

Research Article

Quadratic-Quartic Functional Equations in RN-Spaces

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We obtain the general solution and the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t -norms $f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + 2[f(2x) - 4f(x)] - 6f(y)$.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let (G_1, \cdot) be a group and let $(G_2, *, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad (1.1)$$

for all $x, y \in E$ and some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta \quad (1.2)$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is \mathbb{R} -linear. In 1978, Rassias [3] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–12]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.3)$$

is related to a symmetric biadditive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1.3) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x, x)$ for all x (see [5, 13]). The biadditive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)). \quad (1.4)$$

The Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof for mappings $f : A \rightarrow B$, where A is a normed space and B is a Banach space (see [14]). Cholewa [15] noticed that the theorem of Skof is still true if relevant domain A is replaced an abelian group. In [16], Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.3). Grabiec [17] has generalized the results mentioned above.

In [18], Park and Bae considered the following quartic functional equation

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y) + 6f(y)] - 6f(x). \quad (1.5)$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique symmetric multiadditive mapping $M : X^4 \rightarrow Y$ such that $f(x) = M(x, x, x, x)$ for all x . It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.5), which is called a quartic functional equation (see also [19]). In addition, Kim [20] has obtained the Hyers-Ulam-Rassias stability for a mixed type of quartic and quadratic functional equation.

The Hyers-Ulam-Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [21–26]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M .

The aim of this paper is to investigate the stability of the additive-quadratic functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary continuous t -norms.

In this sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [22, 23, 27–29]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \rightarrow [0, 1]$ such that F is left-continuous and nondecreasing on \mathbb{R} , $F(0) = 0$ and $F(+\infty) = 1$. Also, D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x , that is, $l^-f(x) = \lim_{t \rightarrow x^-} f(t)$. The space Δ^+ is partially

ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in \mathbb{R} . The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (1.6)$$

Definition 1.1 (see [28]). A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (briefly, a continuous t -norm) if T satisfies the following conditions:

- (a) T is commutative and associative;
- (b) T is continuous;
- (c) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t -norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t -norm). Recall (see [30, 31]) that if T is a t -norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i-1}$. It is known [31] that for the Lukasiewicz t -norm, the following implication holds:

$$\lim_{n \rightarrow \infty} (T_L)_{i=1}^\infty x_{n+i-1} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty. \quad (1.7)$$

Definition 1.2 (see [29]). A random normed space (briefly, RN-space) is a triple (X, μ, T) , where X is a vector space, T is a continuous t -norm, and μ is a mapping from X into D^+ such that the following conditions hold:

- (RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$;
- (RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M) , where

$$\mu_x(t) = \frac{t}{t + \|x\|} \quad (1.8)$$

for all $t > 0$, and T_M is the minimum t -norm. This space is called the induced random normed space.

Definition 1.3. Let (X, μ, T) be an RN-space.

- (1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
- (2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

- (3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X .

Theorem 1.4 (see [28]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Gordji et al. establish the stability of cubic, quadratic and additive-quadratic functional equations in RN-spaces (see [32, 33]).

In this paper, we deal with the following functional equation:

$$f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 2[f(2x) - 4f(x)] - 6f(y) \quad (1.9)$$

on RN-spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of (1.9).

In Section 2, we investigate the general solution of the functional equation (1.9) when f is a mapping between vector spaces and in Section 3, we establish the stability of the functional equation (1.9) in RN-spaces.

2. General Solution

We need the following lemma for solution of (1.9). Throughout this section, X and Y are vector spaces.

Lemma 2.1. *If a mapping $f : X \rightarrow Y$ satisfies (1.9) for all $x, y \in X$, then f is quadratic-quartic.*

Proof. We show that the mappings $g : X \rightarrow Y$ defined by $g(x) := f(2x) - 16f(x)$ and $h : X \rightarrow Y$ defined by $h(x) := f(2x) - 4f(x)$ are quadratic and quartic, respectively.

Letting $x = y = 0$ in (1.9), we have $f(0) = 0$. Putting $x = 0$ in (1.9), we get $f(-y) = f(y)$. Thus the mapping f is even. Replacing y by $2y$ in (1.9), we get

$$f(2x + 2y) + f(2x - 2y) = 4[f(x + 2y) + f(x - 2y)] + 2[f(2x) - 4f(x)] - 6f(2y) \quad (2.1)$$

for all $x, y \in X$. Interchanging x with y in (1.9), we obtain

$$f(2y + x) + f(2y - x) = 4[f(y + x) + f(y - x)] + 2[f(2y) - 4f(y)] - 6f(x) \quad (2.2)$$

for all $x, y \in X$. Since f is even, by (2.2), one gets

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 2[f(2y) - 4f(y)] - 6f(x) \quad (2.3)$$

for all $x, y \in X$. It follows from (2.1) and (2.3) that

$$\begin{aligned} & [f(2(x + y)) - 16f(x + y)] + [f(2(x - y)) - 16f(x - y)] \\ & = 2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)] \end{aligned} \quad (2.4)$$

for all $x, y \in X$. This means that

$$g(x + y) + g(x - y) = 2g(x) + 2g(y) \quad (2.5)$$

for all $x, y \in X$. Therefore, the mapping $g : X \rightarrow Y$ is quadratic.

To prove that $h : X \rightarrow Y$ is quartic, we have to show that

$$h(x + 2y) + h(x - 2y) = 4[h(x + y) + h(x - y) + 6h(y)] - 6h(x) \quad (2.6)$$

for all $x, y \in X$. Since f is even, the mapping h is even. Now if we interchange x with y in the last equation, we get

$$h(2x + y) + h(2x - y) = 4[h(x + y) + h(x - y) + 6h(x)] - 6h(y) \quad (2.7)$$

for all $x, y \in X$. Thus, it is enough to prove that h satisfies (2.7). Replacing x and y by $2x$ and $2y$ in (1.9), respectively, we obtain

$$f(2(2x + y)) + f(2(2x - y)) = 4[f(2(x + y)) + f(2(x - y))] + 2[f(4x) - 4f(2x)] - 6f(2y) \quad (2.8)$$

for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$,

$$f(4x) = 20f(2x) - 64f(x) \quad (2.9)$$

for all $x \in X$. By (2.8) and (2.9), we get

$$f(2(2x + y)) + f(2(2x - y)) = 4[f(2(x + y)) + f(2(x - y))] + 32[f(2x) - 4f(x)] - 6f(2y) \quad (2.10)$$

for all $x, y \in X$. By multiplying both sides of (1.9) by 4, we get

$$4[f(2x + y) + f(2x - y)] = 16[f(x + y) + f(x - y)] + 8[f(2x) - 4f(x)] - 24f(y) \quad (2.11)$$

for all $x, y \in X$. If we subtract the last equation from (2.10), we obtain

$$\begin{aligned} h(2x + y) + h(2x - y) &= [f(2(2x + y)) - 4f(2x + y)] + [f(2(2x - y)) - 4f(2x - y)] \\ &= 4[f(2(x + y)) - 4f(x + y)] + 4[f(2(x - y)) - 4f(x - y)] \\ &\quad + 24[f(2x) - 4f(x)] - 6[f(2y) - 4f(y)] \\ &= 4[h(x + y) + h(x - y) + 6h(x)] - 6h(y) \end{aligned} \quad (2.12)$$

for all $x, y \in X$.

Therefore, the mapping $h : X \rightarrow Y$ is quartic. This completes the proof of the lemma. \square

Theorem 2.2. *A mapping $f : X \rightarrow Y$ satisfies (1.9) for all $x, y \in X$ if and only if there exist a unique symmetric multiadditive mapping $M : X^4 \rightarrow Y$ and a unique symmetric bi-additive mapping $B : X \times X \rightarrow Y$ such that*

$$f(x) = M(x, x, x, x) + B(x, x) \quad (2.13)$$

for all $x \in X$.

Proof. Let f satisfy (1.9) and assume that $g, h : X \rightarrow Y$ are mappings defined by

$$g(x) := f(2x) - 16f(x), \quad h(x) := f(2x) - 4f(x) \quad (2.14)$$

for all $x \in X$. By Lemma 2.1, we obtain that the mappings g and h are quadratic and quartic, respectively, and

$$f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x) \quad (2.15)$$

for all $x \in X$.

Therefore, there exist a unique symmetric multiadditive mapping $M : X^4 \rightarrow Y$ and a unique symmetric bi-additive mapping $B : X \times X \rightarrow Y$ such that $(1/12)h(x) = M(x, x, x, x)$ and $(-1/12)g(x) = B(x, x)$ for all $x \in X$ [5, 18]. So

$$f(x) = M(x, x, x, x) + B(x, x) \quad (2.16)$$

for all $x \in X$. The proof of the converse is obvious. \square

3. Stability

Throughout this section, assume that X is a real linear space and (Y, μ, T) is a complete RN-space.

Theorem 3.1. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \quad (3.1)$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned} \lim_{n \rightarrow \infty} T_{i=1}^{\infty} \left(T \left(\rho_{2^{n+i-1}x, 2^{n+i-1}x} \left(2^{2n+i+1}t \right) \right), T \left(\rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x} \left(\frac{2^{2n+i}t}{4} \right), \rho_{0, 2^{n+i-1}x} \left(\frac{2^{2n+i}t}{3} \right) \right) \right) &= 1, \\ \lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y} \left(2^{2n}t \right) &= 1 \end{aligned} \quad (3.2)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q_1 : X \rightarrow Y$ such that

$$\mu_{f(2x)-16f(x)-Q_1(x)}(t) \geq T_{i=1}^{\infty} \left(T \left(\rho_{2^{i-1}x, 2^{i-1}x} \left(2^{i+1}t \right), T \left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x} \left(\frac{2^i t}{4} \right), \rho_{0, 2^{i-1}x} \left(\frac{2^i t}{3} \right) \right) \right) \right) \quad (3.3)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.1), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x,x}(t) \quad (3.4)$$

for all $x \in X$ and all $t > 0$. Letting $y = 2x$ in (3.1), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x,2x}(t) \quad (3.5)$$

for all $x \in X$ and all $t > 0$. Putting $x = 0$ in (3.1), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \quad (3.6)$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (3.6), we see that

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \quad (3.7)$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (3.7) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq T \left(\rho_{x,2x} \left(\frac{t}{2} \right), \rho_{0,x} \left(\frac{2t}{3} \right) \right) \quad (3.8)$$

for all $x \in X$ and all $t > 0$. If we add (3.4) to (3.8), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq T \left(\rho_{x,x}(2t), T \left(\rho_{x,2x} \left(\frac{t}{4} \right), \rho_{0,x} \left(\frac{t}{3} \right) \right) \right). \quad (3.9)$$

Let

$$\psi_{x,x}(t) = T \left(\rho_{x,x}(2t), T \left(\rho_{x,2x} \left(\frac{t}{4} \right), \rho_{0,x} \left(\frac{t}{3} \right) \right) \right) \quad (3.10)$$

for all $x \in X$ and all $t > 0$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \psi_{x,x}(t) \quad (3.11)$$

for all $x \in X$ and all $t > 0$. Let $g : X \rightarrow Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$. Then we conclude that

$$\mu_{g(2x)-4g(x)}(t) \geq \psi_{x,x}(t) \quad (3.12)$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{g(2x)/2^2-g(x)}(t) \geq \psi_{x,x}(2^2t) \quad (3.13)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{g(2^{k+1}x)/2^{2(k+1)}-g(2^kx)/2^{2k}}(t) \geq \psi_{2^kx,2^kx}(2^{2(k+1)}t) \quad (3.14)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. This means that

$$\mu_{g(2^{k+1}x)/2^{2(k+1)}-g(2^kx)/2^{2k}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^kx,2^kx}(2^{k+1}t) \quad (3.15)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > 1/2 + 1/2^2 + \dots + 1/2^n$, it follows that

$$\mu_{g(2^n x)/2^{2n}-g(x)}(t) \geq T_{k=1}^n \left(\mu_{g(2^k x)/2^{2k}-g(2^{k-1}x)/2^{2(k-1)}}\left(\frac{t}{2^k}\right) \right) \geq T_{i=1}^n \left(\psi_{2^{i-1}x,2^{i-1}x}(2^i t) \right) \quad (3.16)$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{g(2^n x)/2^{2n}\}$, we replace x with $2^m x$ in (3.16) to obtain that

$$\mu_{g(2^{n+m}x)/2^{2(n+m)}-g(2^m x)/2^{2m}}(t) \geq T_{i=1}^n \left(\psi_{2^{i+m-1}x,2^{i+m-1}x}(2^{i+2m}t) \right). \quad (3.17)$$

Since the right-hand side of the inequality (3.17) tends to 1 as m and n tend to infinity, the sequence $\{g(2^n x)/2^{2n}\}$ is a Cauchy sequence. Thus we may define $Q_1(x) = \lim_{n \rightarrow \infty} (g(2^n x)/2^{2n})$ for all $x \in X$.

Now we show that Q_1 is a quadratic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.1), respectively, we get

$$\begin{aligned} & \mu_{((g(2^n(2x+y))+g(2^n(2x-y))-4g(2^n(x+y))-4g(2^n(x-y))-2g(2^{n+1}x)+8g(2^n x)+6g(2^n y))/4^n)}(t) \\ & \geq \rho_{(2^n x, 2^n y)}(2^{2n}t). \end{aligned} \quad (3.18)$$

Taking the limit as $n \rightarrow \infty$, we find that Q_1 satisfies (1.9) for all $x, y \in X$. By Lemma 2.1, the mapping $Q_1 : X \rightarrow Y$ is quadratic.

Letting the limit as $n \rightarrow \infty$ in (3.16), we get (3.3) by (3.10).

Finally, to prove the uniqueness of the quadratic mapping Q_1 subject to (3.3), let us assume that there exists another quadratic mapping Q'_1 which satisfies (3.3). Since $Q_1(2^n x) = 2^{2n}Q_1(x), Q'_1(2^n x) = 2^{2n}Q'_1(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.3), it follows that

$$\begin{aligned} &\mu_{Q_1(x)-Q'_1(x)}(2t) \\ &= \mu_{Q_1(2^n x)-Q'_1(2^n x)}(2^{2n+1}t) \\ &\geq T\left(\mu_{Q_1(2^n x)-g(2^n x)}(2^{2n}t), \mu_{g(2^n x)-Q'_1(2^n x)}(2^{2n}t)\right) \\ &\geq T\left(T_{i=1}^\infty\left(T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{2n+i+1}t), T\left(\rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0, 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right)\right)\right), \\ &\quad T_{i=1}^\infty\left(T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{2n+i+1}t), T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0, 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right)\right)\right) \end{aligned} \tag{3.19}$$

for all $x \in X$ and all $t > 0$. Letting $n \rightarrow \infty$ in (3.19), we conclude that $Q_1 = Q'_1$, as desired. \square

Theorem 3.2. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.20}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned} &\lim_{n \rightarrow \infty} T_{i=1}^\infty\left(T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{4n+3i+1}t), \right. \right. \\ &\quad \left. \left. T\left(\rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0, 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right) = 1, \end{aligned} \tag{3.21}$$

$$\lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y}(2^{4n}t) = 1$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q_2 : X \rightarrow Y$ such that

$$\mu_{f(2x)-4f(x)-Q_2(x)}(t) \geq T_{i=1}^\infty\left(T\left(\rho_{2^{2i-1}x, 2^{2i-1}x}(2^{3i+1}t), T\left(\rho_{2^{2i-1}x, 2 \cdot 2^{2i-1}x}\left(\frac{2^{3i}t}{4}\right), \rho_{0, 2^{2i-1}x}\left(\frac{2^{3i}t}{3}\right)\right)\right)\right) \tag{3.22}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.20), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x,x}(t) \tag{3.23}$$

for all $x \in X$ and all $t > 0$. Letting $y = 2x$ in (3.20), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x,2x}(t) \quad (3.24)$$

for all $x \in X$ and all $t > 0$. Putting $x = 0$ in (3.20), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \quad (3.25)$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (3.25), we get

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \quad (3.26)$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (3.26) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq T\left(\rho_{x,2x}\left(\frac{t}{2}\right), \rho_{0,x}\left(\frac{2t}{3}\right)\right) \quad (3.27)$$

for all $x \in X$ and all $t > 0$. If we add (3.23) to (3.27), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right). \quad (3.28)$$

Let

$$\psi_{x,x}(t) = T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right) \quad (3.29)$$

for all $x \in X$ and all $t > 0$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \psi_{x,x}(t) \quad (3.30)$$

for all $x \in X$ and all $t > 0$. Let $h : X \rightarrow Y$ be a mapping defined by $h(x) := f(2x) - 4f(x)$. Then we conclude that

$$\mu_{h(2x)-16h(x)}(t) \geq \psi_{x,x}(t) \quad (3.31)$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{h(2x)/2^4-h(x)}(t) \geq \psi_{x,x}(2^4t) \quad (3.32)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{h(2^{k+1}x)/2^{4(k+1)}-h(2^kx)/2^{4k}}(t) \geq \psi_{2^kx, 2^kx}(2^{4(k+1)}t) \quad (3.33)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. This means that

$$\mu_{h(2^{k+1}x)/2^{4(k+1)}-h(2^kx)/2^{4k}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^kx, 2^kx}(2^{3(k+1)}t) \quad (3.34)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > 1/2 + 1/2^2 + \dots + 1/2^n$, it follows that

$$\begin{aligned} \mu_{h(2^n x)/2^{4n}-h(x)}(t) &\geq T_{k=1}^n \left(\mu_{h(2^k x)/2^{4k}-h(2^{k-1}x)/2^{4(k-1)}}\left(\frac{t}{2^k}\right) \right) \\ &\geq T_{i=1}^n \left(\psi_{2^{i-1}x, 2^{i-1}x}(2^{3i}t) \right) \end{aligned} \quad (3.35)$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{h(2^n x)/2^{4n}\}$, we replace x with $2^m x$ in (3.35) to obtain that

$$\mu_{h(2^{n+m}x)/2^{4(n+m)}-h(2^m x)/2^{4m}}(t) \geq T_{i=1}^n \left(\psi_{2^{i+m-1}x, 2^{i+m-1}x}(2^{3i+4m}t) \right). \quad (3.36)$$

Since the right-hand side of (3.36) tends to 1 as m and n tend to infinity, the sequence $\{h(2^n x)/2^{4n}\}$ is a Cauchy sequence. Thus we may define $Q_2(x) = \lim_{n \rightarrow \infty} (h(2^n x)/2^{4n})$ for all $x \in X$.

Now we show that Q_2 is a quartic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.20), respectively, we get

$$\begin{aligned} &\mu_{(h(2^n(2x+y))+h(2^n(2x-y))-4h(2^n(x+y))-4h(2^n(x-y))-2h(2^{n+1}x)+8h(2^n x)+6h(2^n y))/16^n}(t) \\ &\geq \rho_{2^n x, 2^n y}(2^{4n}t). \end{aligned} \quad (3.37)$$

Taking the limit as $n \rightarrow \infty$, we find that Q_2 satisfies (1.9) for all $x, y \in X$. By Lemma 2.1 we get that the mapping $Q_2 : X \rightarrow Y$ is quartic.

Letting the limit as $n \rightarrow \infty$ in (3.35), we get (3.22) by (3.29).

Finally, to prove the uniqueness of the quartic mapping Q_2 subject to (3.24), let us assume that there exists a quartic mapping Q'_2 which satisfies (3.22). Since $Q_2(2^n x) = 2^{4n}Q_2(x)$ and $Q'_2(2^n x) = 2^{4n}Q'_2(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.22), it follows that

$$\begin{aligned} & \mu_{Q_2(x)-Q'_2(x)}(2t) \\ &= \mu_{Q_2(2^n x)-Q'_2(2^n x)}(2^{4n+1}t) \\ &\geq T\left(\mu_{Q_2(2^n x)-h(2^n x)}(2^{4n}t), \mu_{h(2^n x)-Q'_2(2^n x)}(2^{4n}t)\right), \\ &\geq T\left(T_{i=1}^\infty\left(T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{4n+3i+1}t), T\left(\rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0, 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right)\right), \\ &\quad T_{i=1}^\infty\left(T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{4n+3i+1}t)T\left(\rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0, 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right)\right) \end{aligned} \quad (3.38)$$

for all $x \in X$ and all $t > 0$. Letting $n \rightarrow \infty$ in (3.38), we get that $Q_2 = Q'_2$, as desired. \square

Theorem 3.3. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \quad (3.39)$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned} & \lim_{n \rightarrow \infty} T_{i=1}^\infty\left(T\left(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{4n+3i+1}t), \right. \right. \\ & \quad \left. \left. T\left(\rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0, 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right) = 1, \quad (3.40) \\ & \lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y}(2^{2n}t) = 1 \end{aligned}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique quadratic mapping $Q_1 : X \rightarrow Y$ and a unique quartic mapping $Q_2 : X \rightarrow Y$ such that

$$\begin{aligned} & \mu_{f(x)-Q_1(x)-Q_2(x)}(t) \\ &\geq T\left(T_{i=1}^\infty\left(T\left(\rho_{2^{i-1}x, 2^{i-1}x}\left(\frac{2^i t}{12}\right), T\left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x}\left(\frac{2^i t}{4 \cdot 24}\right), \rho_{0, 2^{i-1}x}\left(\frac{2^i t}{3 \cdot 24}\right)\right)\right)\right)\right), \\ & \quad T_{i=1}^\infty\left(T\left(\rho_{2^{i-1}x, 2^{i-1}x}\left(\frac{2^{3i} t}{24}\right), T\left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x}\left(\frac{2^{3i} t}{4 \cdot 24}\right), \rho_{0, 2^{i-1}x}\left(\frac{2^{3i} t}{3 \cdot 24}\right)\right)\right)\right) \end{aligned} \quad (3.41)$$

for all $x \in X$ and all $t > 0$.

Proof. By Theorems 3.1 and 3.2, there exist a quadratic mapping $Q'_1 : X \rightarrow Y$ and a quartic mapping $Q'_2 : X \rightarrow Y$ such that

$$\begin{aligned}\mu_{f(2x)-16f(x)-Q'_1(x)}(t) &\geq T_{i=1}^{\infty} \left(T \left(\rho_{2^{i-1}x, 2^{i-1}x} \left(2^{i+1}t \right), T \left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x} \left(\frac{2^i t}{4} \right), \rho_{0, 2^{i-1}x} \left(\frac{2^i t}{3} \right) \right) \right) \right), \\ \mu_{f(2x)-4f(x)-Q'_2(x)}(t) &\geq T_{i=1}^{\infty} \left(T \left(\rho_{2^{i-1}x, 2^{i-1}x} \left(2^{3i+1}t \right), T \left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x} \left(\frac{2^{3i} t}{4} \right), \rho_{0, 2^{i-1}x} \left(\frac{2^{3i} t}{3} \right) \right) \right) \right)\end{aligned}\quad (3.42)$$

for all $x \in X$ and all $t > 0$. It follows from the last inequalities that

$$\begin{aligned}\mu_{f(x)+(1/12)Q'_1(x)-(1/12)Q'_2(x)}(t) \\ &\geq T \left(\mu_{f(2x)-16f(x)-Q'_1(x)} \left(\frac{t}{24} \right), \mu_{f(2x)-4f(x)-Q'_2(x)} \left(\frac{t}{24} \right) \right) \\ &\geq T \left(T_{i=1}^{\infty} \left(T \left(\rho_{2^{i-1}x, 2^{i-1}x} \left(\frac{2^i t}{12} \right), T \left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x} \left(\frac{2^i t}{4 \cdot 24} \right), \rho_{0, 2^{i-1}x} \left(\frac{2^i t}{3 \cdot 24} \right) \right) \right) \right) \right), \\ &\quad T_{i=1}^{\infty} \left(T \left(\rho_{2^{i-1}x, 2^{i-1}x} \left(\frac{2^{3i} t}{24} \right), T \left(\rho_{2^{i-1}x, 2 \cdot 2^{i-1}x} \left(\frac{2^{3i} t}{4 \cdot 24} \right), \rho_{0, 2^{i-1}x} \left(\frac{2^{3i} t}{3 \cdot 24} \right) \right) \right) \right)\end{aligned}\quad (3.43)$$

for all $x \in X$ and all $t > 0$. Hence we obtain (3.41) by letting $Q_1(x) = -(1/12)Q'_1(x)$ and $Q_2(x) = (1/12)Q'_2(x)$ for all $x \in X$. The uniqueness property of Q_1 and Q_2 is trivial. \square

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