i = \overline{0, \alpha - 1}$. Without loss of generality, assume that $gu^{(1)}_\gamma \neq gu^{(1)}_j$ for each $\gamma, j \in \{0, 1, 2, \cdots, \alpha\}$ with $\gamma \neq j$. Rename gv_1 as $gu^{(2)}_0$. Since $(gu^{(1)}_0, gu^{(1)}_1) \in E_{ed}(G_r)$, and as $gu^{(2)}_0 \in \Upsilon_1(u^{(1)}_0)$, then by Lemma 1.9, we can find some $gu^{(2)}_1 \in \Upsilon_2(u^{(1)}_1)$ such that

$$\begin{aligned}\rho(gu^{(2)}_0, gu^{(2)}_1) &< \aleph(\Upsilon_1(u^{(1)}_0), \Upsilon_2(u^{(1)}_1))\\ &\leq \psi(\rho(gu^{(1)}_0, gu^{(1)}_1))\rho(gu^{(1)}_0, gu^{(1)}_1)\\ &\quad +Kd(gu^{(1)}_1, \Upsilon_1(u^{(1)}_0))\\ &= \psi(\rho(gu^{(1)}_0, gu^{(1)}_1))\rho(gu^{(1)}_0, gu^{(1)}_1)\\ &< \sqrt{\psi(\rho(gu^{(1)}_0, gu^{(1)}_1))}\rho(gu^{(1)}_0, gu^{(1)}_1)\end{aligned}$$

$$< \rho(gu_0^{(1)}, gu_1^{(1)}).$$

Again, since $(gu_1^{(1)}, gu_2^{(1)}) \in E_{ed}(G_r)$ and $gu_1^{(1)}, \cdots, gu_\alpha^{(1)} \in \Upsilon_2(u_1^{(1)}), \alpha = 1, 2$, by Lemma 1.9, we can find some $gu_2^{(2)} \in \Upsilon_2(u_1^{(1)})$ such that

$$\begin{aligned}
\rho(gu_1^{(2)}, gu_2^{(2)}) &< \aleph(\Upsilon_2(u_1^1), \Upsilon_2(u_2^{(1)})) \\
&\leq \psi(\rho(gu_1^{(1)}, gu_2^{(1)}))\rho(gu_1^{(1)}, gu_2^{(1)}) \\
&\quad + Kd(gu_2^{(1)}, \Upsilon_2(u_1^{(1)})) \\
&= \psi(\rho(gu_1^{(1)}, gu_2^{(1)}))\rho(gu_1^{(1)}, gu_2^{(1)}) \\
&< \sqrt{\psi(\rho(gu_1^{(1)}, gu_2^{(1)}))}\rho(gu_1^{(1)}, gu_2^{(1)}) \\
&< \rho(gu_1^{(1)}, gu_2^{(1)}).
\end{aligned}$$

Thus, we obtain $\{gu_0^{(2)}, gu_1^{(2)}, \cdots, gu_\alpha^{(2)}\}$ of $\alpha + 1$ vertices of $\widetilde{\bigwedge}$ such that $gu_0^{(2)} \in \Upsilon_1(u_0^{(1)})$ and $gu_\rho^{(2)} \in \Upsilon_2(u_\rho^{(1)})$ for $\rho = \overline{1, \alpha}$ with

$$\rho(gu_\rho^{(2)}, gu_{\rho+1}^{(2)}) < \rho(gu_\rho^{(1)}, gu_{\rho+1}^{(1)}),$$

for $\rho = \overline{0, \alpha - 1}$. Because $(gu_\rho^{(1)}, gu_{\rho+1}^{(1)}) \in E_{ed}(G_r)$ for all $\rho = \overline{0, \alpha - 1}$, $(gu_\rho^{(2)}, gu_{\rho+1}^{(2)}) \in E_{ed}(G_r)$ for all $\rho = \overline{0, \alpha - 1}$. Let $gu_\alpha^{(2)} = gv_2$. Thus, the set of points $gv_1 = gu_0^{(2)}, gu_1^{(2)}, \cdots, gu_\alpha^{(2)} = gv_2 \in \Upsilon_2(v_1)$ is a path from gv_1 to gv_2. Relabel gv_2 as $gu_0^{(3)}$. Then, by similar steps as above, we obtain a path $gv_2 = gu_0^{(3)}, gu_1^{(3)}, \cdots, gu_\alpha^{(3)} = gv_3 \in \Upsilon_3(v_2)$ from gv_2 to gv_3. Inductively, it follows that $gv_h = gu_0^{(h+1)}, gu_1^{(h+1)}, \cdots, gu_\alpha^{(h+1)} = gv_{h+1} \in \Upsilon_{h+1}(v_h)$ with

$$\rho(gu_t^{(h+1)}, gu_{t+1}^{(h+1)}) < \rho(gu_t^{(h)}, gu_{t+1}^{(h)}); \tag{2.1}$$

thus, $(gu_t^{(h+1)}, gu_{t+1}^{(h+1)}) \in E_{ed}(G_r)$ for $t = \overline{0, \alpha - 1}$. Consequently, we construct a sequence $\{gv_h\}_{h=1}^\infty$ of points of $\widetilde{\bigwedge}$ with

$$\begin{aligned}
gv_1 = gu_\alpha^{(1)} &= gu_0^{(2)} \in \Upsilon_1(v_0) \\
gv_2 = gu_\alpha^{(2)} &= gu_0^{(3)} \in \Upsilon_2(v_1) \\
gv_3 = gu_\alpha^{(3)} &= gu_0^{(4)} \in \Upsilon_3(v_2) \\
\vdots &= \cdots \\
gv_{h+1} = gu_\alpha^{(h+1)} &= gu_0^{(h+2)} \in \Upsilon_{h+1}(v_h), \forall h \in \mathbb{N}.
\end{aligned}$$

For each $t \in \{0, 1, 2, ..., \alpha - 1\}$, and from (2.1), we see that $\{\rho(gu_t^{(h)}, gu_{t+1}^{(h)})\}_{h=1}^\infty$ is a bounded and decreasing sequence of non-negative real numbers, and so it converges. That is, we can find $\tau^t \geq 0$ such that

$$\lim_{h \longrightarrow \tau^{t+}} \rho(gu_t^{(h)}, gu_{t+1}^{(h)}) = \tau^t.$$

Since ψ is an MT-function, we can find $\varrho^t \in \mathbb{N}$ such that $\psi(\rho(gu_t^{(h)}, gu_{t+1}^{(h)})) < \omega(\tau^t)$ for all $h \geq \varrho^t$, where $\lim_{t \longrightarrow \tau^{t+}} \sup \psi(t) < \omega(\tau^t) < 1$. Now, set

$$\Omega\tau^t = \max\left\{\max_{r = \overline{1, \varrho^t}} \sqrt{\psi(\rho(gu_t^{(r)}, gu_{t+1}^{(r)}))}, \sqrt{\omega(\tau^t)}\right\}.$$

Then, for every $h > \varrho^t$, consider

$$\begin{aligned}
\rho(gu_t^{(h+1)}, gu_{t+1}^{(h+1)}) &< \sqrt{\rho(gu_t^{(h)}, gu_{t+1}^{(h)})}\rho(gu_t^{(h)}, gu_{t+1}^{(h)}) \\
&< \sqrt{\omega(\tau^t)}\rho(gu_t^{(h)}, gu_{t+1}^{(h)}) \\
&\leq \Omega\tau^t\rho(gu_t^{(h)}, gu_{t+1}^{(h)}) \\
&\leq (\Omega\tau^t)^2\rho(gu_t^{(h-1)}, gu_{t+1}^{(h-1)}) \\
&\leq \cdots \\
&\leq (\Omega\tau^t)^n\rho(gu_t^{(1)}, gu_{t+1}^{(1)}).
\end{aligned}$$

Taking $p = \max\{\varrho^t, t = 0, 1, 2, \cdots, \alpha - 1\}$, produces

$$\begin{aligned}
\rho(gv_h, gv_{h+1}) &= \rho(gu_0^{(h+1)}, gu_\alpha^{(h+1)}) \\
&\leq \sum_{t=0}^{\alpha-1}\rho(gu_t^{(h+1)}, gu_{t+1}^{(h+1)}) \\
&< \sum_{t=0}^{\alpha-1}(\Omega\tau^t)^h\rho(gu_t^{(1)}, gu_{t+1}^{(1)}).
\end{aligned}$$

Now, for all $q > h > p$, notice that

$$\begin{aligned}
\rho(gv_h, gv_q) &\leq \rho(gv_h, gv_{h+1}) + \rho(gv_{h+1}, gv_{h+2}) + \cdots + \rho(gv_{q-1}, gv_q) \\
&< \sum_{t=0}^{\alpha-1}(\Omega\tau^t)^h\rho(gu_t^{(1)}, gu_{t+1}^{(1)}) + \cdots + \sum_{t=0}^{\alpha-1}(\Omega\tau^t)^{q-1}\rho(gu_t^{(1)}, gu_{t+1}^{(1)}).
\end{aligned}$$

Since $\Omega\tau^t < 1$ for all $t \in \{0, 1, 2, \cdots, \alpha - 1\}$, it follows that $\{gv_h = gu_\alpha^{(h)}\}$ is a Cauchy sequence. By completeness of $\widetilde{\bigwedge}$, we can find $v^* \in \widetilde{\bigwedge}$ such that $gv_h \longrightarrow gv^*$. Now, availing the fact that $gv_n \in \Upsilon(v_{n-1}) \cap [gv_{n-1}]^\alpha_{G_r}$ for all $n \in \mathbb{N}$, we can find a subsequence $\{gv_{n_\gamma}\}$ such that $(gv_{n_\gamma}, gv^*) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$. Now, for any $p \in \mathbb{N}$,

$$\begin{aligned}
\rho(gv^*, \Upsilon_p(v^*)) &\leq \rho(gv^*, gv_{h+1}) + \rho(gv_{h+1}, \Upsilon_p(v^*)) \\
&\leq \rho(gv^*, gv_{h+1}) + \aleph(\Upsilon_{h+1}(v_h), \Upsilon_p(v^*)) \\
&\leq \rho(gv^*, gv_{h+1}) + \psi(\rho(gv_h, gv^*))\rho(gv_h, gv^*) \\
&\quad + Kd(gv_h, \Upsilon_p(v^*)).
\end{aligned} \tag{2.2}$$

Letting $h \longrightarrow \infty$ in (2.2), yields

$$\rho(gv^*, \Upsilon_p(v^*)) \leq Kd(gv^*, \Upsilon_p(v^*)). \tag{2.3}$$

From (2.3), if $K = 0$, then Lemma 1.8 can be applied to conclude that $gv^* \in \Upsilon_p(v^*)$ for all $p \in \mathbb{N}$. On the other hand, if $K > 0$, assume that $gv^* \notin \Upsilon_p(v^*)$ for some $p \in \mathbb{N}$. So, taking $K = \frac{\rho(gv^*, \Upsilon_p(v^*))}{1+\rho(gv^*, \Upsilon_p(v^*))}$ in (2.3) gives

$$\rho(gv^*, \Upsilon_p(v^*)) \leq \frac{\rho(gv^*, \Upsilon_p(v^*))\rho(gv^*, \Upsilon_p(v^*))}{1 + \rho(gv^*, \Upsilon_p(v^*))}$$

$$< \quad \frac{\rho(gv^*, \Upsilon_p(v^*))\rho(gv^*, \Upsilon_p(v^*))}{\rho(gv^*, \Upsilon_p(v^*))} = \rho(gv^*, \Upsilon_p(v^*)),$$

a contradiction. Consequently, $gv^* \in \bigcap_{p\in\mathbb{N}} \Upsilon_p(v^*)$. $\square$

Example 2.3. For $p \in \mathbb{N}$, let $\widetilde{\bigwedge} = \left\{\frac{1}{3^p}\right\} \cup \{0\} \cup \left[\frac{1}{4^p}, 5\right]$ and $\rho(u, v) = |u - v|$ for all $u, v \in \widetilde{\bigwedge}$. Then, $(\widetilde{\bigwedge}, \rho)$ is a complete MS. Let $G_r = (V_{et}(G_r), E_{ed}(G_r))$ be a directed graph such that $V_{et}(G_r) = \widetilde{\bigwedge}$ and $E_{ed}(G_r) = \left\{(0, 0), \left(\frac{1}{4^p}, 1\right) : p \in \mathbb{N}\right\}$. Let $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be given as $g(u) = 5u$, and $\Upsilon_p : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be given by

$$\Upsilon_p(u) = \begin{cases} \{0\}, & \text{if } u = 0 \\ [0, 4u], & \text{if } u \in \left[\frac{1}{4^p}, 5\right] \\ \{3\}, & \text{if } u = \frac{1}{3^p}, p \in \mathbb{N}. \end{cases}$$

We shall show that Υ_p is a non-linear multivalued G_{rg}-contraction with $\psi(t) = \frac{t}{5}, t \geq 0$, and $K = 5$. Now, notice that if $u = v = 0$ and $gu = gv = 0$, then for all $p, r \in \mathbb{N}$, $\Upsilon_p(u) = \Upsilon_r(v) = \{0\}$, thus $\aleph(\Upsilon_p(u), \Upsilon_r(v)) = 0$. For $(gu, gv) \in E_{ed}(G_r)$ with $u \neq v$, $(u, v) = \left(\frac{1}{4^p}, 1\right)$ for each $p \in \mathbb{N}$. Thus,

$$\begin{aligned} \aleph(\Upsilon_p(u), \Upsilon_r(v)) &= \aleph\left(\Upsilon_p\left(\frac{1}{4^p}\right), \Upsilon_r(1)\right) = \aleph\left(\Upsilon_r(1), \Upsilon_p\left(\frac{1}{4^p}\right)\right) \\ &= \aleph([0, 4u], [0, 4v]) \\ &= |4u - 4v| \leq \frac{4}{5}|5u - 5v| \\ &\leq \frac{4}{5}\rho(5u, 5v) + 5\rho(5v, \Upsilon_p(u)) \\ &\leq \psi(\rho(gu, gv))\rho(gu, gv) + Kd(gv, \Upsilon_p(u)). \end{aligned}$$

Moreover, let $(gu, gv) \in E_{ed}(G_r)$ with $u \neq v$. Then, $(u, v) = \left(\frac{1}{4^p}, 1\right)$ for all $p \in \mathbb{N}$. Thus, $\Upsilon_p(u) = \Upsilon_p\left(\frac{1}{4^p}\right) = [0, 4]$ and $\Upsilon_r(v) = \Upsilon_r(1) = [0, 4v]$. We observe that if $\jmath \in \Upsilon_p(u), \ell \in \Upsilon_r(v)$ and $\rho(g\jmath, g\ell) \leq \rho(gu, gv)$, then $(\jmath, \ell)$ are $\left(0, \frac{1}{4^{p+2}}\right)$ and $\left(0, \frac{1}{4^{p+3}}\right)$. Thus, $(g\jmath, g\ell) \in E_{ed}(G_r)$. Consequently, $\{\Upsilon_p\}_{p\in\mathbb{N}}$ is a non-linear multivalued G_{rg}-contraction. We notice that other conditions of Theorem 2.2 hold obviously. It follows that all conditions of Theorem 1 are obeyed. Thus, g and Υ_p have a CoP $u^* = 0 \in \widetilde{\bigwedge}$ such that $g0 \in \bigcap_{p\in\mathbb{N}} \Upsilon_p(0)$.

In what follows, we show that our result (Theorem 2.2) cannot be followed from some similar ones in the literature. Now, consider [24, Theorem 3.3], if we take $\psi(\rho(u, v)) = \frac{1}{3}$ with $u = \frac{1}{4}, v = 1$ and $p = 1, r = 2$, then $(u, v) = \left(\frac{1}{4}, 1\right) \in E_{ed}(G_r)$, and

$$\begin{aligned} \aleph(\Upsilon_1(u), \Upsilon_2(v)) &= \aleph([0, 1], [0, 4]) = 3 \\ &> \psi\left(\rho\left(\frac{1}{4}, 1\right)\right)\rho\left(\frac{1}{4}, 1\right) + Kd(1, [0, 1]) \\ &= \frac{1}{4}, \end{aligned} \tag{2.4}$$

for all $K \geq 0$. This shows that the mapping Υ_p is not a Berinde graph contractive in the sense of Phikul and Suthep [24, Definition 3.1]. From (2.4), we also observe that Υ_p is not a graph contractive mapping as given by Beg and Butt [4]. Thus, the main results in [4, 24] are not applicable to this illustration.

Theorem 2.4. *Let $(\widetilde{\bigwedge},\rho)$ be a complete MS, $\Upsilon : \widetilde{\bigwedge} \longrightarrow CB(\widetilde{\bigwedge})$ be an MVM and $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ a surjection. Assume further that the following conditions are obeyed:*

(i) we can find $K \geq 0$ such that for all $u, v \in (u \neq v)$, $(gu, gv) \in E_{ed}(G_r)$ yields

$$\aleph(\Upsilon(u), \Upsilon(v)) \leq \psi(\rho(gu, gv))\rho(gu, gv) + Kd(v, \Upsilon u),$$

where $\psi : \mathbb{R}_+ \longrightarrow [0, 1)$ is an MT-function;

(ii) we can find $\alpha \in \mathbb{N}$ and $v_0 \in \widetilde{\bigwedge}$ such that $\Upsilon(v_0) \cap [gv_0]^{\alpha}_{G_r} \neq \emptyset$;

(iii) for any sequence $\{v_n\}$ in $\widetilde{\bigwedge}$, if $v_n \longrightarrow v$ and $v_n \in \Upsilon(v_{n-1}) \cap [v_{n-1}]^{\alpha}_{G_r}$ for all $n \in \mathbb{N}$, we can find a subsequence $\{v_{n_\gamma}\}$ such that $(v_{n_\gamma}, v) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$.

Then g and Υ have a CoP in $\widetilde{\bigwedge}$; that is, we can find $u^* \in \widetilde{\bigwedge}$ such that $gu^* \in \Upsilon(u^*)$.

Proof. Set $\Upsilon_p = \Upsilon$ for all $q \in \mathbb{N}$ in Theorem 2.2. □

The following are further consequences of Theorems 2.2 and 2.4.

Corollary 1. *Let $(\widetilde{\bigwedge},\rho)$ be a complete MS and $\{\Upsilon_p : p \in \mathbb{N}\}$ be a sequence of MVMs from $\widetilde{\bigwedge}$ into $CB(\widetilde{\bigwedge})$. Assume further that:*

(i) if for some $K \geq 0$ and any $u, v \in \widetilde{\bigwedge}(u \neq v)$ such that $(u, v) \in E_{ed}(G_r)$, we have that

$$\aleph(\Upsilon_p(u), \Upsilon_r(v)) \leq \psi(\rho(u, v))\rho(u, v) + Kd(v, \Upsilon_p(u))$$

for all $p, r \in \mathbb{N}$, where $\psi : \mathbb{R}_+$ is an MT-function;

(ii) we can find $\alpha \in \mathbb{N}$ and $v_0 \in \widetilde{\bigwedge}$ such that $\Upsilon_1(v_0) \cap [gv_0]^{\alpha}_{G_r} \neq \emptyset$;

(iii) for any sequence $\{v_n\}$ in $\widetilde{\bigwedge}$, if $v_n \longrightarrow v$ and $v_n \in \Upsilon_n(v_{n-1}) \cap [v_{n-1}]^{\alpha}_{G_r}$ for all $n \in \mathbb{N}$, then we can find a subsequence $\{v_{n_\gamma}\}$ of $\{v_n\}$ such that $(v_{n_\gamma}, v) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$.

Then Υ_p has at least one fixed point in $\widetilde{\bigwedge}$.

Proof. Take $g = I_{\widetilde{\bigwedge}}$, that is, the identity mapping on $\widetilde{\bigwedge}$, in Theorem 2.2. □

The following is a consequence of Theorem 2.2 in the case of single-valued mappings.

Corollary 2. *Let $(\widetilde{\bigwedge},\rho)$ be a complete MS, $\Lambda : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ and $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be a surjection. If $u, v \in \widetilde{\bigwedge}(u \neq v)$ such that $(gu, gv) \in E_{ed}(G_r)$ implies*

$$\rho(\Lambda(u), \Lambda(v)) \leq \psi(\rho(gu, gv))\rho(gu, gv) + Kd(v, \Lambda(u)),$$

for some $K \geq$, where $\psi : \mathbb{R}_+ \longrightarrow [0, 1)$ is an MT-function. If we can find $\alpha \in \mathbb{N}$ and $v_0 \in \widetilde{\bigwedge}$ such that

(i) $\Upsilon(v_0) \cap [gv_0]^{\alpha}_{G_r} \neq \emptyset$;

(ii) for any sequence $\{v_n\}$ in $\widetilde{\bigwedge}$, if $v_n \longrightarrow v$, and $v_n = \Upsilon(v_{n-1}) \cap [v_{n-1}]^{\alpha}_{G_r}$ for all $n \in \mathbb{N}$, then we can find a subsequence $\{v_{n+\gamma}\}$ such that $(v_{n_\gamma}, v) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$.

Then Λ and g have a coincidence in $\widetilde{\bigwedge}$, that is, we can find $u^* \in \widetilde{\bigwedge}$ such that $gu^* = \Lambda(u^*)$.

Proof. Define $\Upsilon : \widetilde{\bigwedge} \longrightarrow CB(\widetilde{\bigwedge})$ by $\Upsilon u = \{\Lambda u\}$ for all $u \in \widetilde{\bigwedge}$, where Λ is a single-valued mapping. Then all conditions of Theorem 2.4 and Corollary 2 coincide. In this case, $\aleph(\Upsilon(u), \Upsilon(v)) = \aleph(\{\Lambda u\}, \{\Lambda v\}) = \rho(\Lambda u, \Lambda v)$. Consequently, we can find $u^* \in \widetilde{\bigwedge}$ such that $gu^* \in \Upsilon u^* = \{\Lambda u^*\}$, which further produces $gu^* = \Lambda u^*$. □

3. Applications in an ϵ-chainable MS and cyclic contractions

Theorem 3.1. *Let $(\widetilde{\bigwedge},\rho)$ be an ϵ-chainable complete MS, $\{\Upsilon_p : p \in \mathbb{N}\}$ be a sequence of MVMs from $\widetilde{\bigwedge}$ into $CB(\widetilde{\bigwedge})$ and $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be a surjection. If we can find an MT-function $\psi : \mathbb{R}_+ \longrightarrow [0,1)$ and a constant $K \geq 0$ such that $0 < \rho(gu, gv) < \epsilon$ implies*

$$\aleph(\Upsilon_p(u), \Upsilon_r(v)) \leq \psi(\rho(gu, gv))\rho(gu, gv) + K\rho(gv, \Upsilon_p(u)),$$

and we can find $\jmath_1 \in \Upsilon_p(u_0)$, $\jmath_2 \in \Upsilon_r(v_0)$ such that $0 < \rho(gu_0, gv_0) < \epsilon$.
Then, we can find $u^* \in \widetilde{\bigwedge}$ such that $gu^* \in \bigcap_{p\in\mathbb{N}} \Upsilon_p(u^*)$.

Proof. Let the graph G_r be given by $V_{et}(G_r) = \widetilde{\bigwedge}$ and $E_{ed}(G_r) = \nabla \cup \{(gu, gv) \in \widetilde{\bigwedge}\times\widetilde{\bigwedge} : 0 < \rho(gu, gv) < \epsilon\}$. Then connectivity of G_r follows from the ϵ-chainable of $(\widetilde{\bigwedge}, \rho)$. If $(gu, gv) \in E_{ed}(G_r)$, then

$$\aleph(\Upsilon_p(u), \Upsilon_r(v)) \leq \psi(\rho(gu, gv))\rho(gu, gv) + Kd(gv, \Upsilon_p(u)).$$

Now, take $\jmath \in \Upsilon_p(u)$, $\ell \in \Upsilon_r(v)$ and $\rho(g\jmath, g\ell) \leq \rho(gu, gv)$. Since $(gu, gv) \in E_{ed}(G_r)$, then $0 < \rho(gu, gv) < \epsilon$. Observe that if $g\jmath \neq g\ell$, for each $\jmath, \ell \in \widetilde{\bigwedge}$, then $0 < \rho(g\jmath, g\ell) \leq \rho(gu, gv) < \epsilon$, so that $(g\jmath, g\ell) \in E_{ed}(G_r)$. Thus, for each $p \in \mathbb{N}$, $\{\Upsilon_p\}$ is a sequence of non-linear multivalued G_{rg}-contraction. Notice also that if $v_n \longrightarrow v$ and $\rho(v_n, v_{n+1}) < \epsilon$ for all $n \in \mathbb{N}$ with $v_n \in \Upsilon_n(v_{n-1}) \cap [v_{n-1}]^{\alpha}_{G_r}$, then we can find a natural number $\eta(\epsilon)$ such that $\rho(v_n, v) < \epsilon$ for all $n \geq \eta(\epsilon)$. It follows that we can find a subsequence $\{v_{n_\gamma}\}$ of $\{v_n\}$ such that $(v_{n_\gamma}, v) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$. Moreover, since $\jmath_1 \in \Upsilon_p(u_0)$ and $\jmath_2 \in \Upsilon_r(v_0)$ such that $0 < \rho(gu_0, gv_0) < \epsilon$, then $(gu_0, gv_0) \in E_{ed}(G_r)$. Consequently, Theorem 2.2 can be applied to find $u^* \in \widetilde{\bigwedge}$ such that $gu^* \in \bigcap_{p\in\mathbb{N}} \Upsilon_p$. □

The idea of cyclic contractions was introduced by Kirk et. al. [15]. Later on, Rus [27] brought up the concepts of cyclic representations consistent with [15]. Let $\widetilde{\bigwedge}$ be a non-empty set, α be a natural number and $\{\widehat{A_i}\}_{i=1}^{\alpha}$ be a non-empty closed subset of $\widetilde{\bigwedge}$ with $\xi : \bigcup_{i=1}^{\alpha} \widehat{A_i} \longrightarrow \bigcup_{i=1}^{\alpha} \widehat{A_i}$ as an operator. Then $\widetilde{\bigwedge} = \bigcup_{i=1}^{\alpha} \widehat{A_i}$ is called a cyclic representation of $\widetilde{\bigwedge}$ with respect to ξ, if

$$\xi(\widehat{A_1}) \subset \widehat{A_2}, \cdots, \xi(\widehat{A_{\alpha-1}}) \subset \widehat{A_\alpha}, \xi(\widehat{A_\alpha}) \subset \widehat{A_1},$$

and the operator ξ is called a cyclic operator (see [19]).

In what follows, we initiate the idea of cyclic representations for sequence of MVMs by following [24]. Let $\widetilde{\bigwedge}$ be a non-empty set, $\{\widehat{A_i}\}_{i=1}^{\alpha}$ be a non-empty closed subset of $\widetilde{\bigwedge}$ for each $\alpha \in \mathbb{N}$ and $\{\Upsilon_p : p \in \mathbb{N}\}$ be a sequence of MVMs from $\widetilde{\bigwedge}$ into $2^{\widetilde{\bigwedge}}$. Then $\widetilde{\bigwedge} = \bigcup_{i=1}^{\alpha} \widehat{A_i}$ is called a cyclic representation of $\widetilde{\bigwedge}$ with respect to Υ_p, $p \in \mathbb{N}$, if

$$\Upsilon_p : \widehat{A_i} \longrightarrow CB(\widehat{A_{i+1}}), i = \overline{1, \alpha}, \widehat{A_{\alpha+1}} = \widehat{A_1},$$

and Υ_p is called a sequence of multivalued operators.

Theorem 3.2. *Let $(\widetilde{\bigwedge}, \rho)$ be a complete MS, α be a positive integer, $\{\widehat{A_i}\}_{i=1}^{\alpha}$ be a non-empty closed subset of $\widetilde{\bigwedge}$, $\Phi = \bigcup_{i=1}^{\alpha} \widehat{A_i}$, $\{\Upsilon_p : p \in \mathbb{N}\}$ be a sequence of MVMs from $\widetilde{\bigwedge}$ into 2^{Φ} and $g : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ be a surjection. Suppose that $\bigcup_{i=1}^{\alpha} \widehat{A_i}$ is a cyclic representation of Φ with respect to $\{\Upsilon_p\}_{p=1}^{\infty}$. If we can find an MT-function $\psi : \mathbb{R}_+ \longrightarrow [0,1)$ and a constant $K \geq 0$ such that for $g(u) \neq g(v)$,*

$$\aleph(\Upsilon_p(u), \Upsilon_r(v)) \leq \psi(\rho(g(u), g(v)))\rho(g(u), g(v)) + Kd(g(v), \Upsilon_p(u))$$

for $g(u) \in \widehat{A}_i, g(v) \in \widehat{A}_{i+1}, \widehat{A}_{\alpha+1} = \widehat{A}_1$.
Then we can find $u^* \in \widetilde{\bigwedge}$ *such that* $g(u^*) \in \bigcap_{p\in\mathbb{N}} \Upsilon_p(u^*)$.

Proof. Given that $\widehat{A}_i, i = \overline{1,\alpha}$ are closed in $\widetilde{\bigwedge}$, it follows that (Φ, ρ) is a complete MS. Given a graph G_r consisting of $V_{et}(G_r) = \Phi$ and $E_{ed}(G_r) = \nabla \cup \{(g(u), g(v)) \in \nabla \times \nabla : u \in \widehat{A}_{i+1}, i = \overline{1,\alpha}, \widehat{A}_{\alpha+1} = \widehat{A}_1\}$, let $g(u), g(v) \in \Phi$ be such that $(g(u), g(v)) \in E_{ed}(G_r) \Leftrightarrow (u, v) \in E_{ed}(G_r)$ with $g(u) \neq g(v)$. Then, $g(u) \in \widehat{A}_i, g(v) \in \widehat{A}_{i+1}$, for each $i = \overline{1,\alpha}$. It follows that

$$\aleph(\Upsilon_p(u), \Upsilon_r(v)) \leq \psi(\rho(g(u), g(v)))\rho(g(u), g(v)) + Kd(g(v), \Upsilon_p(u)).$$

Let $\jmath \in \Upsilon_p(u), \ell \in \Upsilon_r(v)$ and $\rho(g\jmath, g\ell) \leq \rho(gu, gv)$. Then $\jmath \in \Upsilon_p(u) \subseteq \widehat{A}_{i+1}, \ell \in \Upsilon_r(v) \subseteq \widehat{A}_{i+2}$; thus, $(g\jmath, g\ell) \in E_{ed}(G_r)$. Thus, $\{\Upsilon_p\}_{p=1}^{\infty}$ is a non-linear multivalued G_{rg}-contraction. Suppose further that $\{v_n\}$ is a sequence in Φ with $\rho(v_n, v) \longrightarrow 0$ as $n \longrightarrow \infty$, where $v_n \in \Upsilon_n(v_{n-1}) \cap [v_{n-1}]_{G_r}^{\alpha}$ for all $n \in \mathbb{N}$, and $(gv_{n-1}, gv_n) \in E_{ed}(G_r)$ for all $n \in \mathbb{N}$. Clearly, infinitely many terms of $\{v_n\}$ are contained in $\widehat{A}_i$, thus, we can produce a subsequence $\{v_{n_\gamma}\}$such that $\rho(v_{n_\gamma}, v) \longrightarrow 0$ for all $\gamma \in \mathbb{N}$. Since $\widehat{A}_i$ is closed for each $i = \overline{1,\alpha}$, then $v \in \bigcap_{i=1}^{\alpha} \widehat{A}_i$. It follows from the definition of $E_{ed}(G_r)$ that $(v_{n_\gamma}, v) \in E_{ed}(G_r)$ for all $\gamma \in \mathbb{N}$. Consequently,Theorem 2.2 can be applied to find $u^* \in \widetilde{\bigwedge}$ such that $g(u^*) \in \bigcap_{p\in\mathbb{N}} \Upsilon_p(u^*)$. □

4. Applications to integral equations

Integral equations are found to be of great usefulness in studying dynamical systems and stochastic processes. Some examples are in the areas of oscillation problems, sweeping processes, granular systems, control problems and so on. In a like manner, integral equations arise in several problems in mathematical physics, bio-mathematics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, fuzzy set arithmetic, traffic problems, to mention but a few. Usually, the first most concerning problem in the study of differential or integral equations is the conditions for the existence of its solutions. Along this lane, many authors have proposed different fixed point approaches to obtain existence results for differential or integral equations in abstract spaces (see, e.g. [1, 20–22]).

In this section, we examine new conditions for the existence of a unique solution to a more general version of the integral equation analyzed in [28], given as

$$u(t) = h(t) + \int_a^b \Gamma(t, s) f(s, g(u(s)))ds, \ t \in [a, b] = \varpi, \tag{4.1}$$

where $f : \varpi \times \mathbb{R} \longrightarrow \mathbb{R}, \Gamma : \varpi^2 \longrightarrow \mathbb{R}_+, h : \varpi \longrightarrow \mathbb{R}$ and $g : \varpi \longrightarrow \mathbb{R}$ are given continuous functions. Note that if, in (4.1), $h(t) = 0$ and $g = I_{\widetilde{\bigwedge}}$, which is the identity mapping on $\widetilde{\bigwedge}$, then Problem (4.1) represents an integral reformulation of physical phenomena such as the motion of a spring that is under the influence of a frictional force or a damping force. For some articles modeling real-life problems, via integral/differential equations, see [10–14] and the references therein.

Let $\widetilde{\bigwedge} = C(\varpi, \mathbb{R})$ be the set of all real-valued continuous functions defined on ϖ, and let $\rho(u, v) = \max_{t\in\varpi} |u(t) - v(t)|$. Then $(\widetilde{\bigwedge}, \rho)$ is a complete MS. Define the mapping $\Upsilon : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ by

$$\Upsilon u(t) = h(t) + \int_a^b \Gamma(t, s) f(s, g(u(s)))ds, \ t \in \varpi, \tag{4.2}$$

and $\eta : \widetilde{\bigwedge} \longrightarrow \widetilde{\bigwedge}$ by $\eta u = gu$, with $(\eta u)(t) = (gu)(t)$, for all $t \in \varpi$. Then, finding a solution of (4.1) is equivalent to showing that Υ and g have a CoP.
Now, we investigate the existence of solution of (4.1) given the following hypotheses.

Theorem 4.1. *Given the surjective function $g \in C(\varpi, \mathbb{R})$ and $f : \varpi \times \mathbb{R} \longrightarrow \mathbb{R}$ obeying:*

(i) for all $t \in \varpi$,
$$|f(s, g(u(s))) - f(s, g(v(s)))| \leq |g(u(s)) - g(v(s))|,$$
(ii) we can find a function $\lambda_* : \mathbb{R}_+ \longrightarrow [0, 1)$ such that
$$\max_{t\in\varpi} \int_a^b \Gamma(t, s)ds \leq \lambda_*, \ \forall t \in \mathbb{R}_+.$$

Then Problem (4.1) *has a unique solution in $\widetilde{\bigwedge}$.*

Proof. Let $u, v \in \widetilde{\bigwedge}$. Then, we have
$$\begin{aligned}
|\Upsilon u(t) - \Upsilon v(t)| &= \left| \int_a^b \Gamma(t, s)(f(s, g(u(s))) - f(s, g(v(s))))ds \right| \\
&\leq \int_a^b \Gamma(t, s)\, |f(s, g(u(s))) - f(s, g(v(s)))|\, ds \\
&\leq \int_a^b \Gamma(t, s)|g(u(s)) - g(v(s))|ds \\
&\leq \int_a^b \Gamma(t, s)|(\eta u)(s) - (\eta v)(s)|ds \\
&\leq \int_a^b \Gamma(t, s)\rho(\eta u, \eta v)ds \\
&\leq \rho(\eta u, \eta v) \max_{t\in\varpi} \int_a^b \Gamma(t, s)ds \\
&\leq \lambda_*(\rho(\eta u, \eta v))\rho(\eta u, \eta v) \\
&\leq \lambda_*(\rho(\eta u, \eta v))\rho(\eta u, \eta v) + Kd(\eta v, \Upsilon u),
\end{aligned}$$
for all $K \geq 0$. This implies that for each $u, v \in \widetilde{\bigwedge}$, we get $\rho(\Upsilon u, \Upsilon v) \leq \lambda_*(\rho(\eta u, \eta v))\rho(\eta u, \eta v)$. Thus, by applying Corollary 2 with the graph $G_r = G_{r0}$, where $E_{ed}(G_{r0}) = \widetilde{\bigwedge} \times \widetilde{\bigwedge}$, we can find $u^* \in \widetilde{\bigwedge}$ such that $\Upsilon u^* = \eta u^*$, where $(\eta u^*)(t) = (gu^*)(t)$ for each $t \in \varpi$. Thus, u^* is the CoP of Υ and g, which corresponds to the solution of Problem (4.1). □

An open problem

As a future assignment, we suggest the following: a discretized population balance for continuous systems at steady state can be modeled via the following integral equation:
$$g(t) = \frac{\sigma}{2(1 + 2\sigma)} \int_a^b g(t - x)g(x)dx + e^{-t}. \tag{4.3}$$
It is not known whether the existence criteria for the solution of (4.3) can be examined using any of the results obtained in this work. The advantage of analyzing this type of problem is that it will allow us to examine the existence criteria of several non-linear physical phenomena.

5. Conclusions

In this note, a new type of sequence of multivalued contractions under the name non-linear multivalued G_{rg}-contractions on an MS with a graph is introduced (see Definition 2.1). CoP theorems (see Theorem 2.2 and Theorem 2.4) of a single-valued mapping and the new sequence of multivalued mappings were examined via appropriate hypotheses. A comparative illustration (Example 2.3) was constructed to authenticate our assumptions and establish some links between the obtained results herein and their analogues in the literature. Some significant results in an ϵ-chainable MS and cyclic contractions were derived (see Corollaries 1, 2 and Theorems 3.1, 3.2) as some consequences of our findings. From an application view-point, one of the special cases of our theorems was used to investigate novel criteria for solving a more general Fredholm-type integral equation. As a future exercise, an open problem regarding a discretized population balance model whose solution may be discussed using any of the ideas put forward here was unveiled.

Conflict of interest

The authors declare that there are no conflicts of interest.

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