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A FIXED POINT APPROACH TO THE CAUCHY-RASSIAS STABILITY OF GENERAL JENSEN TYPE QUADRATIC-QUADRATIC MAPPINGS

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Abstract. In this paper, we investigate the Cauchy-Rassias stability in Banach spaces and also the Cauchy-Rassias stability using the alternative fixed point for the functional equation:

$$f(\frac{sx+ty}{2}+rz)+f(\frac{sx+ty}{2}-rz)+f(\frac{sx-ty}{2}+rz)+f(\frac{sx-ty}{2}-rz)$$
$$=s^2f(x)+t^2f(y)+4r^2f(z)$$

for any fixed nonzero integers s, t, r with $r \neq \pm 1$.

1. Introduction

Ulam [29] raised the following problem concerning the stability of homomorphisms: Give conditions in order for a linear mapping near an approximately linear mapping to exist? The following theorem which is called the Cauchy-Rassias stability is a generalized solution to this problem.

Theorem 1.1. *Let E be a real normed space, F be a real Banach space and $f : E \to F$ be a mapping such that for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Assume that there exist constants $\varepsilon \geq 0$ and $p \geq 0$ with $p \neq 1$ such that*

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p) \quad (x, y \in E).$$

Then there exists a unique $\mathbb{R}$-linear mapping $T : E \to F$ satisfying

$$\|f(x)-T(x)\| \leq \varepsilon\|x\|^p/(1-2^{p-1}) \quad (x \in E).$$

The above Cauchy-Rassias stability in real Banach spaces was obtained by Hyers [10] for the case $p = 0$, by Rassias [24] for the case $p \in (0, 1)$, and by Gajda [9] for the case $p > 1$. In particular, Rassias and Šemrl [25] gave an

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example to show that it does not occur for the case $p = 1$. Also, Trif [28] studied the Cauchy-Rassias stability of the Jensen type functional equation. In addition, Park [17] studied the Cauchy-Rassias stability of modified Trif functional equations associated with homomorphisms in Banach module over C*-algebras.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to symmetric bi-additive function and is called a quadratic functional equation and every solution of the quadratic equation (1.1) is said to be a quadratic function particulary. It is well known that a function f between two real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$, where

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y))$$

for all x (see [1, 13]). Skof proved Hyers-Ulam stability problem for the quadratic functional equation (1.1) for a class of functions $f : A \to B$, where A is normed space and B is a Banach space (see [27]). In 1992, Czerwik [7] proved the Cauchy-Rassias stability of the equation (1.1) (see also [18, 19]). Recently, Park, Hong, and Kim [19] have investigated the Cauchy-Rassias stability of the Jensen type quadratic-quadratic equation:

$$f(\frac{x+y}{2}+z)+f(\frac{x+y}{2}-z)+f(\frac{x-y}{2}+z)+f(\frac{x-y}{2}-z) = f(x)+f(y)+4f(z)$$

in Banach spaces. Several functional equations have been investigated in [2, 3, 11, 14, 16, 22, 23].

We recall some basic facts concerning Jensen type quadratic-quadratic mapping.

Definition 1.2. An even mapping $Q : X \to Y$ is called quadratic-quadratic if Q satisfies $Q(0) = 0$ and the functional equation (1.1). We note that (1.1) is equivalent to the Jensen quadratic equation

$$2f(\frac{z+w}{2}) + 2f(\frac{z-w}{2}) = f(z) + f(w) \tag{1.2}$$

for $z = x + y, w = x - y$. An even mapping $Q : X \to Y$ is called Jensen type quadratic-quadratic mapping if Q satisfies $Q(0) = 0$ and the functional equation (1.2).

Now we introduce the general Jensen type quadratic-quadratic functional equation:

(1.3)

$$\begin{aligned} &f(\frac{sx+ty}{2}+rz) + f(\frac{sx+ty}{2}-rz) + f(\frac{sx-ty}{2}+rz) + f(\frac{sx-ty}{2}-rz) \\ &= s^2 f(x) + t^2 f(y) + 4r^2 f(z) \end{aligned}$$

for any fixed nonzero integers s, t, r with $r \neq \pm 1$. Afterward, we investigate the Cauchy-Rassias stability in Banach spaces and also the Cauchy-Rassias stability using the alternative fixed point.

2. General Jensen type quadratic-quadratic functional equation

Let X and Y be real vector spaces. We here present the general solution of (1.3).

Theorem 2.1. *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and the functional equation* (1.3)*, then the mapping f satisfies* (1.2). *Therefore, any even mapping f satisfies* (1.3) *and $f(0) = 0$ is a Jensen type quadratic-quadratic mapping.*

Proof. Letting $y = z = 0$ in (1.3) and using $f(0) = 0$, we get

$$f(\frac{s}{2}x) = \frac{s^2}{4}f(x) \tag{2.1}$$

for all $x \in X$. Setting $x = y = 0$ in (1.3) and using the evenness of f, we obtain

$$f(rz) = r^2 f(z) \tag{2.2}$$

for all $z \in X$. So

$$f(\frac{sr}{2}x) = \frac{s^2 r^2}{4}f(x) \tag{2.3}$$

for all $x \in X$. Replacing x, y and z by rx, 0 and $\frac{s}{2}z$ in (1.3), respectively, we have

$$f(\frac{sr}{2}x + \frac{sr}{2}z) + f(\frac{sr}{2}x - \frac{sr}{2}z) = s^2 f(rx) + 4r^2 f(\frac{s}{2}z) \tag{2.4}$$

for all $x, z \in X$. But since $s, r \neq 0$, it follows from (2.1), (2.2), (2.3) and (2.4) that

$$f(x + z) + f(x - z) = 2f(x) + 2f(z) \tag{2.5}$$

for all $x, z \in X$. Now, we substitute $x = \frac{x+y}{2}$ and $z = \frac{x-y}{2}$ in (2.5), we lead to

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y)$$

for all $x, y \in X$.

Therefore f satisfies (1.2). □

3. Cauchy-Rassias stability in Banach spaces

From this point on, let X be a real vector space and let Y be a Banach space. Before taking up the main subject, for the given mapping $f : X \to Y$ we define the difference operator $\Delta_f : X \times X \times X \to Y$ by

$$\begin{aligned}\Delta_f(x, y, z) := f(\frac{sx + ty}{2} + rz) + f(\frac{sx + ty}{2} - rz) + f(\frac{sx - ty}{2} + rz)\\ + f(\frac{sx - ty}{2} - rz) - s^2 f(x) - t^2 f(y) - 4r^2 f(z)\end{aligned}$$

for all $x, y, z \in X$ and any fixed nonzero integers s, t, r with $r \neq \pm 1$.

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \times X \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(z) := \sum_{i=\frac{1-j}{2}}^{\infty} \frac{1}{r^{2ij}} \varphi(0, 0, r^{ij} z) < \infty, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{r^{2nj}} \varphi(r^{nj}x, r^{nj}y, r^{nj}z) = 0 \tag{3.2}$$

for all $x, y, z \in X$. Suppose that $f : X \to Y$ be an even mapping with $f(0) = 0$ satisfies

$$\|\Delta_f(x, y, z)\| \leq \varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(z) - Q(z)\| \leq \frac{1}{4r^{1+j}} \widetilde{\varphi}\left(\frac{z}{r^{\frac{1-j}{2}}}\right) \tag{3.4}$$

for all $z \in X$.

Proof. For $j = 1$. Setting $x = y = 0$ in (3.3) and using the evenness of f, we obtain

$$\|4f(rz) - 4r^2 f(z)\| \leq \varphi(0, 0, z) \tag{3.5}$$

for all $z \in X$. So

$$\|f(z) - \frac{1}{r^2} f(rz)\| \leq \frac{1}{4r^2} \varphi(0, 0, z) \tag{3.6}$$

for all $z \in X$. Replacing z by rz in (3.6) and dividing by r^2 and summing the resulting inequality with (3.6), we get

$$\|f(z) - \frac{1}{r^4} f(r^2 z)\| \leq \frac{1}{4r^2} \left(\varphi(0, 0, z) + \frac{\varphi(0, 0, rz)}{r^2}\right) \tag{3.7}$$

for all $z \in X$. Hence

$$\|\frac{1}{r^{2k}} f(r^k z) - \frac{1}{r^{2m}} f(r^m z)\| \leq \frac{1}{4r^2} \sum_{i=k}^{m-1} \frac{1}{r^{2i}} \varphi(0, 0, r^i z) \tag{3.8}$$

for all nonnegative integers m and k with $m > k$ and for all $z \in X$. It follows from (3.1) and (3.8) that the sequence $\{\frac{1}{r^{2n}} f(r^n z)\}$ is a Cauchy sequence for all $z \in X$. Since Y is complete, the sequence $\{\frac{1}{r^{2n}} f(r^n z)\}$ converges. Therefore, one can define the mapping $Q : X \to Y$ by

$$Q(z) := \lim_{n\to\infty} \frac{1}{r^{2n}} f(r^n z)$$

for all $z \in X$. By (3.2) for $j = 1$ and (3.3),

$$\begin{aligned}\|\Delta_Q(x,y,z)\| &= \lim_{n\to\infty} \frac{1}{r^{2n}} \|\Delta_f(r^n x, r^n y, r^n z)\| \\ &\le \lim_{n\to\infty} \frac{1}{r^{2n}} \varphi(r^n x, r^n y, r^n z) = 0\end{aligned}$$

for all $x, y, z \in X$. So $\Delta_Q(x,y,z) = 0$. By Theorem 2.1, the mapping $Q : X \to Y$ is a Jensen type quadratic-quadratic mapping. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.8), we get the inequality (3.4) for $j = 1$.

Now, let $Q' : X \to Y$ be another Jensen type quadratic-quadratic mapping satisfying (1.3) and (3.4). So

$$\begin{aligned}\|Q(z) - Q'(z)\| &= \frac{1}{r^{2n}} \|Q(r^n z) - Q'(r^n z)\| \\ &\le \frac{1}{r^{2n}} (\|Q(r^n z) - f(r^n z)\| + \|Q'(r^n z) - f(r^n z)\|) \\ &\le \frac{1}{2r^2 r^{2n}} \widetilde{\varphi}(r^n z)\end{aligned}$$

which tends to zero as $n \to \infty$ for all $z \in X$. So we can conclude that $Q(z) = Q'(z)$ for all $z \in X$. This proves the uniqueness of Q.

Also, for $j = -1$, it follows from (3.5) that

$$\|f(z) - r^2 f(\frac{z}{r})\| \le \frac{1}{4} \varphi(0, 0, \frac{z}{r}) \tag{3.9}$$

for all $z \in X$. Hence

$$\|r^{2k} f(\frac{z}{r^k}) - r^{2m} f(\frac{z}{r^m})\| \le \frac{1}{4} \sum_{i=k}^{m-1} r^{2i} \varphi(0, 0, \frac{z}{r^{i+1}}) \tag{3.10}$$

for all nonnegative integers m and k with $m > k$ and for all $z \in X$. It follows from (3.10) that the sequence $\{r^{2n} f(\frac{z}{r^n})\}$ is a Cauchy sequence for all $z \in X$. Since Y is complete, the sequence $\{r^{2n} f(\frac{z}{r^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(z) := \lim_{n\to\infty} r^{2n} f(\frac{z}{r^n})$$

for all $z \in X$. By (3.2) for $j = -1$ and (3.3),

$$\|\Delta_Q(x,y,z)\| = \lim_{n\to\infty} r^{2n} \|\Delta_f(\frac{x}{r^n}, \frac{y}{r^n}, \frac{z}{r^n})\| \le \lim_{n\to\infty} r^{2n} \varphi(\frac{x}{r^n}, \frac{y}{r^n}, \frac{z}{r^n}) = 0$$

for all $x, y, z \in X$. So $\Delta_Q(x,y,z) = 0$. By Theorem 2.1, the mapping $Q : X \to Y$ is a Jensen type quadratic-quadratic mapping. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.10), we get the inequality (3.4) for $j = -1$.

The rest of the proof is similar to the proof of the previous section. □

4. Cauchy-Rassias stability using alternative fixed point

Recently, Cădariu and Radu [4] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such a clever idea, they could present another proof for the Hyers-Ulam stability of that equation [5, 6]. In this section, by using the idea of Cădariu and Radu, we will prove the Cauchy-Rassias stability of the general Jensen type quadratic-quadratic functional equation (1.3) (see also [12, 15, 21]).

Theorem 4.1 (The alternative of fixed point [8, 26])**.** *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L. *Then for each given* $x \in \Omega$, *either* $d(T^n x, T^{n+1}x) = \infty$ *for all* $n \geq 0$, *or other exists a natural number* n_0 *such that*

- $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
- *the sequence* $\{T^n x\}$ *is convergent to a fixed point* y^* *of* T;
- y^* *is the unique fixed point of* T *in the set* $\Lambda = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

Utilizing the above mentioned fixed point alternative, we now obtain our main result, i.e., the Cauchy-Rassias stability of the functional equation (1.3).

Theorem 4.2. *Suppose that* $j \in \{-1, 1\}$ *be fixed, and let* $f : X \to Y$ *an even function with* $f(0) = 0$ *for which there exists a function* $\varphi : X \times X \times X \to [0, \infty)$ *such that*

$$\lim_{n\to\infty} \frac{1}{r^{2nj}} \varphi(r^{nj}x, r^{nj}y, r^{nj}z) = 0, \tag{4.1}$$

$$\|\Delta_f(x, y, z)\| \leq \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in X$. *If there exists* $L < 1$ *such that the function* φ *has the property*

$$\varphi(0, 0, \frac{z}{r}) \leq L r^2 \varphi(0, 0, \frac{z}{r^2}) \tag{4.3}$$

for all $z \in X$, *then there exists a unique Jensen type quadratic-quadratic function* $Q : X \to Y$ *such that, we have the inequality*

$$\|f(z) - Q(z)\| \leq \frac{L^{\frac{j+1}{2}}}{4(1-L)} \varphi(0, 0, \frac{z}{r}) \tag{4.4}$$

for all $z \in X$.

Proof. Consider the set $\Omega = \{g |\ g : X \to Y,\ g(0) = 0\}$, and introduce the generalized metric

$$d(g, h) = d_\varphi(g, h) = \inf\{K \in (0, \infty) :\ \|g(z) - h(z)\| \leq K\varphi(0, 0, \frac{z}{r}),\ z \in X\}$$

on Ω. It is easy to see that (Ω, d) is complete.

Now we define a function $T : \Omega \to \Omega$ by $Tg(z) = \frac{1}{r^{2j}}\, g(r^j z)$ for all $z \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g,h) < K &\Rightarrow \|g(z) - h(z)\| \le K\varphi(0,0,\frac{z}{r}) && \text{for all } z \in X \\ &\Rightarrow \|\frac{1}{r^{2j}} g(r^j z) - \frac{1}{r^{2j}} h(r^j z)\| \le \frac{1}{r^{2j}}\ K\ \varphi(0,0,r^{j-1}z) && \text{for all } z \in X \\ &\Rightarrow \|\frac{1}{r^{2j}} g(r^j z) - \frac{1}{r^{2j}} h(r^j z)\| \le L\ K\ \varphi(0,0,\frac{z}{r}) && \text{for all } z \in X \\ &\Rightarrow d(Tg,Th) \le L\ K. \end{aligned}$$

Hence we see that

$$d(Tg,Th) \le L\ d(g,h)$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. Putting $x = y = 0$ in (3.3) and using the evenness of f, we get

$$\|4f(rz) - 4r^2 f(z)\| \le \varphi(0,0,z) \tag{4.5}$$

for all $z \in X$. Now, by using (4.3) for $z := rz$, we obtain that

$$\|f(z) - \frac{1}{r^2} f(rz)\| \le \frac{1}{4r^2}\ \varphi(0,0,z) \le \frac{L}{4}\ \varphi(0,0,\frac{z}{r})$$

for all $z \in X$, that is, $d(f,Tf) \le \frac{L}{4} < \infty$.

If we substitute $z := \frac{z}{r}$ in (4.5), we see that

$$\|f(z) - r^2 f(\frac{z}{r})\| \le \frac{1}{4}\ \varphi(0,0,\frac{z}{r})$$

for all $z \in X$, that is, $d(f,Tf) \le \frac{1}{4} < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$Q(z) = \lim_{n\to\infty} \frac{1}{r^{2nj}} f(r^{nj} z) \tag{4.6}$$

for all $z \in X$, since $\lim_{n\to\infty} d(T^n f, Q) = 0$.

Also, if we replace x, y and z by $r^{nj}\ x$, $r^{nj}\ y$ and $r^{nj}\ z$ in (2.30), respectively, and divide by r^{2nj}. Then it follows from (4.1) and (4.6) that

$$\begin{aligned} \|\Delta_Q(x,y,z)\| &= \lim_{n\to\infty} \frac{1}{r^{2nj}} \|\Delta_f(r^{nj}x, r^{nj}y, r^{nj}z)\| \\ &\le \lim_{n\to\infty} \frac{1}{r^{2nj}} \varphi(r^{nj}x, r^{nj}y, r^{nj}z) = 0 \end{aligned}$$

for all $x, y, z \in X$, so $\Delta_Q(x,y,z) = 0$. By Theorem 2.1, the function Q is a Jensen type quadratic-quadratic function.

According to the fixed point alterative, since Q is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f,g) < \infty\}, Q$ is the unique function such that

$$\|f(z) - Q(z)\| \le K\ \varphi(0,0,\frac{z}{r})$$

for all $z \in X$ and $K > 0$. Again using the fixed point alterative, gives

$$d(f,Q) \leq \frac{1}{1-L} d(f,Tf) \leq \frac{L^{\frac{j+1}{2}}}{4(1-L)}$$

so we conclude that

$$\|f(z) - Q(z)\| \leq \frac{L^{\frac{j+1}{2}}}{4(1-L)} \varphi(0,0,\frac{z}{r})$$

for all $z \in X$. This completes the proof. □

Corollary 4.3. *Let $\varepsilon, p_1, p_2, p_3 \geq 0$ be real numbers such that $p_1, p_2, p_3 < 2$ or $p_1, p_2, p_3 > 2$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|\Delta_f(x,y,z)\| \leq \varepsilon\ (\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3})$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic function $Q : X \to Y$ such that

$$\|f(z) - Q(z)\| \leq \frac{\varepsilon}{4|r^2 - r^p|} \|z\|^{p_3} \tag{4.7}$$

for all $z \in X$.

Proof. In Theorem 4.2, put $\varphi(x,y,z) := \varepsilon\ (\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3})$ for all $x, y, z \in X$. Then the relation (4.1) is true for $p_1, p_2, p_3 < 2$ or $p_1, p_2, p_3 > 2$ and also the inequality (4.3) holds with $L = r^{(p_3-2)j}$. So from (4.4), we get (4.7). □

Corollary 4.4. *Assume that $\theta \geq 0$ be fixed. Let $f : X \to Y$ be an even function such that*

$$\|\Delta_f(x,y,z)\| \leq \theta$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic function $Q : X \to Y$ such that

$$\|f(z) - Q(z)\| \leq \frac{\theta}{12(r^2 - 1)}$$

holds for all $z \in X$.

Proof. Letting $p_3 = 0$ and $\varepsilon = \frac{\theta}{3}$ and applying Corollary 4.3, we get the result. □

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