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Non-Archimedean $\mathcal{L}$-fuzzy normed spaces and stability of functional equations

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ABSTRACT

Lee et al. considered the following quadratic functional equation

$$f(lx+y)+f(lx-y)=2l^2f(x)+2f(y)$$

and proved the Hyers–Ulam–Rassias stability of the above functional equation in classical Banach spaces.

In this paper, we prove the Hyers–Ulam–Rassias stability of the above quadratic functional equation in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh in 1965 [1]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [2–12]. One of the problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. Saadati and J. Park [13], respectively, introduced and studied a notion of intuitionistic fuzzy metric (normed) spaces and then Deschrijver et al. and Saadati generalized the concept of intuitionistic fuzzy metric (normed) spaces and introduced and studied a notion of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces [14,15].

On the other hand, the study of stability problems for functional equations is related to a question of Ulam [16] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [17]. Subsequently, the result of Hyers was generalized by Aoki [18] for additive mappings and by Th.M. Rassias [19] for linear mappings by considering an unbounded Cauchy difference. The paper by Th.M. Rassias has provided a lot of influences in the development of what we now call the *Hyers–Ulam–Rassias stability* of functional equations. For more information on such problems, we refer the interested readers to [20–32].

Throughout this paper, assume that l is a fixed positive integer. In this paper, we prove the Hyers–Ulam–Rassias stability of the quadratic functional equation

$$f(lx+y)+f(lx-y)=2l^2f(x)+2f(y) \qquad (1.1)$$

in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

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2. Preliminaries

In this section, we recall some definitions and results for our main results in this paper.

A *triangular norm* (briefly, a *t-norm*) is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative, monotone and has 1 as the unit element. Basic examples are the Lukasiewicz t-norm T_L, $T_L(a, b) = \max\{a + b - 1, 0\}$ for all $a, b \in [0, 1]$ and the t-norms T_P, T_M, T_D, where $T_P(a, b) := ab$, $T_M(a, b) := \min\{a, b\}$,

$$T_D(a, b) := \begin{cases} \min\{a, b\}, & \text{if } \max\{a, b\} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

A t-norm T is said to be *of Hadžić type* (we denote this by $T \in \mathcal{H}$) ([33]) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$, where $x_T^{(n)}$ is defined by

$$x_T^{(1)} = x, \qquad x_T^{(n)} = T(x_T^{(n-1)}, x), \quad \forall n \geq 2,\ x \in [0, 1].$$

Other important triangular norms are as follows (see [34]).

(1) The *Sugeno–Weber family* $\{T_\lambda^{SW}\}_{\lambda\in[-1,\infty]}$, is defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and

$$T_\lambda^{SW}(x, y) = \max\left\{0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right\}$$

if $\lambda \in (-1, \infty)$.

(2) The *Domby family* $\{T_\lambda^{D}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{D}(x, y) = \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda}}$$

if $\lambda \in (-1, \infty)$.

(3) The *Aczel–Alsina family* $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{AA}(x, y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}$$

if $\lambda \in (-1, \infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-ary operation taking, for all $(x_1, \ldots, x_n) \in [0, 1]^n$, the value $T(x_1, \ldots, x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \qquad \mathrm{T}_{i=1}^n x_i = T(\mathrm{T}_{i=1}^{n-1} x_i, x_n) = T(x_1, \ldots, x_n).$$

A t-norm T can also be extended to a countable operation taking, for any sequence $\{x_n\}_{n\in N}$ in $[0, 1]$, the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i.$$

Proposition 2.1 (*[34]*).

(1) *For $T \geq T_L$ the following implication holds:*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

(2) *If T is of Hadžić type, then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1$$

for every sequence $(x_n)_{n\in N}$ in $[0, 1]$ such that $\lim_{n\to\infty} x_n = 1$.

(3) *If $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n)^\alpha < \infty.$$

(4) *If $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

3. $\mathcal{L}$-fuzzy normed spaces

In what follows, we shall adopt the usual terminology, notation and some definitions introduced by Saadati et al. [35].

Definition 3.1 (*[36]*). Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$*-fuzzy set* in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the *degree* (in L) to which u is an element of $\mathcal{A}$.

Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, \quad x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice (see [37,38]).

Definition 3.2 (*[39]*). An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ in the universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

In the last section, t-norms on $([0, 1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(1, x) = x$ for all $x \in [0, 1]$. This definition can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$.

Definition 3.3. A *triangular norm* (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (: boundary condition);
(ii) $(\forall (x, y) \in L^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (: commutativity);
(iii) $(\forall (x, y, z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (: associativity);
(iv) $(\forall (x, x', y, y') \in L^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (: monotonicity).

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be *continuous* if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively,

$$\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y).$$

For examples, $\mathcal{T}(x, y) = \min(x, y)$ and $\mathcal{T}(x, y) = xy$ are two continuous t-norms on $[0, 1]$. The t-norm $\wedge$ defined by

$$\wedge(x, y) = \begin{cases} x & \text{if } x \leq_L y \\ y & \text{if } y \leq_L x \end{cases}$$

is a continuous t-norm.

A t-norm $\mathcal{T}$ can also be defined recursively as an $(n + 1)$-ary operation ($n \in \mathbf{N}$) by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, \dots, x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, \dots, x_n), x_{n+1})$$

for all $n \geq 2$ and $x_i \in L$.

Definition 3.4. (1) A *negator* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.
(2) If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negator*.
(3) The negator N_s on $([0, 1], \leq)$ defined as $N_s(x) = 1 - x$ for all $x \in [0, 1]$ is called the *standard negator* on $([0, 1], \leq)$.

In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 3.5. (1) The triple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathcal{L}$*-fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s \in]0, +\infty[$,
(a) $\mathcal{M}(x, y, t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(d) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)$;
(e) $\mathcal{M}(x, y, \cdot) :]0, +\infty[\to L$ is continuous.
In this case, $\mathcal{M}$ is called an $\mathcal{L}$*-fuzzy metric*.
(2) If $\mathcal{M} = \mathcal{M}_{M,N}$ is an intuitionistic fuzzy set (see Definition 3.2), then the 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy metric space*.

Example 3.6. Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times]0, \infty[$ defined as follows:

$$\mathcal{M}_{M,N}(x, t) = \left(\frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)} \right)$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 3.7. Let $X = \mathbb{N}$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2b_2)$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times]0, \infty[$ defined as follows:

$$\mathcal{M}_{M,N}(x, t) = \begin{cases} \left(\dfrac{x}{y}, \dfrac{y-x}{y}\right) & \text{if } x \le y \\ \left(\dfrac{y}{x}, \dfrac{x-y}{x}\right) & \text{if } y \le x \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 3.8. (1) The triple $(V, \mathcal{P}, \mathcal{T})$ is said to be an $\mathcal{L}$-*fuzzy normed space* if V is vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0, +\infty)$,
(a) $\mathcal{P}(x, t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \le_L \mathcal{P}(x + y, t + s)$;
(e) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;
(f) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.
In this case, $\mathcal{P}$ is called an $\mathcal{L}$-*fuzzy norm*.
(2) If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set (see Definition 3.2), then the 3-tuple $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space*.

Example 3.9. Let $(V, \|\cdot\|)$ be a normed space. Denote $\mathcal{T}(a, b) = (a_1b_1, \min(a_2+b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $V \times (0, +\infty)$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right)$$

for all $t \in \mathbb{R}^+$. Then $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an intuitionistic fuzzy normed space.

Definition 3.10. (1) A sequence $(x_n)_{n\in\mathbb{N}}$ in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ is called a *Cauchy sequence* if, for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \ge n_0$,

$$\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon),$$

where $\mathcal{N}$ is a negator on $\mathcal{L}$.
(2) A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *convergent* to $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$, which is denoted by $x_n \xrightarrow{\mathcal{P}} x$, if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.
(3) An $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in V is convergent.

Note that, if $\mathcal{P}$ is an $\mathcal{L}$-fuzzy norm on V, then the following are satisfied:
(1) $\mathcal{P}(x, t)$ is nondecreasing with respect to t for all $x \in V$.
(2) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in V$ and $t \in (0, +\infty)$.

Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space. If we define

$$\mathcal{M}(x, y, t) = \mathcal{P}(x - y, t)$$

for all $x, y \in V$ and $t > 0$, then $\mathcal{M}$ is an $\mathcal{L}$-fuzzy metric on V, which is called the $\mathcal{L}$-*fuzzy metric* induced by the $\mathcal{L}$-fuzzy norm $\mathcal{P}$.

Definition 3.11. Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$.
(1) For all $t > 0$, we define the *open ball* $B(x, r, t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as follows:

$$B(x, r, t) = \{y \in V : \mathcal{P}(x - y, t) >_L \mathcal{N}(r)\}$$

and define the *unit ball* of V by

$$B(0, r, 1) = \{x : \mathcal{P}(x, 1) >_L \mathcal{N}(r)\}.$$

(2) A subset $A \subseteq V$ is said to be *open* if, for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$.
(3) Let $\tau_{\mathcal{P}}$ denote the family of all open subsets of V. Then $\tau_{\mathcal{P}}$ is called the *topology induced by the* $\mathcal{L}$-*fuzzy norm* $\mathcal{P}$.

Note that, in the case of an intuitionistic fuzzy normed space, this topology is the same as the topology induced by intuitionistic fuzzy metric which is Hausdorff (see Remark 3.3 and Theorem 3.5 of [40]).

Definition 3.12. Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. A subset A of V is said to be $\mathcal{L}F$-*bounded* if there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x, t) >_L \mathcal{N}(r)$ for all $x \in A$.

Note that, in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$, every compact set is closed and $\mathcal{L}F$-bounded (see Remark 3.10 of [40]).

4. Non-Archimedean $\mathcal{L}$-fuzzy normed spaces

In 1897, Hensel [41] introduced a field with a valuation in which it does not have the Archimedean property.

Definition 4.1. Let $\mathcal{K}$ be a field. A *non-Archimedean absolute value* on $\mathcal{K}$ is a function $|\cdot| : \mathcal{K} \to [0, +\infty)$ such that, for any $a, b \in \mathcal{K}$,

(i) $|a| \geq 0$ and the equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq \max\{|a|, |b|\}$ (the strict triangle inequality).

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \in \mathcal{K}$ such that $|a_0| \neq 0, 1$.

Definition 4.2. A *non-Archimedean $\mathcal{L}$-fuzzy normed space* is a triple $(V, \mathcal{P}, \mathcal{T})$, where V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0, +\infty)$,

(a) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(b) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, \max\{t, s\})$;
(e) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;
(f) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

Example 4.3. Let $(X, \|.\|)$ be a non-Archimedean normed linear space. Then the triple $(X, \mathcal{P}, \min)$, where

$$\mathcal{P}(x, t) = \begin{cases} 0, & \text{if } t \leq \|x\|; \\ 1, & \text{if } t > \|x\|, \end{cases}$$

is a non-Archimedean $\mathcal{L}$-fuzzy normed space in which $L = [0, 1]$.

Example 4.4. Let $(X, \|\cdot\|)$ be is a non-Archimedean normed linear space. Denote $\mathcal{T}_M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $X \times]0, +\infty[$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right)$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ is a non-Archimedean intuitionistic fuzzy normed space.

5. $\mathcal{L}$-Fuzzy Hyers–Ulam–Rassias stability

Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$.

In this section, we prove the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1).

Next, we define an $\mathcal{L}$-fuzzy approximately quadratic mapping. Let Ψ be an $\mathcal{L}$-fuzzy set on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing,

$$\Psi(cx, cx, t) \geq_L \Psi\left(x, x, \frac{t}{|c|}\right), \quad \forall x \in X, \quad c \neq 0$$

and

$$\lim_{t\to\infty} \Psi(x, y, t) = 1_{\mathcal{L}}, \quad \forall x, y \in X, \quad t > 0.$$

Definition 5.1. A mapping $f : X \to Y$ is said to be *Ψ-approximately quadratic* if

$$\mathcal{P}(f(lx + y) + f(lx - y) - 2l^2 f(x) - 2f(y), t) \geq_L \Psi(x, y, t), \quad \forall x, y \in X,\ t > 0. \tag{5.1}$$

The following is one of our main results in this section.

Theorem 5.2. *Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$. Let $f : X \to Y$ be a Ψ-approximately quadratic mapping and $f(0) = 0$. If there exist an $\alpha \in \mathbb{R}(\alpha > 0)$ and an integer k, $k \geq 2$ with $|l^k| < \alpha$, $|l| \neq 1$ and $l \neq 0$ such that*

$$\Psi(l^{-k}x, l^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x \in X,\ t > 0, \tag{5.2}$$

and

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|l|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), t) \geq \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|l|^{ki}}\right), \quad \forall x \in X,\ t > 0, \tag{5.3}$$

where

$$\mathcal{M}(x, t) := \mathcal{T}(\Psi(x, 0, t), \Psi(lx, 0, t), \dots, \Psi(l^{k-1}x, 0, t)), \quad \forall x \in X,\ t > 0.$$

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\begin{aligned}\mathcal{P}(f(l^j x) - l^{2j} f(x), t) &\geq_L \mathcal{M}_j(x, t)\\ &:= \mathcal{T}(\Psi(x, 0, t), \dots, \Psi(l^{j-1}x, 0, t)).\end{aligned} \tag{5.4}$$

Putting $y = 0$ in (5.1), we obtain

$$\mathcal{P}(2f(lx) - 2l^2 f(x), t) \geq_L \Psi(x, 0, t), \quad \forall x \in X,\ t > 0,$$

and

$$\mathcal{P}(f(lx) - l^2 f(x), t) \geq_L \Psi(x, 0, 2t) \geq_L \Psi(x, 0, t), \quad \forall x \in X,\ t > 0.$$

This proves (5.4) for $j = 1$. Let (5.4) hold for some $j > 1$. Replacing y by 0 and x by $l^j x$ in (5.1), we get

$$\mathcal{P}(f(l^{j+1}x) - l^2 f(l^j x), t) \geq_L \Psi(l^j x, 0, t), \quad \forall x \in X,\ t > 0.$$

Since $|l| \leq 1$, it follows that

$$\begin{aligned}\mathcal{P}(f(l^{j+1}x) - l^{2(j+1)} f(x), t) &\geq_L \mathcal{T}(\mathcal{P}(f(l^{j+1}x) - l^2 f(l^j x), t), \mathcal{P}(l^2 f(l^j x) - l^{2(j+1)} f(x), t))\\ &= \mathcal{T}\left(\mathcal{P}(f(l^{j+1}x) - l^2 f(l^j x), t), \mathcal{P}\left(f(l^j x) - l^{2j} f(x), \frac{t}{|l|^2}\right)\right)\\ &\geq_L \mathcal{T}(\mathcal{P}(f(l^{j+1}x) - l^2 f(l^j x), t), \mathcal{P}(f(l^j x) - l^{2j} f(x), t))\\ &\geq_L \mathcal{T}(\Psi(l^j x, 0, t), \mathcal{M}_j(x, t))\\ &= \mathcal{M}_{j+1}(x, t), \quad \forall x \in X,\ t > 0.\end{aligned}$$

Thus (5.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(l^k x) - l^{2k} f(x), t) \geq_L \mathcal{M}(x, t), \quad \forall x \in X,\ t > 0. \tag{5.5}$$

Replacing x by $l^{-(kn+k)}x$ in (5.5) and using inequality (5.2), we obtain

$$\begin{aligned}\mathcal{P}\left(f\left(\frac{x}{l^{kn}}\right) - l^{2k} f\left(\frac{x}{l^{kn+k}}\right), t\right) &\geq_L \mathcal{M}\left(\frac{x}{l^{kn+k}}, t\right)\\ &\geq_L \mathcal{M}(x, \alpha^{n+1} t) \quad \forall x \in X,\ t > 0,\ n \geq 0\end{aligned}$$

and so

$$\begin{aligned}\mathcal{P}\left((l^{2k})^n f\left(\frac{x}{(l^k)^n}\right) - (l^{2k})^{n+1} f\left(\frac{x}{(l^k)^{n+1}}\right), t\right) &\geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{|(l^{2k})^n|} t\right)\\ &\geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{|(l^k)^n|} t\right), \quad \forall x \in X,\ t > 0,\ n \geq 0.\end{aligned}$$

Hence it follows that

$$\mathcal{P}\left((l^{2k})^n f\left(\frac{x}{(l^k)^n}\right) - (l^{2k})^{n+p} f\left(\frac{x}{(l^k)^{n+p}}\right), t\right) \geq_L \mathcal{T}_{j=n}^{n+p}\left(\mathcal{P}\left((l^{2k})^j f\left(\frac{x}{(l^k)^j}\right) - (l^{2k})^{j+p} f\left(\frac{x}{(l^k)^{j+p}}\right), t\right)\right)$$
$$\geq_L \mathcal{T}_{j=n}^{n+p}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|(l^k)^j|}t\right), \quad \forall x \in X,\ t > 0,\ n \geq 0.$$

Since $\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}(x, \frac{\alpha^{j+1}}{|(l^k)^j|}t) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\{(l^{2k})^n f(\frac{x}{(l^k)^n})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence we can define a mapping $Q : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((l^{2k})^n f\left(\frac{x}{(l^k)^n}\right) - Q(x), t\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0. \tag{5.6}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\mathcal{P}\left(f(x) - (l^{2k})^n f\left(\frac{x}{(l^k)^n}\right), t\right) = \mathcal{P}\left(\sum_{i=0}^{n-1}(l^{2k})^i f\left(\frac{x}{(l^k)^i}\right) - (l^{2k})^{i+1} f\left(\frac{x}{(l^k)^{i+1}}\right), t\right)$$
$$\geq_L \mathcal{T}_{i=0}^{n-1}\left(\mathcal{P}\left((l^{2k})^i f\left(\frac{x}{(l^k)^i}\right) - (l^{2k})^{i+1} f\left(\frac{x}{(l^k)^{i+1}}\right), t\right)\right)$$
$$\geq_L \mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|l^k|^i}\right)$$

and so

$$\mathcal{P}(f(x) - Q(x), t) \geq_L \mathcal{T}\left(\mathcal{P}\left(f(x) - (l^{2k})^n f\left(\frac{x}{(l^k)^n}\right), t\right), \mathcal{P}\left((l^{2k})^n f\left(\frac{x}{(l^k)^n}\right) - Q(x), t\right)\right)$$
$$\geq_L \mathcal{P}\left(\mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|l^k|^i}\right), \mathcal{P}\left((l^{2k})^n f\left(\frac{x}{(l^k)^n}\right) - Q(x), t\right)\right). \tag{5.7}$$

Taking the limit as $n \to \infty$ in (5.7), we obtain

$$\mathcal{P}(f(x) - Q(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|l^k|^i}\right),$$

which proves (5.3). As $\mathcal{T}$ is continuous, from a well-known result in $\mathcal{L}$-fuzzy (probabilistic) normed space (see [42], Chapter 12), it follows that

$$\lim_{n\to\infty} \mathcal{P}((l^{2k})^n f(l^{-kn}(lx+y)) + (l^{2k})^n f(l^{-kn}(lx-y)) - 2(l^{2k})^n f(l^{-kn}(x)) - 2(l^{2k})^n f(l^{-kn}(y)), t)$$
$$= \mathcal{P}(Q(lx+y) + Q(lx-y) - 2l^2 Q(x) - 2Q(y), t)$$

for almost all $t > 0$.

On the other hand, replacing x, y by $l^{-kn}x, l^{-kn}y$ in Eqs. (5.1) and (5.2), we get

$$\mathcal{P}((l^{2k})^n f(l^{-kn}(lx+y)) + (l^{2k})^n f(l^{-kn}(lx-y)) - 2(l^{2k})^n f(l^{-kn}(x)) - 2(l^{2k})^n f(l^{-kn}(y)), t)$$
$$\geq_L \Psi\left(l^{-kn}x, l^{-kn}y, \frac{t}{|l^{2k}|^n}\right)$$
$$\geq_L \Psi\left(x, y, \frac{\alpha^n t}{|l^k|^n}\right), \quad \forall x, y \in X,\ t > 0.$$

Since $\lim_{n\to\infty} \Psi(x, y, \frac{\alpha^n t}{|l^k|^n}) = 1_{\mathcal{L}}$, we infer that Q is a quadratic mapping.

For the uniqueness of Q, let $Q' : X \to Y$ be another quadratic mapping such that

$$\mathcal{P}(Q'(x) - f(x), t) \geq_L \mathcal{M}(x, t), \quad \forall x \in X,\ t > 0.$$

Then we have, for all $x, y \in X$ and $t > 0$,

$$\mathcal{P}(Q(x) - Q'(x), t) \geq_L \mathcal{T}\left(\mathcal{P}\left(Q(x) - (l^{2k})^n f\left(\frac{x}{(l^k)^n}\right), t\right), \mathcal{P}\left((l^{2k})^n f\left(\frac{x}{(l^k)^n}\right) - Q'(x), t\right), t\right).$$

Therefore, from (5.6), we conclude that $Q = Q'$. This completes the proof. □

Corollary 5.3. *Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$ under a t-norm $\mathcal{T} \in \mathcal{H}$. Let $f : X \to Y$ be a Ψ-approximately quadratic mapping. If there exist an $\alpha \in \mathbb{R}(\alpha > 0)$, $|l| \neq 1$, $l \neq 0$ and an integer k, $k \geq 2$ with $|l^k| < \alpha$ such that*

$$\Psi(l^{-k}x, l^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x \in X,\ t > 0,$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), t) \geq_L \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|l|^{ki}}\right), \quad \forall x \in X,\ t > 0,$$

where

$$\mathcal{M}(x, t) := \mathcal{T}(\Psi(x, 0, t), \Psi(lx, 0, t), \ldots, \Psi(l^{k-1}x, 0, t)), \quad \forall x \in X,\ t > 0.$$

Proof. Since

$$\lim_{n\to\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|l|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

and $\mathcal{T}$ is of Hadžić type, it follows from Proposition 2.1 that

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|l|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0.$$

Now, if we apply Theorem 5.2, we get the conclusion. □

Now, we give an example to validate the main result as follows:

Example 5.4. Let $(X, \|.\|)$ be a non-Archimedean Banach space, $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) in which

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \quad \forall x \in X,\ t > 0,$$

and let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ be a complete non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) (see Example 4.4). Define

$$\Psi(x, y, t) = \left(\frac{t}{1+t}, \frac{1}{1+t}\right).$$

It is easy to show that (5.2) holds for $\alpha = 1$ (note that $|l| \neq 1$, $l \neq 0$). Also, since

$$\mathcal{M}(x, t) = \left(\frac{t}{1+t}, \frac{1}{1+t}\right),$$

we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}_{M,j=n}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|l|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} \mathcal{T}_{M,j=n}^{m} \mathcal{M}\left(x, \frac{t}{|l|^{kj}}\right)\right) \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t + |l^k|^n}, \frac{|l^k|^n}{t + |l^k|^n}\right) \\
&= (1, 0) = 1_{L^*}, \quad \forall x \in X,\ t > 0.
\end{aligned}$$

Let $f : X \to Y$ be a Ψ-approximately quadratic mapping. Therefore, all the conditions of Theorem 5.2 hold and so there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} \left(\frac{t}{t + |l^k|}, \frac{|l^k|}{t + |l^k|}\right).$$

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References

[1] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.
[2] M. Amini, R. Saadati, Topics in fuzzy metric space, J. Fuzzy Math. 4 (2003) 765–768.
[3] A. George, P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets and Systems 64 (1994) 395–399.

[4] A. George, P. Veeramani, On some result of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (1997) 365–368.
[5] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 485–489.
[6] V. Gregori, S. Romaguera, On completion of fuzzy metric spaces, Fuzzy Sets and Systems 130 (2002) 399–404.
[7] V. Gregori, S. Romaguera, Characterizing completable fuzzy metric spaces, Fuzzy Sets and Systems 144 (2004) 411–420.
[8] S.B. Hosseini, R. Saadati, M. Amini, Alexandroff theorem in fuzzy metric spaces, Math. Sci. Res. J. 8 (2004) 357–361.
[9] C. Hu, $\mathcal{C}$-structure of FTS V: fuzzy metric spaces, J. Fuzzy Math. 3 (1995) 711–721.
[10] R. Lowen, Fuzzy Set Theory, Kluwer Academic Publishers, Dordrecht, 1996.
[11] R. Saadati, S.M. Vaezpour, Some results on fuzzy Banach spaces, J. Appl. Math. Comput. 17 (2005) 475–484.
[12] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 314–334.
[13] R. Saadati, J. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals 27 (2006) 331–344.
[14] G. Deschrijver, D. O'Regan, R. Saadati, S.M. Vaezpour, L-Fuzzy Euclidean normed spaces and compactness, Chaos Solitons Fractals 42 (2009) 40–45.
[15] R. Saadati, On the $\mathcal{L}$-Fuzzy topological spaces, Chaos Solitons Fractals 37 (2008) 1419–1426.
[16] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964, Chapter VI, Science Editions.
[17] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941) 222–224.
[18] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64–66.
[19] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[20] S. Abbaszadeh, Intuitionistic fuzzy stability of a quadratic and quartic functional equation, Int. J. Nonlinear Anal. Appl. 1 (2) (2010) 100–124.
[21] Y.J. Cho, C. Park, R. Saadati, Functional inequalities in non-Archimedean Banach spaces, Appl. Math. Lett. 10 (2010) 1238–1242.
[22] P. Găvruta, L. Găvruta, A new method for the generalized Hyers–Ulam–Rassias stability, Int. J. Nonlinear Anal. Appl. 1 (2) (2010) 11–18.
[23] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.
[24] S. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.
[25] J. Lee, J. An, C. Park, On the stability of quadratic functional equations, Abstract Appl. Anal. 2008 (2008) 8. Article ID 628178.
[26] Th.M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
[27] R. Saadati, Y.J. Cho, J. Vahidi, The stability of the quartic functional equation in various spaces, Comput. Math. Appl. (2010) doi:10.1016/j.camwa.2010.07.034.
[28] R. Saadati, S.M. Vaezpour, Y. Cho, A note on the "On the stability of cubic mappings and quadratic mappings in random normed spaces", J. Inequal. Appl. 2009 (2009). Article ID 214530.
[29] R. Saadati, C. Park, J.M. Rassias, Gh. Sadeghi, Stability of a quartic functional equation in various random normed spaces, Abstr. Appl. Anal. (in press).
[30] R. Saadati, S.M. Vaezpour, C. Park, The stability of the cubic functional equation in various spaces, Math. Commun. (in press).
[31] S. Shakeri, Intuitionistic Fuzzy stability of Jensen type mapping, J. Nonlinear Sci. Appl. 2 (2009) 105–112.
[32] S. Shakeri, R. Saadati, C. Park, Stability of the quadratic functional equation in non-Archimedean $\mathcal{L}-$fuzzy normed spaces, Int. J. Nonlinear Anal. Appl. 1 (2) (2010) 72–83.
[33] O. Hadžić, E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic, Dordrecht, 2001.
[34] O. Hadžić, E. Pap, M. Budincević, Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces, Kybernetica 38 (2002) 363–381.
[35] R. Saadati, A. Razani, H. Adibi, A common fixed point theorem in $\mathcal{L}$-fuzzy metric spaces, Chaos Solitons Fractals 33 (2007) 358–363.
[36] J. Goguen, $\mathcal{L}$-Fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[37] G. Deschrijver, C. Cornelis, E.E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, IEEE Trans. Fuzzy Syst. 12 (2004) 45–61.
[38] G. Deschrijver, E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems 133 (2003) 227–235.
[39] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[40] J. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals 22 (2004) 1039–1046.
[41] K. Hensel, Uber eine neue Begrundung der Theorie der algebraischen Zahlen Jahres, Deutsch. Math. Verein 6 (1897) 83–88.
[42] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Elsevier, North Holand, New York, 1983.