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Comment on "Approximate ternary Jordan derivations on Banach ternary algebras" [Bavand Savadkouhi *et al.* J. Math. Phys. 50, 042303 (2009)]

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Let A be a Banach ternary algebra over $\mathbb{C}$ and X a ternary Banach A-module. A $\mathbb{C}$-linear mapping $D:(A,[\ ]_A)\to(X,[\ ]_X)$ is called a ternary Jordan derivation if $D([xxx]_A)=[D(x)xx]_X+[xD(x)x]_X+[xxD(x)]_X$ for all $x\in A$. [Bavand Savadkouhi *et al.*, J. Math. Phys. **50**, 042303 (2009)] investigated ternary Jordan derivations on Banach ternary algebras, associated with the following functional equation: $f((x+y+z)/4)+f((3x-y-4z)/4)+f((4x+3z)/4)=2f(x)$, and proved the generalized Ulam–Hyers stability of ternary Jordan derivations on Banach ternary algebras. The mapping f in Lemma 2.2 of Bavand Savadkouhi *et al.* is identically zero and all of the results are trivial. In this note, we correct the statements of the results and the proofs. © *2010 American Institute of Physics.* [doi:10.1063/1.3299295]

I. INTRODUCTION

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley[1] who introduced the notion of cubic matrix, which in turn was generalized by Kapranov *et al.*[2] in 1990.

The comments on physical applications of ternary structures can be found in Refs. 3–16.

Let A be a Banach ternary algebra and X a Banach space. Then X is called a ternary Banach A-module if module operations $A\times A\times X\to X$, $A\times X\times A\to X$, and $X\times A\times A\to X$ are $\mathbb{C}$-linear in each variable and if they satisfy

$$[[xab]_X cd]_X=[x[abc]_A d]_X=[xa[bcd]_A]_X,$$

$$[[axb]_X cd]_X=[a[xbc]_X d]_X=[ax[bcd]_A]_X,$$

$$[[abx]_X cd]_X=[a[bxc]_X d]_X=[ab[xcd]_X]_X,$$

$$[abc]_A\ xd]_X=[a[bcx]_X d]_X=[ab[cxd]_X]_X,$$

$$[[abc]_A dx]_X=[a[bcd]_A x]_X=[ab[cdx]_X]_X$$

for all $x\in X$ and all $a,b,c,d\in A$, and

$$\max\{\|[xab]_X\|,\|[axb]_X\|,\|[abx]_X\|\}\le\|a\|\|b\|\|x\|$$

for all $x\in X$ and all $a,b\in A$.

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Let $(A,[\ ]_A)$ be a Banach ternary algebra over $\mathbb{C}$ and $(X,[\ ]_X)$ a ternary Banach A-module. A $\mathbb{C}$-linear mapping $D:(A,[\ ]_A)\to(X,[\ ]_X)$ is called a ternary derivation if

$$D([xyz]_A) = [D(x)yz]_X + [xD(y)z]_X + [xyD(z)]_X$$

for all $x,y,z\in A$.

A $\mathbb{C}$-linear mapping $D:(A,[\ ]_A)\to(X,[\ ]_X)$ is called a ternary Jordan derivation if

$$D([xxx]_A) = [D(x)xx]_X + [xD(x)x]_X + [xxD(x)]_X$$

for all $x\in A$.

The stability of functional equations was first introduced by Ulam[17] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2,d) and a positive number ϵ, does there exist a $\delta>0$, such that if a function $f:G_1\to G_2$ satisfies the inequality $d(f(xy),f(x)f(y))<\delta$ for all $x,y\in G1$, then there exists a homomorphism $T:G_1\to G_2$, such that $d(f(x),T(x))<\epsilon$ for all $x\in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers[18] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1950, Aoki[19] was the second author to treat this problem for additive mappings (see also Ref. 20). In 1978, Rassias[21] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences.

On the other hand Rassias[22–24] generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms (see also Refs. 25–31).

During the last decades, several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in Refs. 6, 19, and 32–45.

The main purpose of the present paper is to prove the Ulam–Hyers stability of ternary Jordan derivations on Banach ternary algebras associated with the following functional equation:

$$f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right) = 2f(x). \tag{1.1}$$

II. TERNARY JORDAN DERIVATIONS ON BANACH TERNARY ALGEBRAS

In this section, we investigate ternary Jordan derivations on Banach ternary algebras.

Throughout this section, assume that $(A,[\ ]_A)$ is a Banach ternary algebra and $(X,[\ ]_X)$ is a ternary Banach A-module.

Lemma 2.1: (Reference 46) Let V and W be linear spaces and let $f:V\to W$ be an additive mapping, such that $f(\mu x)=\mu f(x)$ for all $x\in V$ and all $\mu\in\mathbb{T}^1:=\{\lambda\in\mathbb{C};|\lambda|=1\}$. Then the mapping $f:V\to W$ is $\mathbb{C}$-linear.

Lemma 2.2: Let $f:A\to X$ be a mapping such that

$$f\left(\frac{\mu x+y+z}{4}\right) + f\left(\frac{3\mu x-y-4z}{4}\right) + f\left(\frac{4\mu x+3z}{4}\right) = 2\mu f(x) \tag{2.1}$$

for all $x,y,z\in A$. Then $f:A\to X$ is $\mathbb{C}$-linear.

Proof: Letting $x=y=z=0$ and $\mu=1$ in (2.1), we get $f(0)=0$.

Letting $x=z=0$, $\mu=1$ and replacing y by $4y$ in (2.1), we get

$$f(y) + f(-y) = 0$$

for all $y\in A$. So $f(-y)=-f(y)$ for all $y\in A$.

Letting $x=0$ and $\mu=1$ in (2.1), we get

$$f\left(\frac{y+z}{4}\right)+f\left(\frac{-y-4z}{4}\right)+f\left(\frac{3z}{4}\right)=0 \tag{2.2}$$

for all $y,z\in A$.

Letting $y=2z$ in (2.2), we get

$$2f\left(\frac{3z}{4}\right)=f\left(\frac{3z}{2}\right) \tag{2.3}$$

for all $z\in A$.

Replacing $3z$ by z in (2.3), we have

$$2f\left(\frac{z}{4}\right)=f\left(\frac{z}{2}\right) \tag{2.4}$$

for all $z\in A$.

Replacing z by $4z$ in (2.4), we get

$$2f(z)=f(2z) \tag{2.5}$$

for all $z\in A$.

Replacing z by $2z$ in (2.5), we have

$$4f(z)=f(4z)$$

for all $z\in A$.

Letting in (2.2), $(y+z)/4=w_1$, and $(-y-4z)/4=w_2$, we get

$$f(w_1)+f(w_2)-f(w_1+w_2)=0.$$

So f is additive.

Letting $y=z=0$ in (2.1), we get

$$f\left(\frac{\mu x}{4}\right)+f\left(\frac{3\mu x}{4}\right)+f\left(\frac{4\mu x}{4}\right)=2\mu f(x). \tag{2.6}$$

It follows from (2.6) that

$$f(\mu x)=\mu f(x)$$

for all $y\in A$ and all $\mu\in\mathbb{T}^1$. So by Lemma 2.1, the mapping $f:A\to X$ is $\mathbb{C}$-linear. ■

Theorem 2.3: *Let $p\neq 1$ and θ be non-negative real numbers, and let $f:A\to X$ be a mapping such that*

$$f\left(\frac{\mu x+y+z}{4}\right)+f\left(\frac{3\mu x-y-4z}{4}\right)+f\left(\frac{4\mu x+3z}{4}\right)=2\mu f(x) \tag{2.7}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$, and

$$\|f([yyy]_A)-[f(y)yy]_X-[yf(y)y]_X-[yyf(y)]_X\|\leq\theta\|y\|^{3p} \tag{2.8}$$

for all $y\in A$. Then the mapping $f:A\to X$ is a ternary Jordan derivation.

Proof: Assume $p<1$. By Lemma 2.2, the mapping $f:A\to X$ is $\mathbb{C}$-linear. It follows from (2.8) that

$$\begin{aligned}&\|f([yyy]_A)-[f(y)yy]_X-[yf(y)y]_X-[yyf(y)]_X\|\\&=\frac{1}{n^3}\|f([(ny)(ny)(ny)]_A)-[f(ny)(ny)(ny)]_X-[(ny)f(ny)(ny)]_X-[(ny)(ny)f(ny)]_X\|\\&\le\frac{\theta}{n^3}n^{3p}\|y\|^{3p}\end{aligned}$$

for all $y\in A$. Since $p<1$, by letting n tend to ∞ in the last inequality, we obtain

$$f([yyy]_A)=[f(y)yy]_X+[yf(y)y]_X+[yyf(y)]_X$$

for all $y\in A$. Hence the mapping $f:A\to X$ is a ternary Jordan derivation.

Similarly, one obtains the result for the case $p>1$. ■

We prove the Ulam–Hyers stability of the functional equation (1.1) controlled by the mixed-type product-sum function,

$$(x,y)\to\theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3}+\|x\|^p+\|y\|^p+\|z\|^p),$$

introduced by Rassias (see Ref. 36).

Theorem 2.4: *Let p,p_1,p_2,p_3 be real numbers such that $p<1$, $p_1+p_2+p_3<1$, and $\theta>0$. Suppose that $f:A\to X$ satisfies*

$$\begin{aligned}&\left\|f\left(\frac{\mu x+y+z}{4}\right)+f\left(\frac{3\mu x-y-4z}{4}\right)+f\left(\frac{4\mu x+3z}{4}\right)-2\mu f(x)\right\|\\&\le\theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3}+\|x\|^p+\|y\|^p+\|z\|^p)\end{aligned}\tag{2.9}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$, and that

$$\|f([xxx]_A)-[f(x)xx]_X-[xf(x)x]_X-[xxf(x)]_X\|\le\theta\|x\|^{3p}\tag{2.10}$$

for all $x\in A$. Then there exists a unique ternary Jordan derivation $D:A\to X$ satisfying

$$\|f(x)-D(x)\|\le2\theta\frac{2^p}{2-2^p}\|x\|^p\tag{2.11}$$

for all $x\in A$.

Proof: Setting $\mu=1$ and $x=y=z=0$ in (2.9), we get $f(0)=0$. Let us take $\mu=1$, $z=0$, and $y=x$ in (2.9). Then we obtain

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|\le2\theta\|x\|^p\tag{2.12}$$

for all $x\in A$. In (2.12), replacing $x/2$ by x and then dividing by 2, we get

$$\left\|f(x)-\tfrac{1}{2}f(2x)\right\|\le2^p\theta\|x\|^p\tag{2.13}$$

for all $x\in A$. We easily prove that by induction that

$$\left\|f(x)-\frac{1}{2^n}f(2^nx)\right\|\le2\theta\|x\|^p\sum_{i=1}^{n}2^{i(p-1)}.\tag{2.14}$$

In order to show that the functions $D_n(x)=(1/2^n)f(2^nx)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, it follows from (2.14) that

$$\left\| \frac{1}{2^m} f(2^m x) - \frac{1}{2^{m+n}} f(2^{m+n} x) \right\| \leq 2\theta \|x\|^p \sum_{i=m+1}^{m+n} 2^{i(p-1)}$$

for all positive integers m,n. Hence by the Cauchy criterion the limit $D(x) = \lim_{n\to\infty} D_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.14) we see that

$$\|f(x) - D(x)\| \leq 2\theta \|x\|^p \sum_{i=1}^{\infty} 2^{i(p-1)}$$

and (2.11) holds for all $x \in A$. Now, we have

$$\begin{aligned}
&\left\| D\left(\frac{\mu x + y + z}{4}\right) + D\left(\frac{3\mu x - y - 4z}{4}\right) + D\left(\frac{4\mu x + 3z}{4}\right) - 2\mu D(x) \right\| \\
&= \lim_{n\to\infty} \frac{1}{2^n} \left\| f\left(\frac{2^n \mu x + 2^n y + 2^n z}{4}\right) + f\left(\frac{3 \cdot 2^n \mu x - 2^n y - 4 \cdot 2^n z}{4}\right) \right. \\
&\quad \left. + f\left(\frac{4 \cdot 2^n \mu x + 3 \cdot 2^n z}{4}\right) - 2\mu f(2^n x) \right\|_A \leq \lim_{n\to\infty} \frac{1}{2^n} \theta(\|2^n x\|^{p_1} \|2^n y\|^{p_2} \|2^n z\|^{p_3} \\
&\quad + \|2^n x\|^p + \|2^n y\|^p + \|2^n z\|^p) = \lim_{n\to\infty} 2^{n(p_1+p_2+p_3-1)} \theta(\|x\|^{p_1} \|y\|^{p_2} \|z\|^{P_3}) \\
&\quad + \lim_{n\to\infty} 2^{n(p-1)} \theta(\|x\|^p + \|y\|^p + \|z\|^P) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Hence,

$$D\left(\frac{\mu x + y + z}{4}\right) + D\left(\frac{3\mu x - y - 4z}{4}\right) + D\left(\frac{4\mu x + 3z}{4}\right) = 2\mu D(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.2, D is $\mathbb{C}$-linear.

On the other hand

$$\begin{aligned}
&\|D([xxx]_A) - [D(x)xx]_X - [xD(x)x]_X - [xxD(x)]_X\| \\
&= \lim_{n\to\infty} \frac{1}{8^n} \|f([(2^n x)(2^n x)(2^n x)]_A) - [f(2^n x)(2^n x)(2^n x)]_X \\
&\quad - [(2^n x) f(2^n x)(2^n x)]_X - [(2^n x)(2^n x) f(2^n x)]_X\| \\
&\leq \lim_{n\to\infty} \frac{\theta}{8^n} \|2^n x\|^{3p} = \lim_{n\to\infty} \theta 8^{n(p-1)} \|x\|^{3p} = 0
\end{aligned}$$

for all $x \in A$, which means that

$$D([xxx]_A) = [D(x)xx]_X + [xD(x)x]_X + [xxD(x)]_X.$$

Therefore, we conclude that D is a ternary Jordan derivation.

Suppose that there exists another ternary Jordan derivation $D' : A \to X$ satisfying (2.11). Since $D'(x) = 1/2^n D'(2^n x)$, we see that

$$\|D(x) - D'(x)\| = \frac{1}{2^n}\|D(2^n x) - D'(2^n x)\|$$
$$\leq \frac{1}{2^n}(\|f(2^n x) - D(2^n x)\| + \|f(2^n x) - D'(2^n x)\|)$$
$$\leq 4\theta\frac{2^p}{2-2^p}2^{n(p-1)}\|x\|^p,$$

which tends to zero as $n \to \infty$ for all $x \in A$. Therefore, $D' = D$ as claimed and the proof of the theorem is complete. ∎

Theorem 2.5: *Let p, p_1, p_2, p_3 be real numbers, such that $p > 1$, $p_1 + p_2 + p_3 > 1$, and $\theta > 0$. Suppose that $f: A \to X$ satisfies*

$$\left\| f\left(\frac{\mu x + y + z}{4}\right) + f\left(\frac{3\mu x - y - 4z}{4}\right) + f\left(\frac{4\mu x + 3z}{4}\right) - 2\mu f(x)\right\|$$
$$\leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p) \tag{2.15}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$, and that

$$\|f([xxx]_A) - [f(x)xx]_X - [xf(x)x]_X - [xxf(x)]_X\| \leq \theta\|x\|^{3p}$$

for all $x \in A$. Then there exists a unique ternary Jordan derivation $D: A \to X$ satisfying

$$\|D(x) - f(x)\| \leq 2\theta\frac{2^p}{2^p - 2}\|x\|^p$$

for all $x \in A$.

Proof: Setting $\mu = 1$ and $x = y = z = 0$ in (2.15), we get $f(0) = 0$. Let us take $\mu = 1$, $z = 0$, and $y = x$ in (2.15). Then we obtain

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 2\theta\|x\|^p \tag{2.16}$$

for all $x \in A$. By induction, we get

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \leq 2\theta\|x\|^p \sum_{i=0}^{n-1} 2^{i(1-p)}. \tag{2.17}$$

In order to show that the functions $D_n(x) = 2^n f(x/2^n)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, if follows from (2.17) that

$$\left\|2^{m+n} f\left(\frac{x}{2^{m+n}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq 2\theta\|x\|^p \sum_{i=m}^{m+n-1} 2^{i(1-p)}$$

for all positive integers m, n.

The rest of the proof is similar to the proof of Theorem 2.4. ∎

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[1] A. Cayley, Am. J. Math. **4**,1 (1881).
[2] M. Kapranov, I. M. Gelfand, and A. Zelevinskii, *Discrimininants, Resultants and Multidimensional Determinants* (Birkhäuser, Berlin, 1994).
[3] V. Abramov, R. Kerner, and B. Le Roy, J. Math. Phys. **38**, 1650 (1997).
[4] M. Amyari and M. S. Moslehian, Lett. Math. Phys. **77**, 1 (2006).

[5] M. Amyari, C. Baak and M.S. Moslehian, Taiwan. J. Math. **11**, 1417 (2007).
[6] C. Baak and M. S. Moslehian, Nonlinear Anal. Theory, Methods Appl. **63**, 42 (2005).
[7] F. Bagarello and G. Morchio, J. Stat. Phys. **66**, 849 (1992).
[8] N. Bazunova, A. Borowiec, and R. Kerner, Lett. Math. Phys. **67**, 195 (2004).
[9] R. Haag and D. Kastler, J. Math. Phys. **5**, 848 (1964).
[10] R. Kerner, http://arxiv.org/list/math-ph/0011.
[11] R. Kerner, Class. Quantum Grav. **14**, A203 (1997).
[12] M. S. Moslehian, Acta Appl. Math. **100**, 187 (2008).
[13] G. L. Sewell, *Quantum Mechanics and Its Emergent Macrophysics* (Princeton University Press, Princeton, NJ, 2002).
[14] L. Takhtajan, Commun. Math. Phys. **160**, 295 (1994).
[15] L. Vainerman and R. Kerner, J. Math. Phys. **37**, 2553 (1996).
[16] H. Zettl, Adv. Math. **48**, 117 (1983).
[17] S. M. Ulam, *Problems in Modern Mathematics* (Wiley, New York, 1940), Chap. VI.
[18] D. H. Hyers, Proc. Natl. Acad. Sci. U.S.A. **27**, 222 (1941).
[19] T. Aoki, J. Math. Soc. Jpn. **2**, 64 (1950).
[20] D. G. Bourgin, Duke Math. J. **16**, 385 (1949).
[21] Th. M. Rassias, Proc. Am. Math. Soc. **72**, 297 (1978).
[22] J. M. Rassias, Bull. Sci. Math. **108**, 445 (1984).
[23] J. M. Rassias, J. Funct. Anal. **46**, 126 (1982).
[24] J. M. Rassias, Glas. Mat. Ser. III **34**(2), 243 (1999).
[25] J. M. Rassias, Discuss. Math. **14**, 101 (1994).
[26] J. M. Rassias, *Geometry, Analysis and Mechanics* (World Scientific, River Edge, NJ, 1994), pp. 365–375.
[27] J. M. Rassias, *Approximation Theory and Numerical Analysis* (World Scientific, River Edge, NJ, 1994), pp. 241–249.
[28] J. M. Rassias, Discuss. Math. **12**, 95 (1992).
[29] J. M. Rassias, J. Approx. Theory **57**, 268 (1989).
[30] J. M. Rassias, Discuss. Math. **7**, 193 (1985).
[31] J. M. Rassias and H. M. Kim, J. Math. Phys. **49**, 063507 (2008).
[32] H. Khodaei and Th. M. Rassias, Int. J. Nonlinear Anal. Appl. **1**, 22 (2010).
[33] P. W. Cholewa, Aequ. Math. **27**, 76 (1984).
[34] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type* (Hadronic, Palm Harbor, FL, 2003).
[35] Z. Gajda, Int. J. Math. Math. Sci. **14**, 431 (1991).
[36] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables* (Birkhäuser, Boston, 1998).
[37] Th. M. Rassias, Acta Appl. Math. **62**(1), 23 (2000).
[38] Th. M. Rassias, J. Math. Anal. Appl. **251**, 264 (2000).
[39] Th. M. Rassias, J. Math. Anal. Appl. **246**, 352 (2000).
[40] K. Ravi, M. Arunkumar and J. M. Rassias, Int. J. Math. Stat. **3**, 36 (2008).
[41] R. Badora, Math. Inequal. Appl. **9**, 167 (2006).
[42] T. Miura, G. Hirasawa, and S. E. Takahasi, J. Math. Anal. Appl. **319**, 522 (2006).
[43] G. Isac and Th. M. Rassias, J. Approx. Theory **72**, 131 (1993).
[44] C. Park, J. Math. Anal. Appl. **327**, 101 (2007).
[45] O. Hatori and J. Wada, Tokyo J. Math. **15**, 223 (1992).
[46] C. Park, Bull. Braz. Math. Soc. N. S. **36**, 79 (2005).