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On density of transitive operator algebras ☆

Jaeseong Heo

Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea

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ABSTRACT

In this paper we study the transitive algebra question by considering the invariant subspace problem relative to von Neumann algebras. We prove that the algebra (not necessarily $*$) generated by a pair of sums of two unitary generators of $L(\mathbf{F}_\infty)$ and its commutant is strong-operator dense in $\mathcal{B}(\mathcal{H})$. The relations between the transitive algebra question and the invariant subspace problem relative to some von Neumann algebras are discussed.

1. Introduction

This paper is concerned with the transitive algebra question and the invariant subspace problem relative to von Neumann algebras [3,6]. Many theorems and open problems concerning the invariant subspace problem and the transitive algebra question remain to be solved. The classical invariant subspace problem on a Hilbert space asks if every bounded operator on a Hilbert space has a non-trivial invariant subspace. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. The invariant subspace problem can be restated as follows: for any operator T in $\mathcal{B}(\mathcal{H})$, is there a non-trivial orthogonal projection P in $\mathcal{B}(\mathcal{H})$ which is invariant under T, that is, $PTP = TP$ and $P \neq 0, I$? See the monograph [10] for a general discussion of the invariant subspace problem, transitive algebra question and related topics. Throughout this paper, "generate" will mean only "algebraically generate", so that an algebra need not be a $*$-algebra.

The finite dimensional Hilbert space case immediately follows from the fact that every matrix in a finite dimensional vector space is unitarily equivalent to an upper triangular matrix. The Jordan canonical form theorem for matrices can be regarded as exhibiting matrices as direct sums of their restrictions to certain invariant subspaces. In a non-separable case, the invariant subspace problem always has a positive answer since any non-zero vector spans a non-trivial closed invariant subspace. However, the separable infinite dimensional case is still open, although there are many partial positive answers to it. Questions concerning the existence of invariant subspaces for certain operators provide many examples and theorems. In particular, Lomonosov's remarkable theorem on hyperinvariant subspaces has generated many results concerning invariant subspaces of compact operators extending to the hyperinvariant case. Brown's results on the existence of non-trivial invariant subspaces for subnormal operators [2] and hyponormal operators with thick spectra have been much generalized and have resulted in the development of many other results.

Concerning algebras whose invariant subspaces have invariant complementary subspaces, Kadison [8] raised the following question which is well-known as the transitive algebra question: *if a subalgebra (not necessarily $*$-subalgebra) $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ has no non-trivial common invariant subspace in $\mathcal{H}$, is $\mathcal{B}$ strong-operator dense in $\mathcal{B}(\mathcal{H})$?* If $\mathcal{H}$ is finite dimensional, it follows

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E-mail address: hjs@hanyang.ac.kr.

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from Burnside's theorem that the only transitive algebra is the full matrix algebra of all linear transformations. In the infinite dimensional case, the von Neumann's double commutant theorem shows that every selfadjoint transitive algebra is strong-operator dense in $\mathcal{B}(\mathcal{H})$. In fact, the invariant subspace problem asks if even a singly generated algebra acting on a separable Hilbert space can be transitive. Let $W(T)$ be the strong-operator closed unital algebra generated by T and let $W^*(T)$ be the strong-operator closed unital algebra generated by T and T^*. Then $W^*(T)$ is a von Neumann algebra (a strong-operator closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$). Concerning the single generation of all von Neumann algebras, we can pose the following invariant subspace problem relative to a von Neumann algebra: Does every operator T in $\mathcal{B}(\mathcal{H})$ have a non-trivial invariant projection in $W^*(T)$? It is well known that von Neumann algebras are generated (linearly) by their projections. If $W^*(T)$ has a non-trivial center, then projections in the center commute with T and thus are invariant under T. Therefore, we will limit our discussion to cases where the center of $W^*(T)$ is trivial.

There have been many attempts to solve the invariant subspace problem relative to von Neumann algebras including Arveson [1], Fang, Hadwin and Ravichandran [4], Haagerup and Schultz [6], Pearcy and Salinas [9], Radjavi and Rosenthal [10]. In particular, Haagerup and Schultz [6] recently showed that if $\mathcal{M}$ is a II_1-factor and if for any operator $T \in \mathcal{M}$, its Brown's spectral distribution measure is not concentrated in one point, then T has a non-trivial closed invariant subspace affiliated with $\mathcal{M}$. That is, there is a projection P in $\mathcal{M}$ such that $P \neq 0, I$ and $PTP = TP$. Recently, Fang, Hadwin and Ravichandran [4] gave a positive answer to the transitive algebra question in cases where a transitive algebra containing a finite W^*-algebra is 2-fold transitive.

However, only partial solutions have been proposed on the transitive algebra question, so we will investigate whether, for some operators $T_1, \ldots, T_n$ in a II_1-factor $\mathcal{M}$, there is a non-trivial projection $P \in \mathcal{M}$ such that $PT_iP = T_iP$ $(1 \leqslant i \leqslant n)$. In this paper, we will focus on the transitive algebra question related to the invariant subspace problem relative to the free group von Neumann algebra. This paper is organized in the following fashion. The second section contains main results of this paper. In Section 2, we prove that the algebra generated by the set $\{\lambda_{a_1} + \lambda_{a_2}, \lambda_{a_3} + \lambda_{a_4}\}$ and the commutant $R(\mathbf{F}_\infty)$ of $L(\mathbf{F}_\infty)$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$ where a_1, a_2, a_3 and a_4 are four of generators of $\mathbf{F}_\infty$. In the third section, we explicitly construct the invariant subspace of some operator using a similar model as upper triangular matrix models for circular free Poisson operators [3]. Furthermore, the relation between the transitive algebra question and the invariant subspace problem relative to finite von Neumann algebras is discussed with some remarks and examples.

2. Transitive operator algebras

With regards to the transitive algebra question, Kadison suggested the idea that some selfadjoint maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$ and some elements not in the subalgebra might generate a non-trivial (strong-operator closed) transitive algebra. Arveson proved in [1] that Kadison's original idea does not work. That is, if $\mathcal{A}$ is a transitive subalgebra of $\mathcal{B}(\mathcal{H})$ which contains a selfadjoint maximal abelian von Neumann algebra, then $\mathcal{A}$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$. Inspired by the invariant subspace problem affiliated with a von Neumann algebra [6], Fang, Hadwin and Ravichandran [4] studied the transitive algebra $\mathfrak{A}$ generated by some set of operators in a finite von Neumann algebra and its commutant. We also consider a type II_1 factor $\mathcal{M}$ and a transitive set of some operators in the commutant $\mathcal{M}'$ (in [7], we have already used a transitive set consisting of two unitaries to prove the strong-operator density of the algebra generated by the Thompson group factor together with two unitaries). Then we investigate whether these two sets can provide some examples of non-trivial transitive algebras. For example, if T in a factor $\mathcal{M}$ has no non-trivial invariant projection in $\mathcal{M}$, then T and the commutant $\mathcal{M}'$ will generate a transitive algebra (even with a non-trivial commutant). In this section, we study the transitivity in a factor of type II_1.

Definition 2.1. We call a subset (or a subalgebra) $\mathcal{X}$ of a II_1-factor $\mathcal{M}$ *transitive with respect to* $\mathcal{M}$ if $\mathcal{X}$ has no non-trivial common invariant projections in $\mathcal{M}$. In this case, we simply say that $\mathcal{X}$ is *transitive in* $\mathcal{M}$.

We can consider algebras generated by special kinds of operators. If $\mathfrak{A}$ is a transitive algebra generated by selfadjoint operators, then $\mathfrak{A}$ is a von Neumann algebra and must be equal to $\mathcal{B}(\mathcal{H})$. What then is the situation if $\mathfrak{A}$ is generated by isometries or normal operators? In spite of a great deal of interest in this question, transitive algebras other than $\mathcal{B}(\mathcal{H})$ have not been discovered, yet. In this section we will investigate whether a transitive subset in a standard finite von Neumann algebra $\mathcal{M}$ on $L^2(\mathcal{M}, \tau)$ together with the commutant $\mathcal{M}'$ generate a non-trivial strong-operator closed transitive algebra in $\mathcal{B}(\mathcal{H})$.

Proposition 2.2. *If $\mathcal{X}$ is a selfadjoint subset of a factor $\mathcal{M}$ with a trivial relative commutant, then $\mathcal{X}$ together with the commutant $\mathcal{M}'$ generates $\mathcal{B}(\mathcal{H})$ in the strong-operator topology.*

Proof. The proof follows from the double commutant theorem of von Neumann. □

Let G be a discrete group with the identity e and let $\mathcal{H}$ be the Hilbert space $l^2(G)$ with the usual inner product. We shall assume that G is countable, so that $\mathcal{H}$ is separable. For each $g \in G$, let λ_g denote the left translation of functions in $\mathcal{H}$ by g^{-1}. Then the map $\lambda : g \mapsto \lambda_g$ is a faithful unitary representation of G on the Hilbert space $\mathcal{H}$. Let $L(G)$ be the

Neumann algebra generated by $\{\lambda_g\colon g \in G\}$. Similarly, let ρ_g be the right translation by g on $\mathcal{H}$ and $R(G)$, the Neumann algebra generated by $\{\rho_g\colon g \in G\}$. Then $L(G)' = R(G)$ and $R(G)' = L(G)$. The function χ_g that is 1 at g and 0 elsewhere is a cyclic trace vector for $L(G)$ (and $R(G)$). In general, $L(G)$ and $R(G)$ are finite von Neumann algebras with tracial state $\tau(\sum_g \alpha_g \lambda_g) = \alpha_e$. They are factors (of type II_1) precisely when each conjugacy class in G (other than that of e) is infinite. In this case we say that G is an infinite conjugacy class (i.c.c.) group.

Let $\mathbf{F}_\infty$ be the free group with countably infinitely generators a_j $(j = 1, 2, \ldots)$. Throughout this section, we assume that $\mathcal{H}$ is the Hilbert space $l^2(\mathbf{F}_\infty)$ with the basis $\{\chi_g\colon g \in \mathbf{F}_\infty\}$ unless specified otherwise. If X is an element in $L(\mathbf{F}_\infty)$ (or $R(\mathbf{F}_\infty)$), then there is an element x in $l^2(\mathbf{F}_\infty)$ corresponding to it. Sometimes we write X as λ_x, the operator induced by the left multiplication by x on $l^2(\mathbf{F}_\infty)$ (or ρ_x, the right multiplication by x^*). In particular, if $x = \chi_g$ for some $g \in \mathbf{F}_\infty$, then we will use λ_g instead of λ_{χ_g} and ρ_g instead of ρ_{χ_g}. As usual, we denote τ the unique trace on $L(\mathbf{F}_\infty)$, and τ' on $R(\mathbf{F}_\infty)$.

Now we consider two sums $\lambda_{a_1} + \lambda_{a_2}$ and $\lambda_{a_3} + \lambda_{a_4}$ of pairs of unitary generators in the free group von Neumann algebra $L(\mathbf{F}_\infty)$ where a_1, a_2, a_3 and a_4 are four of the free generators of $\mathbf{F}_\infty$.

Theorem 2.3. *Let $R(\mathbf{F}_\infty)$ and a_j $(1 \leqslant j \leqslant 4)$ be as above. The set $\{\lambda_{a_1} + \lambda_{a_2}, \lambda_{a_3} + \lambda_{a_4}\}$ together with the commutant $R(\mathbf{F}_\infty)$ generates $\mathcal{B}(\mathcal{H})$ as a von Neumann algebra.*

Proof. We will first show that the algebra generated by $\{\lambda_{a_1} + \lambda_{a_2}, \lambda_{a_3} + \lambda_{a_4}, R(\mathbf{F}_\infty)\}$ contains an operator of rank one. Let

$$T = \frac{1}{4}\bigl\{(\lambda_{a_1} + \lambda_{a_2})(\rho_{a_1} + \rho_{a_2}) + (\lambda_{a_3} + \lambda_{a_4})(\rho_{a_3} + \rho_{a_4})\bigr\}. \tag{1}$$

Applying χ_e in the both sides, we obtain that

$$T\chi_e = \frac{1}{4}(4\chi_e + \chi_{a_1 a_2^{-1}} + \chi_{a_2 a_1^{-1}} + \chi_{a_3 a_4^{-1}} + \chi_{a_4 a_3^{-1}}), \tag{2}$$

so that we compute the norm $\|T\chi_e\| = (1 + \frac{1}{2^2})^{\frac{1}{2}}$. Put $T\chi_e = \chi_e + \xi_1$ where

$$\xi_1 = \frac{1}{4}(\chi_{a_1 a_2^{-1}} + \chi_{a_2 a_1^{-1}} + \chi_{a_3 a_4^{-1}} + \chi_{a_4 a_3^{-1}}). \tag{3}$$

Clearly, any summand of $T\xi_1$ is different from each other. Hence we get $\|T^2\chi_e\| = (1 + \frac{1}{2^2} + \frac{1}{2^3})^{\frac{1}{2}}$. A tedious computation shows that $\{\|T^n\chi_e\|\}$ is an increasing sequence of bounded positive numbers and that

$$\lim_{n\to\infty} \|T^n\chi_e\| = \left(1 + \frac{1}{2}\right)^{\frac{1}{2}}. \tag{4}$$

Let ζ be the limit of $T^n\chi_e$ in $\mathcal{H}$. To prove that the strong-operator closure of the algebra generated by $\{\lambda_{a_1} + \lambda_{a_2}, \lambda_{a_3} + \lambda_{a_4}, R(\mathbf{F}_\infty)\}$ is the whole algebra $\mathcal{B}(\mathcal{H})$, we will show that the strong-operator limit of T^n is bounded and that the limit of $T^n\xi$ lies in the linear span of ζ for any unit vector $\xi \in \mathcal{H}$. Let g be any element of $\mathbf{F}_\infty$ of length k. If k is odd, then $T^n\chi_g$ do not contains χ_e as a summand for every positive integer n. Hence we can see that the limit of the norm $\|T^n\chi_g\|$ tends to zero as n goes to ∞.

Suppose that k is even, that is, $k = 2m$. Let g be an element of $\mathbf{F}_\infty$ of length $k = 2m$ such that $T^m\chi_g$ contains χ_e as a summand. For example, take $g = a_1^{-m} a_2^m$. Then we have

$$\lim_{n\to\infty} \|T^n\chi_h\| \leqslant \lim_{n\to\infty} \|T^n\chi_g\| = \frac{1}{4^m}\left(1 + \frac{1}{2}\right)^{\frac{1}{2}} \tag{5}$$

where h is any element of length k. Let $\mathcal{T}_m$ be a subset of elements g in $\mathbf{F}_\infty$ of length $k = 2m$ such that $T^m\chi_g$ contains χ_e as a summand. Then one can see that the order of $\mathcal{T}_m$ is 2^{3m-1}. Letting

$$\xi_m = 2^{-\frac{3m-1}{2}} \sum_{g\in\mathcal{T}_m} \chi_g, \tag{6}$$

we obtain that ξ_m is a unit vector. A direct computation shows that

$$\lim_{n\to\infty} \|T^n\xi_m\| = 2^{-\frac{m+1}{2}}\left(1 + \frac{1}{2}\right)^{\frac{1}{2}}. \tag{7}$$

Moreover, we know that this limit is the largest number among the limits of $\|T^n\xi\|$ where ξ is an l^2-convex combination of elements of length m. Here the l^2-convex combination implies that the sum of squares of moduli of coefficients is 1.

In general, we take any unit vector $\xi \in \mathcal{H}$. Then we can write $\xi = \sum_{g\in\mathbf{F}_\infty} \alpha_g \lambda_g$ with $\sum_{g\in\mathbf{F}_\infty} |\alpha_g|^2 = 1$. Let $\mathcal{S}$ be the support of ξ, that is, $\mathcal{S} = \{g \in \mathbf{F}_\infty\colon \alpha_g \neq 0,\ \xi = \sum \alpha_g \lambda_g\}$. Let $\mathcal{S}_{i_k}$ be the subset of $\mathcal{S}$ such that every element of $\mathcal{S}_{i_k}$ has length i_k. Hence we have $\mathcal{S} = \mathcal{S}_{i_1} \cup \mathcal{S}_{i_2} \cup \cdots$ and $\xi = \sum_{g_1 \in \mathcal{S}_{i_1}} \alpha_{g_1}\lambda_{g_1} + \sum_{g_2 \in \mathcal{S}_{i_2}} \alpha_{g_2}\lambda_{g_2} + \cdots$. For sufficiently large n, we have that

$$\|T^n\xi\| \leqslant \left\|T^n\left(\sum_{g_1\in\mathcal{S}_{i_1}} \alpha_{g_1}\lambda_{g_1}\right)\right\| + \left\|T^n\left(\sum_{g_2\in\mathcal{S}_{i_2}} \alpha_{g_2}\lambda_{g_2}\right)\right\| + \cdots \tag{8}$$

$$\leqslant \|T^n\xi_{i_1}\| + \|T^n\xi_{i_2}\| + \cdots \tag{9}$$

$$\leqslant \|T^n\chi_e\| + \|T^n\xi_1\| + \|T^n\xi_2\| + \cdots. \tag{10}$$

Taking the limit on both sides, we obtain

$$\lim_{n\to\infty}\|T^n\xi\| \leqslant \lim_{n\to\infty}\left(\|T^n\chi_e\| + \sum_{j=1}^{\infty}\|T^n\xi_j\|\right) \tag{11}$$

$$\leqslant \left(1+\frac{1}{2}\right)^{\frac{1}{2}} \frac{1}{1-2^{-\frac{1}{2}}}. \tag{12}$$

From the computation, we can see that $\lim_{n\to\infty} T^n\xi$ lies in the linear span of ζ, which implies that the strong limit T_∞ of $\{T^n\}$ is of rank one.

To complete the proof, we only have to show that the set $\{\lambda_{a_1}+\lambda_{a_2}, \lambda_{a_3}+\lambda_{a_4}, R(\mathbf{F}_\infty)\}$ is transitive in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(l^2(\mathbf{F}_\infty))$ since the transitive algebra containing a non-zero finite rank operator is strongly dense in $\mathcal{B}(\mathcal{H})$ [10]. Let ζ be the limit of the sequence $\{T^n\chi_e\}$ as n goes to ∞. Take a sequence $\{\zeta_n\}$ in the group algebra $\mathbf{CF}_\infty$ converging to ζ in the l^2 sense. We can also consider each ζ_n as an operator in $R(\mathbf{F}_\infty)$ acting on $\mathcal{H}$ by the right multiplication. Since there are invertible operators strongly converging to each ζ_n, we may assume that each ζ_n is an invertible operator in $R(\mathbf{F}_\infty)$. Then we see that for any vector $\xi \in \mathcal{H}$, $\zeta_n T_\infty(\xi)$ tends to χ_e as n tends to ∞ where T_∞ is the strong-operator limit of $\{T^n\}$. This means that the set $\{\lambda_{a_1}+\lambda_{a_2}, \lambda_{a_3}+\lambda_{a_4}, R(\mathbf{F}_\infty)\}$ is transitive in $\mathcal{B}(\mathcal{H})$, which completes the proof. □

Remark 2.4. Replacing $R(\mathbf{F}_\infty)$ with $R(\mathbf{F}_4)$ in Theorem 2.3, the same result also holds, that is, the algebra generated by $\{\lambda_{a_1}+\lambda_{a_2}, \lambda_{a_3}+\lambda_{a_4}\}$ and the commutant $R(\mathbf{F}_4)$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$ where a_i's are generators of $\mathbf{F}_4$ and $\mathcal{H} = l^2(\mathbf{F}_4)$.

The following proposition is originated from the discussion with Professor L. Ge [5]. We would like to thank him for permitting me to put it into this paper.

Proposition 2.5. *(See [5].) Let a and b be two of free generators of $\mathbf{F}_\infty$ and let $\mathcal{H}$ be the Hilbert space $l^2(\mathbf{F}_\infty)$. Then the algebra generated by $\{\lambda_a, \lambda_b\}$ and the commutant $R(\mathbf{F}_\infty)$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$.*

Proof. If $P \in L(\mathbf{F}_\infty)$ is an invariant projection under λ_a and λ_b, that is, $P\lambda_a P = \lambda_a P$ and $P\lambda_b P = \lambda_b P$, then P commutes with λ_a and λ_b since

$$\|\lambda_a P - P\lambda_a\|_2^2 = \tau\big((\lambda_a P - P\lambda_a)^*(\lambda_a P - P\lambda_a)\big) = 0, \tag{13}$$

$$\|\lambda_b P - P\lambda_b\|_2^2 = \tau\big((\lambda_b P - P\lambda_b)^*(\lambda_b P - P\lambda_b)\big) = 0. \tag{14}$$

Since the operators λ_a and λ_b are free in $L(\mathbf{F}_\infty)$, they can't have a common non-trivial invariant projection in $L(\mathbf{F}_\infty)$. This implies that $\{\lambda_a, \lambda_b\}$ is transitive in $L(\mathbf{F}_\infty)$. Moreover, since a and b are free elements in $\mathbf{F}_\infty$, the set $\{a^j b^j\colon\ j = 1, 2, \ldots\}$ forms a free subset in $\mathbf{F}_\infty$. Put

$$T_n = \frac{1}{n}\sum_{j=1}^{n} \lambda_{a^j b^j}\rho_{a^j b^j}. \tag{15}$$

To prove that $\{\lambda_a, \lambda_b\}$ together with $R(\mathbf{F}_\infty)$ generates $\mathcal{B}(\mathcal{H})$, we only have to show that T_n tends to the one dimensional projection onto the unit vector χ_e because a strong-operator closed transitive algebra in $\mathcal{B}(\mathcal{H})$ containing a one dimensional projection must be $\mathcal{B}(\mathcal{H})$ itself (see [10] for more information). Note that T_n is uniformly bounded for each n since $\lambda_{a^j b^j}\rho_{a^j b^j}$ $(j = 1, \ldots, n)$ is a unitary operator in $\mathcal{B}(\mathcal{H})$ and T_n is the convex combination of n unitary operators.

It is clear that $T_n\chi_e = \chi_e$ for any n. Now we will show that $T_n(\chi_g)$ tends to zero when n tends to infinity for every $g \in \mathbf{F}_\infty \setminus \{e\}$. In this case,

$$T_n\chi_g = \frac{1}{n}\sum_{j=1}^{n}\lambda_{a^j b^j}\rho_{a^j b^j}\chi_g = \frac{1}{n}\sum_{j=1}^{n}\chi_{a^j b^j g b^{-j} a^{-j}} \tag{16}$$

if the following relations hold:

$$a^j b^j g b^{-j} a^{-j} = a^k b^k g b^{-k} a^{-k} \quad \text{for } j \neq k, \tag{17}$$

$$a^l b^l g b^{-l} a^{-l} = a^h b^h g b^{-h} a^{-h} \quad \text{for } l \neq h, \tag{18}$$

where $1 \leqslant j, k, l, h \leqslant n$, then we have that

$$g = b^{-j} a^{k-j} b^k g b^{-k} a^{j-k} b^j = b^{-l} a^{h-l} b^h g b^{-h} a^{l-h} b^l. \tag{19}$$

This implies that the element g commutes with $b^{-j}a^{k-j}b^k$ and $b^{-l}a^{h-l}b^h$. Since every subgroup of a free group is also a free group, the centralizer of g is a free group. Hence the centralizer of g must be a cyclic group, denoted by $\mathbf{Z}(g)$. But $b^{-j}a^{k-j}b^k$ and $b^{-l}a^{h-l}b^h$ are both contained in the infinite cyclic group $\mathbf{Z}(g)$. Therefore, there are some non-zero integers m, m' such that

$$\left(b^{-j}a^{k-j}b^k\right)^m = \left(b^{-l}a^{h-l}b^h\right)^{m'}. \tag{20}$$

This can happen if and only if $j = l$ and $k = h$. Among these elements $a^j b^j g b^{-j} a^{-j}$ $(j = 1, \ldots, n)$, there are at most two elements equal to each other. Then we obtain

$$\|T_n \chi_g\|^2 = \left\| \frac{1}{n} \sum_{j=1}^{n} \chi_{a^j b^j g b^{-j} a^{-j}} \right\|^2 \tag{21}$$

$$\leqslant (n-2)\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 = \frac{n+2}{n^2}, \tag{22}$$

so that $\lim_{n\to\infty} \|T_n \chi_g\| = 0$. Therefore, T_n tends strongly to the one dimensional orthogonal projection onto χ_e. This completes the proof. □

3. On the algebra generated by an operator and von Neumann algebra

Let (A, ϕ) be a non-commutative probability space where A is a unital algebra and ϕ is a unital linear functional, and let $(a_i)_{i\in\mathbf{I}}$ be a family of random variables in (A, ϕ). Let $\mathbf{C}\langle X_i\colon\ i \in \mathbf{I}\rangle$ be a non-commutative polynomial ring with non-commuting variables X_i $(i \in \mathbf{I})$. The joint distribution of $(a_i)_{i\in\mathbf{I}}$ is a linear functional $\mu : \mathbf{C}\langle X_i\colon\ i \in \mathbf{I}\rangle \to \mathbf{C}$ defined by $\mu(f(\langle X_i\colon\ i \in \mathbf{I}\rangle)) = \phi(f(\langle X_i\colon\ i \in \mathbf{I}\rangle))$ for all $f \in \mathbf{C}\langle X_i\colon\ i \in \mathbf{I}\rangle$. Voiculescu introduced a non-commutative probability theory whose basic objects inherit the asymptotic properties of families of random matrices. The basic idea in applications of the free probability theory to von Neumann algebras is to model some elements, especially generators of a von Neumann algebra, by large random matrices with entries from a classical probability space. He proved that the von Neumann algebra generated by a free semicircular family $(X_j)_{j\in J}$ is isomorphic to $L(\mathbf{F}_{|J|})$ associated to a free group with $|J|$ generators [11]. From Voiculescu's free probability theory, we see that the joint distribution of a circular system can be approximated by the joint distributions of corresponding systems of independent Gaussian random matrices.

Dykema and Haagerup [3] found that Voiculescu's matrix model leads to upper triangular models for the circular operator and the DT-operator. Using upper triangular realizations of the circular free Poisson element, they prove that the circular operator and each circular free Poisson operator which arise naturally in the free probability theory have a continuous family of invariant subspaces relative to the von Neumann algebra which it generates. To do this, they introduced an invariant subspace $\mathcal{H}_r$ $(r \geqslant 0)$ for an operator $T \in \mathcal{B}(\mathcal{H})$:

$$\mathcal{H}_r(T) = \overline{\left\{\xi \in \mathcal{H} \mid \limsup_{k\to\infty} \left\|T^k \xi\right\|^{1/k} \leqslant r\right\}}. \tag{23}$$

If T has the upper triangular form satisfying the condition on the spectra as in Proposition 3.1 in [3], then the subspace $\mathcal{H}_r$ is a non-trivial invariant subspace for T. We can get a similar result as in [3] if such a condition on the spectra can be replaced by a norm condition.

Theorem 3.1. *Let* $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}$ *act on* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. *Suppose that* T_{11} *is invertible and that* $\|T_{33}\| < \|T_{11}^{-1}\|^{-1}$. *Then there is an invariant subspace* $\mathcal{K}$ *for* T *such that* $\mathcal{H}_1 \subseteq \mathcal{K} \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$.

Proof. We define an analogue of the subspace $\mathcal{H}_r(T)$ by

$$\mathcal{K}_0 = \overline{\left\{\eta \in \mathcal{H} \mid \limsup_{k\to\infty} \left\|\left(T^*\right)^n \eta\right\|^{\frac{1}{n}} \leqslant \|T_{33}\|\right\}}. \tag{24}$$

For any $\xi \in \mathcal{H}_3$, $T^*\xi = T_{33}^*\xi$, so that $(T^*)^n\xi = (T_{33}^*)^n\xi$. Thus we have

$$\left\|\left(T^*\right)^n\xi\right\|^{\frac{1}{n}} = \left\|\left(T_{33}^*\right)^n\xi\right\|^{\frac{1}{n}} \leqslant \left\|\left(T_{33}^*\right)^n\right\|^{\frac{1}{n}}\|\xi\|^{\frac{1}{n}} \leqslant \left\|T_{33}^*\right\|\|\xi\|^{\frac{1}{n}}. \tag{25}$$

Taking lim sup on both sides, we get $\xi \in \mathcal{K}_0$, which implies $\mathcal{H}_3 \subset \mathcal{K}_0$.

We take $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{K}_0$ and may assume that $\|\xi\| = 1$ if necessary after replacing ξ by $\xi/\|\xi\|$. Let $\epsilon > 0$ be such that $\|T_{33}\| + \epsilon < \|T_{11}^{-1}\|^{-1}$ and let n be so large that $\|(T^*)^n\eta\|^{1/n} \leqslant \|T_{33}\| + \epsilon$. Then we have that

$$\left|\langle\xi,\eta\rangle\right| = \left|\left\langle T_{11}^n T_{11}^{-n}\xi,\eta\right\rangle\right| \leqslant \left\|T_{11}^{-n}\xi\right\|\left\|\left(T^*\right)^n\eta\right\| \tag{26}$$

$$\leqslant \left\|T_{11}^{-1}\right\|^n\left(\|T_{33}\| + \epsilon\right)^n \tag{27}$$

$$\leqslant \frac{(\|T_{33}\| + \epsilon)^n}{\|T_{11}^{-1}\|^{-n}}, \tag{28}$$

which goes to 0 as $n \to \infty$. Furthermore, we see $T^*(\mathcal{K}_0) \subset \mathcal{K}_0$ since

$$\limsup_{n\to\infty}\left\|\left(T^*\right)^n\left(T^*\eta\right)\right\|^{\frac{1}{n}} = \limsup_{n\to\infty}\left\|\left(T^*\right)^{n+1}\eta\right\|^{\frac{1}{n+1}\cdot\frac{n+1}{n}} \leqslant \|T_{33}\|. \tag{29}$$

Since $\mathcal{K}_0$ is invariant under T^*, the orthogonal complement of $\mathcal{K}_0$ is a T-invariant subspace containing $\mathcal{H}_1$ and contained in $\mathcal{H}_1 \oplus \mathcal{H}_2$. □

In the second section we investigated the transitivity of the set consisting of sums of two unitary operators in a von Neumann algebra. The motivation follows from the fact that any operator T in a von Neumann algebra $\mathcal{M}$ are expressed as

$$T = 4\|T\|\left(U + V - \frac{1}{2}I\right) \tag{30}$$

where U and V are unitary operators in $\mathcal{M}$. Hence the summation of two unitary operators and the identity operator can give any operator (up to a multiple of constant) as wanted. However, the relations between two unitary operators can be very complicated. If $\mathcal{M}$ is a finite von Neumann algebra, we can omit the term $\frac{1}{2}I$. The following proposition may be well-known, but we will give a proof for reader's convenience.

Proposition 3.2. *If $\mathcal{M}$ is a finite von Neumann algebra, then any element T in $\mathcal{M}$ can be expressed by a sum of only two unitaries in $\mathcal{M}$.*

Proof. We may assume that with $\|T\| \leqslant 2$ by dividing T by $\|T\|$ if necessary. Let $V|T|$ be the polar decomposition of T where V is a partial isometry with a support projection V^*V and a range projection VV^*. Since V^*V and VV^* are equivalent and $\mathcal{M}$ is a finite von Neumann algebra, $I - V^*V$ and $I - VV^*$ are also equivalent. Thus there is a partial isometry V_1 in $\mathcal{M}$ such that

$$V_1^*V_1 = I - V^*V \quad \text{and} \quad V_1V_1^* = I - VV^*. \tag{31}$$

Since the image of $|T|$ is contained in the image of V^*V, we obtain $V_1|T| = 0$. Furthermore, $|T|$ can be expressed by a sum of two unitaries: $|T| = W_+ + W_-$ where

$$W_+ = \frac{1}{2}|T| + i\sqrt{I - \frac{1}{4}|T|^2}, \tag{32}$$

$$W_- = \frac{1}{2}|T| - i\sqrt{I - \frac{1}{4}|T|^2}. \tag{33}$$

Then we get the decomposition

$$T = V|T| = (V + V_1)|T| = U(W_+ + W_-) = U_1 + U_2 \tag{34}$$

where $U = V + V_1$ is a unitary and $U_1 = UW_+$, $U_2 = UW_-$. □

We denote by A_θ the C^*-algebra generated by two unitaries U and V with the irrational rotation relation $UV = \omega VU$ where $\omega = e^{2\pi i\theta}$ and θ is irrational in [0, 1]. If θ is an irrational number, then A_θ is called the irrational rotation C^*-algebra. Let τ be the canonical tracial state on A_θ and let π_τ be the GNS representation associated with the tracial state τ. We denote by W_θ the weak closure of $\pi_\tau(A_\theta)$. If θ is irrational, then W_θ is isomorphic to the hyperfinite II_1-factor $\mathcal{R}$.

Theorem 3.3. *Let B_θ be the von Neumann subalgebra of W_θ generated by $\pi_\tau(U + V)$. Then $B_\theta = W_\theta$, so that W_θ is singly generated.*

Proof. By symmetry, it is enough to show that $\pi_\tau(U)$ belongs to B_θ. For simplicity, we will write U and V instead of $\pi_\tau(U)$ and $\pi_\tau(V)$, respectively. We first observe that

$$(U+V)\big(U^*+V^*\big)=2I+\omega U^*V+UV^*, \tag{35}$$

$$\big(U^*+V^*\big)(U+V)=2I+U^*V+\omega UV^* \tag{36}$$

belong to B_θ. Since $I\in B_\theta$ and $1-\omega\neq 0$, it follows that U^*V is in B_θ. Therefore, we obtain that $(U+V)U^*V=U(I+U^*V)UV^*\in B_\theta$ so that $U(I+U^*V)(U^*V)^n\in B_\theta$ for all $n\in\mathbf{N}$. Hence we can infer that

$$U\big(I+(-1)^{k(n)}\big(U^*V\big)^n\big)\in B_\theta$$

where $k(n)=1$ if n is odd and $k(n)=-1$ if n is even. Since $(U^*V)^n\to 0$ weakly as n tends to infinity, it follows that $U\in B_\theta$. □

Remark 3.4. By Haagerup and Schultz' result [6], $U+V$ has a non-trivial invariant projection $P\in\mathcal{R}$. However, $P\mathcal{R}P$ is the hyperfinite II_1-factor with a single generator $P(U+V)P$. Applying again the Haagerup's result, $P(U+V)P$ has a non-trivial invariant projection $Q\in P\mathcal{R}P$ with $Q\neq P$. Furthermore, $Q\in\mathcal{R}$ is invariant under $U+V$, that is, Q satisfies $Q(U+V)Q=(U+V)Q$. By continuing this process, we obtain that $U+V$ has a family of infinitely many non-trivial invariant projections P in $\mathcal{R}$.

In [7], we showed that the adjoint $\lambda^*_{x_0}$ is contained the strong-operator closure of the algebra (algebraically) generated by λ_{x_0} and the commutant $L(F)'$ where x_0 is one of generators in the Thompson group F and $L(F)$ is the von Neumann algebra associated with the left regular representation λ of F. Fang, Hadwin and Ravichandran [4] obtained a general result containing our result.

Proposition 3.5. *(See [4].) If $\mathcal{M}$ is a finite von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ is an algebra* ($\mathfrak{A}$ *is not necessarily transitive*) *containing $\mathcal{M}'$, then T^* is in the strong closure of $\mathfrak{A}$ for any normal operator $T\in\mathcal{M}\cap\mathfrak{A}$.*

Example 3.6. From Proposition 3.5, we can easily see the following:

(1) λ^*_a of λ_a is in the strong closure of the algebra generated by λ_a and the commutant $L(\mathbf{F}_\infty)'$ where a is one of generators of the free group $\mathbf{F}_\infty$ on countably infinite many generators.
(2) U^* is in the strong closure of the algebra generated by U and the commutant of the hyperfinite II_1-factor $\mathcal{R}$ where U, V are generators of $\mathcal{R}$ with the irrational rotation relation.

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References

[1] W. Arveson, A density theorem for operator algebras, Duke Math. J. 23 (1967) 635–648.
[2] S. Brown, Some invariant subspaces for subnormal operators, Integral Equations Operator Theory 1 (1978) 310–333.
[3] K. Dykema, U. Haagerup, Invariant subspaces of Voiculescu's circular operator, Geom. Funct. Anal. 11 (2001) 693–741.
[4] J. Fang, D. Hadwin, M. Ravichandran, On transitive algebras containing a standard finite von Neumann subalgebra, J. Funct. Anal. 252 (2007) 581–602.
[5] L. Ge, Private communication.
[6] U. Haagerup, H. Schultz, Invariant subspaces for operators in a general II_1-factor, preprint, 2007.
[7] J. Heo, On the Thompson group factor, J. Operator Theory 53 (2005) 185–195.
[8] R. Kadison, On the orthogonalization of operator representations, Amer. J. Math. 78 (1955) 600–621.
[9] C. Pearcy, N. Salinas, An invariant subspace theorem, Michigan Math. J. 20 (1973) 21–31.
[10] H. Radjavi, P. Rosenthal, Invariant Subspaces, Springer-Verlag, Berlin, 1973.
[11] D. Voiculescu, Limit laws for random matrices and free products, Invent. Math. 104 (1991) 201–220.