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Functional inequalities in non-Archimedean Banach spaces

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ABSTRACT

In this work, we prove the generalized Hyers–Ulam stability of the following functional inequality:

$$\|f(x)+f(y)+f(z)\| \leq \left\|kf\left(\frac{x+y+z}{k}\right)\right\|, \quad |k| < |3|,$$

in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field $\mathcal{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r|+|s|, \quad \forall r,s \in \mathcal{K}.$$

A field $\mathcal{K}$ is called a *valued field* if $\mathcal{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a condition stronger than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|,|s|\}, \quad \forall r,s \in \mathcal{K},$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$.

Throughout this work, we assume that the base field is a non-Archimedean field and hence call it simply a field.

Definition 1.1 (*[1]*). Let $\mathcal{X}$ be a vector space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is called a *non-Archimedean norm* if the following conditions hold:

(i) $\|x\| = 0$ if and only if $x = 0$ for all $x \in \mathcal{X}$;
(ii) $\|rx\| = |r|\|x\|$ for all $r \in \mathcal{K}$ and $x \in \mathcal{X}$;
(iii) the strong triangle inequality holds:

$$\|x+y\| \leq \max\{\|x\|,\|y\|\}, \quad \forall x,y \in \mathcal{X}.$$

Then $(\mathcal{X},\|\cdot\|)$ is called a *non-Archimedean normed space*.

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Definition 1.2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $\mathcal{X}$.

(1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a Cauchy sequence iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
(2) The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there are a positive integer N and $x \in \mathcal{X}$ such that

$$\|x_n - x\| \leq \varepsilon, \quad \forall n \geq N.$$

Then the point $x \in \mathcal{X}$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.
(3) If every Cauchy sequence in X converges, then the non-Archimedean normed space $\mathcal{X}$ is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [2] concerning the stability of group homomorphisms. Hyers [3] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [4] for additive mappings and by Rassias [5] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [5] has had a lot of influence in the development of what we call *generalized Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias approach.

The stability problems for several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7–15,5,16–22]). In 2007, Park et al. [23] investigated three-variable functional inequalities and proved the generalized Hyers–Ulam stability of three-variable functional inequalities in Banach spaces. Also, the stability problems in non-Archimedean Banach space are studied by Moslehian and Rassias [24], Moslehian and Sadeghi [1,25], Mirmostafaee [26] and Najati and Moradlou [27].

In this work, we prove that if f satisfies the functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \left\| kf\left(\frac{x+y+z}{k}\right)\right\|, \quad |k| < |3|, \tag{1.1}$$

then f is additive, and prove the generalized Hyers–Ulam stability of the functional inequality (1.1) in non-Archimedean Banach spaces.

Throughout this work, assume that $\mathcal{X}$ is a non-Archimedean normed space and that $\mathcal{Y}$ is a non-Archimedean Banach space. Let $|2| \neq 1$; also we assume that $2 \neq 0$ in $\mathcal{K}$ (i.e. the characteristic of $\mathcal{K}$ is not 2).

2. Generalized Hyers–Ulam stability of the functional inequality (1.1)

Let k be a fixed integer greater than 3 and let $|k| < |3|$.

Proposition 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\|_{\mathcal{Y}} \leq \left\| kf\left(\frac{x+y+z}{k}\right)\right\|_{\mathcal{Y}}, \quad \forall x, y, z \in \mathcal{X}. \tag{2.1}$$

Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$\|3f(0)\|_{\mathcal{Y}} \leq \|kf(0)\|_{\mathcal{Y}}.$$

Since $|3| > |k|, f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.1), we get

$$\|f(x) + f(-x)\|_{\mathcal{Y}} \leq \|kf(0)\|_{\mathcal{Y}} = 0, \quad \forall x \in \mathcal{X}.$$

Hence $f(-x) = -f(x)$ for all $x \in \mathcal{X}$.

Letting $z = -x - y$ in (2.1), we get

$$\begin{aligned} \|f(x) + f(y) - f(x+y)\|_{\mathcal{Y}} &= \|f(x) + f(y) + f(-x-y)\|_{\mathcal{Y}} \\ &\leq \|kf(0)\|_{Y} \\ &= 0, \quad \forall x, y \in \mathcal{X}. \end{aligned}$$

Thus we have

$$f(x+y) = f(x) + f(y), \quad \forall x, y \in \mathcal{X}.$$

This completes the proof. □

Now, we prove the generalized Hyers–Ulam stability of the functional inequality (1.1).

Theorem 2.2. *Let $r < 1, \theta$ be nonnegative real numbers and $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\|_{\mathcal{Y}} \le \left\| kf\left(\frac{x+y+z}{k}\right)\right\|_{\mathcal{Y}} + \theta(\|x\|_{\mathcal{X}}^r + \|y\|_{\mathcal{X}}^r + \|z\|_{\mathcal{X}}^r), \quad \forall x, y, z \in \mathcal{X}. \tag{2.2}$$

Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\|_{\mathcal{Y}} \le \frac{2 + |2|^r}{|2|}\theta\|x\|_{\mathcal{X}}^r, \quad \forall x \in \mathcal{X}. \tag{2.3}$$

Proof. Letting $y = x$ and $z = -2x$ in (2.2), we get

$$\|2f(x) - f(2x)\|_{\mathcal{Y}} = \|2f(x) + f(-2x)\|_{\mathcal{Y}} \le (2 + |2|^r)\theta\|x\|_{\mathcal{X}}^r, \quad \forall x \in \mathcal{X}, \tag{2.4}$$

and so

$$\|f(x) - 2f\left(\frac{x}{2}\right)\|_{\mathcal{Y}} \le \frac{2 + |2|^r}{|2|^r}\theta\|x\|_{\mathcal{X}}^r, \quad \forall x \in \mathcal{X}.$$

Hence we have

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} \le \frac{2 + |2|^r}{|2|^{(r-1)n+1}}\theta\|x\|_{\mathcal{X}}^r, \quad \forall m, n \ge 1\ (m > l),\ x \in \mathcal{X}. \tag{2.5}$$

It follows from (2.5) that the sequence $\left\{2^k f\left(\frac{x}{2^k}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a non-Archimedean Banach space, the sequence $\left\{2^k f\left(\frac{x}{2^k}\right)\right\}$ converges. So one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right), \quad \forall x \in \mathcal{X}.$$

Now, let $T : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (2.3). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\|_{\mathcal{Y}} \\ &\le \max\left\{\left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|_{\mathcal{Y}}, \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|_{\mathcal{Y}}\right\} \\ &\le \frac{2 + |2|^r}{|2|^{(r-1)q+1}}\theta\|x\|_{\mathcal{X}}^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in \mathcal{X}$. So we can conclude that $A(x) = T(x)$ for all $x \in \mathcal{X}$. This proves the uniqueness of A.

It follows from (2.2) that

$$\begin{aligned} \|A(x) + A(y) + A(y)\|_{\mathcal{Y}} &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right)\right\|_{\mathcal{Y}} \\ &\le \lim_{n\to\infty}\left\|2^n kf\left(\frac{x+y+z}{2^n k}\right)\right\|_{\mathcal{Y}} + \lim_{n\to\infty}\frac{|2|^n\theta}{|2|^{nr}}(\|x\|_{\mathcal{X}}^r + \|y\|_{\mathcal{X}}^r + \|z\|_{\mathcal{X}}^r) \\ &= \left\|kA\left(\frac{x+y+z}{k}\right)\right\|_{\mathcal{Y}}, \quad \forall x, y, z \in \mathcal{X}, \end{aligned}$$

and so

$$\|A(x) + A(y) + A(z)\|_{\mathcal{Y}} \le \left\|kA\left(\frac{x+y+z}{k}\right)\right\|_{Y}, \quad \forall x, y, z \in \mathcal{X}.$$

By Proposition 2.1, the mapping $A : \mathcal{X} \to \mathcal{Y}$ is additive. This completes the proof. □

Theorem 2.3. *Let $r > 1, \theta$ be nonnegative real numbers and $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping satisfying (2.2). Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - A(x)\|_{\mathcal{Y}} \le \frac{2 + |2|^r}{|2|}\theta\|x\|_{\mathcal{X}}^r, \quad \forall x \in \mathcal{X}.$$

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_{\mathcal{Y}} \leq \frac{2 + |2|^r}{|2|} \theta \|x\|_{\mathcal{X}}^r, \quad \forall x \in \mathcal{X}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.4. *Let $r < \frac{1}{3}$, θ be nonnegative real numbers and $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\|_{\mathcal{Y}} \leq \left\| kf\left(\frac{x+y+z}{k}\right) \right\|_{\mathcal{Y}} + \theta \cdot \|x\|_{\mathcal{X}}^r \cdot \|y\|_{\mathcal{X}}^r \cdot \|z\|_{\mathcal{X}}^r, \quad \forall x, y, z \in \mathcal{X}. \tag{2.6}$$

Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\|_{\mathcal{Y}} \leq \frac{|2|^r \theta}{|2|^{3r}} \|x\|_{\mathcal{X}}^{3r}, \quad \forall x \in X.$$

Proof. Letting $y = x$ and $z = -2x$ in (2.6), we get

$$\|2f(x) - f(2x)\|_{\mathcal{Y}} = \|2f(x) + f(-2x)\|_Y \leq |2|^r \theta \|x\|_{\mathcal{X}}^{3r}, \quad \forall x \in \mathcal{X}, \tag{2.7}$$

and so

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_{\mathcal{Y}} \leq \frac{|2|^r}{|2|^{3r}} \theta \|x\|_{\mathcal{X}}^{3r}, \quad \forall x \in \mathcal{X}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.5. *Let $r > \frac{1}{3}$, θ be positive real numbers and $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping satisfying (2.6). Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - A(x)\|_{\mathcal{Y}} \leq \frac{|2|^r \theta}{|2|} \|x\|_{\mathcal{X}}^{3r}, \quad \forall x \in \mathcal{X}.$$

Proof. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_{\mathcal{Y}} \leq \frac{|2|^r}{|2|} \theta \|x\|_{\mathcal{X}}^{3r}, \quad \forall x \in \mathcal{X}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

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