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Research Article

On the Stability of a Generalized Quadratic and Quartic Type Functional Equation in Quasi-Banach Spaces

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We establish the general solution of the functional equation $f(nx+y)+f(nx-y)=n^2f(x+y)+n^2f(x-y)+2(f(nx)-n^2f(x))-2(n^2-1)f(y)$ for fixed integers n with $n\neq 0,\pm1$ and investigate the generalized Hyers-Ulam stability of this equation in quasi-Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\left\|f(x+y)-f(x)-f(y)\right\|\leq\delta \tag{1.1}$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\left\|f(x)-T(x)\right\|\leq\delta \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. In 1978, Th. M. Rassias [3] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to a symmetric biadditive mapping [4–7]. It is natural that this functional equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x,x)$ for all x (see [4, 7]). The biadditive mapping B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.4}$$

The generalized Hyers-Ulam stability problem for the quadratic functional equation (1.3) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [8]). Cholewa [9] noticed that the theorem of Skof is still true if relevant domain A is replaced by an abelian group. In [10] , Czerwik proved the generalized Hyers-Ulam stability of the functional equation (1.3). Grabiec [11] has generalized these results mentioned above.

In [12], Park and Bae considered the following quartic functional equation:

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y) + 6f(y)) - 6f(x). \tag{1.5}$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique symmetric multiadditive mapping $D : X \times X \times X \times X \to Y$ such that $f(x) = D(x,x,x,x)$ for all x. It is easy to show that $f(x) = x^4$ satisfies the functional equation (1.5), which is called a quartic functional equation (see also [13]).

In addition, Kim [14] has obtained the generalized Hyers-Ulam stability for a mixed type of quartic and quadratic functional equation between two real linear Banach spaces. Najati and Zamani Eskandani [15] have established the general solution and the generalized Hyers-Ulam stability for a mixed type of cubic and additive functional equation, whenever f is a mapping between two quasi-Banach spaces (see also [16, 17]).

Now we introduce the following functional equation for fixed integers n with $n \neq 0, \pm 1$:

$$f(nx+y) + f(nx-y) = n^2 f(x+y) + n^2 f(x-y) + 2f(nx) - 2n^2 f(x) - 2\left(n^2-1\right) f(y) \tag{1.6}$$

in quasi-Banach spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of the functional equation (1.6). In the present paper we investigate the general solution of the functional equation (1.6) when f is a mapping between vector spaces, and we establish the generalized Hyers-Ulam stability of this functional equation whenever f is a mapping between two quasi-Banach spaces.

We recall some basic facts concerning quasi-Banach space and some preliminary results.

Definition 1.1 (See [18, 19]). Let X be a real linear space. A quasinorm is a real-valued function on X satisfying the following.

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from the condition (3) that

$$\left\| \sum_{i=1}^{2m} x_i \right\| \leq M^m \sum_{i=1}^{2m} \|x_i\|, \qquad \left\| \sum_{i=1}^{2m+1} x_i \right\| \leq M^{m+1} \sum_{i=1}^{2m+1} \|x_i\| \tag{1.7}$$

for all $m \geq 1$ and all $x_1, x_2, \ldots, x_{2m+1} \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X. The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \tag{1.8}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [19] (see also [18]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. In [20], Tabor has investigated a version of Hyers-Rassias-Gajda theorem (see [3, 21]) in quasi-Banach spaces.

2. General Solution

Throughout this section, X and Y will be real vector spaces. We here present the general solution of (1.6).

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies the functional equation* (1.6)*, then f is a quadratic and quartic mapping.*

Proof. Letting $x = y = 0$ in (1.6), we get $f(0) = 0$. Setting $x = 0$ in (1.6), we get $f(y) = f(-y)$ for all $y \in X$. So the mapping f is even. Replacing x by $x + y$ in (1.6) and then x by $x - y$ in (1.6), we get

$$\begin{aligned} &f(nx + (n+1)y) + f(nx + (n-1)y) \\ &\quad = n^2 f(x + 2y) + n^2 f(x) + 2f(nx + ny) - 2n^2 f(x + y) - 2\left(n^2 - 1\right) f(y), \end{aligned} \tag{2.1}$$

$$\begin{aligned} &f(nx - (n-1)y) + f(nx - (n+1)y) \\ &\quad = n^2 f(x) + n^2 f(x - 2y) + 2f(nx - ny) - 2n^2 f(x - y) - 2\left(n^2 - 1\right) f(y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. Interchanging x and y in (1.6) and using the evenness of f, we get the relation

$$f(x+ny)+f(x-ny) = n^2f(x+y)+n^2f(x-y)+2f(ny)-2n^2f(y)-2\left(n^2-1\right)f(x) \tag{2.3}$$

for all $x, y \in X$. Replacing y by ny in (1.6) and then using (2.3), we have

$$f(nx+ny)+f(nx-ny) = n^4f(x+y)+n^4f(x-y)+2f(ny)+2f(nx)-2n^4f(x)-2n^4f(y) \tag{2.4}$$

for all $x, y \in X$. If we add (2.1) to (2.2) and use (2.4), then we have

$$\begin{aligned} &f(nx+(n+1)y)+f(nx-(n+1)y)+f(nx+(n-1)y)+f(nx-(n-1)y) \\ &\quad= n^2f(x+2y)+n^2f(x-2y)+2n^2\left(n^2-1\right)f(x+y)+2n^2\left(n^2-1\right)f(x-y) \\ &\qquad+4f(ny)+4f(nx)+\left(-4n^4+2n^2\right)f(x)+\left(-4n^4-4n^2+4\right)f(y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. Replacing y by $x+y$ in (1.6) and then y by $x-y$ in (1.6) and using the evenness of f, we obtain that

$$f((n+1)x+y)+f((n-1)x-y) = n^2f(2x+y)+n^2f(y)+2f(nx)-2n^2f(x)-2\left(n^2-1\right)f(x+y), \tag{2.6}$$

$$f((n+1)x-y)+f((n-1)x+y) = n^2f(2x-y)+n^2f(y)+2f(nx)-2n^2f(x)-2\left(n^2-1\right)f(x-y) \tag{2.7}$$

for all $x, y \in X$. Interchanging x with y in (2.6) and (2.7) and using the evenness of f, we get the relations

$$f(x+(n+1)y)+f(x-(n-1)y) = n^2f(x+2y)+n^2f(x)+2f(ny)-2n^2f(y)-2\left(n^2-1\right)f(x+y), \tag{2.8}$$

$$f(x-(n+1)y)+f(x+(n-1)y) = n^2f(x-2y)+n^2f(x)+2f(ny)-2n^2f(y)-2\left(n^2-1\right)f(x-y) \tag{2.9}$$

for all $x, y \in X$. Replacing y by $(n+1)y$ in (1.6) and then y by $(n-1)y$ in (1.6), we have

$$f(nx+(n+1)y)+f(nx-(n+1)y) = n^2 f(x+(n+1)y)+n^2 f(x-(n+1)y)+2f(nx)-2n^2 f(x)-2\left(n^2-1\right)f((n+1)y), \tag{2.10}$$

$$f(nx+(n-1)y)+f(nx-(n-1)y) = n^2 f(x+(n-1)y)+n^2 f(x-(n-1)y)+2f(nx)-2n^2 f(x)-2\left(n^2-1\right)f((n-1)y) \tag{2.11}$$

for all $x, y \in X$. Replacing x by y in (1.6), we obtain

$$f((n+1)y)+f((n-1)y) = n^2 f(2y)-2\left(2n^2-1\right)f(y)+2f(ny) \tag{2.12}$$

for all $y \in X$. Adding (2.10) to (2.11) and using (2.8), (2.9), and (2.12), we get

$$\begin{aligned} &f(nx+(n+1)y)+f(nx-(n+1)y)+f(nx+(n-1)y)+f(nx-(n-1)y) \\ &\quad = n^4 f(x+2y)+n^4 f(x-2y)-2n^2\left(n^2-1\right)f(x+y)-2n^2\left(n^2-1\right)f(x-y) \\ &\qquad +4f(ny)+4f(nx)-2n^2\left(n^2-1\right)f(2y)+\left(2n^4-4n^2\right)f(x)+\left(4n^4-12n^2+4\right)f(y) \end{aligned} \tag{2.13}$$

for all $x, y \in X$. By (2.5) and (2.13), we obtain

$$f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)+2f(2y)-8f(y)-6f(x) \tag{2.14}$$

for all $x, y \in X$. Interchanging x and y in (2.14) and using the evenness of f, we get the relation

$$f(2x+y)+f(2x-y) = 4f(x+y)+4f(x-y)+2f(2x)-8f(x)-6f(y) \tag{2.15}$$

for all $x, y \in X$.

Now we show that (2.15) is a quadratic and quartic functional equation. To get this, we show that the mappings $g : X \to Y$, defined by $g(x) = f(2x)-16f(x)$, and $h : X \to Y$, defined by $h(x) = f(2x)-4f(x)$, are quadratic and quartic, respectively.

Replacing y by $2y$ in (2.15) and using the evenness of f, we have

$$f(2x+2y)+f(2x-2y) = 4f(2y+x)+4f(2y-x)+2f(2x)-8f(x)-6f(2y) \tag{2.16}$$

for all $x, y \in X$. Interchanging x with y in (2.16) and then using (2.15), we obtain by the evenness of f

$$\begin{aligned} &f(2x+2y)+f(2x-2y) \\ &\quad = 4f(2x+y)+4f(2x-y)+2f(2y)-8f(y)-6f(2x) \\ &\quad = 16f(x+y)+16f(x-y)+2f(2x)+2f(2y)-32f(x)-32f(y) \end{aligned} \tag{2.17}$$

for all $x, y \in X$. By (2.17), we have

$$[f(2x+2y)-16f(x+y)]+[f(2x-2y)-16f(x-y)]=2[f(2x)-16f(x)]+2[f(2y)-16f(y)] \tag{2.18}$$

for all $x, y \in X$. This means that

$$g(x+y)+g(x-y)=2g(x)+2g(y) \tag{2.19}$$

for all $x, y \in X$. Thus the mapping $g : X \to Y$ is quadratic.

To prove that $h : X \to Y$ is quartic, we have to show that

$$h(2x+y)+h(2x-y)=4h(x+y)+4h(x-y)+24h(x)-6h(y) \tag{2.20}$$

for all $x, y \in X$. Replacing x and y by $2x$ and $2y$ in (2.15), respectively, we get

$$f(4x+2y)+f(4x-2y)=4f(2x+2y)+4f(2x-2y)+2f(4x)-8f(2x)-6f(2y) \tag{2.21}$$

for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$ and $g : X \to Y$ is a quadratic mapping, we have

$$f(4x)=20f(2x)-64f(x) \tag{2.22}$$

for all $x \in X$. So it follows from (2.15), (2.21), and (2.22) that

$$\begin{aligned} &h(2x+y)+h(2x-y) \\ &\quad = [f(4x+2y)-4f(2x+y)]+[f(4x-2y)-4f(2x-y)] \\ &\quad = 4[f(2x+2y)-4f(x+y)]+4[f(2x-2y)-4f(x-y)] \\ &\qquad +24[f(2x)-4f(x)]-6[f(2y)-4f(y)] \\ &\quad = 4h(x+y)+4h(x-y)+24h(x)-6h(y) \end{aligned} \tag{2.23}$$

for all $x, y \in X$. Thus $h : X \to Y$ is a quartic mapping. □

Theorem 2.2. *A mapping $f : X \to Y$ satisfies* (1.6) *if and only if there exist a unique symmetric multiadditive mapping $D : X \times X \times X \times X \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that*

$$f(x) = D(x, x, x, x) + B(x, x) \tag{2.24}$$

for all $x \in X$.

Proof. We first assume that the mapping $f : X \to Y$ satisfies (1.6). Let $g, h : X \to Y$ be mappings defined by

$$g(x) := f(2x) - 16f(x), \qquad h(x) := f(2x) - 4f(x) \tag{2.25}$$

for all $x \in X$. By Lemma 2.1, we achieve that the mappings g and h are quadratic and quartic, respectively, and

$$f(x) := \frac{1}{12}h(x) - \frac{1}{12}g(x) \tag{2.26}$$

for all $x \in X$. Thus there exist a unique symmetric multiadditive mapping $D : X \times X \times X \times X \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that $D(x, x, x, x) = (1/12)h(x)$ and $B(x, x) = -(1/12)g(x)$ for all $x \in X$ (see citead, ki). So

$$f(x) = D(x, x, x, x) + B(x, x) \tag{2.27}$$

for all $x \in X$.

Conversely assume that

$$f(x) = D(x, x, x, x) + B(x, x) \tag{2.28}$$

for all $x \in X$, where the mapping $D : X \times X \times X \times X \to Y$ is symmetric multi-additive and $B : X \times X \to Y$ is bi-additive. By a simple computation, one can show that the mappings D and B satisfy the functional equation (1.6), so the mapping f satisfies (1.6). □

3. Generalized Hyers-Ulam Stability of (1.6)

From now on, let X and Y be a quasi-Banach space with quasi-norm $\|\cdot\|_X$ and a p-Banach space with p-norm $\|\cdot\|_Y$, respectively. Let M be the modulus of concavity of $\|\cdot\|_Y$. In this section, using an idea of Găvruta [22], we prove the stability of (1.6) in the spirit of Hyers, Ulam, and Rassias. For convenience we use the following abbreviation for a given mapping

$f : X \to Y$:

$$\Delta f(x,y) = f(nx+y) + f(nx-y) - n^2 f(x+y) - n^2 f(x-y) \\ - 2f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f(y) \tag{3.1}$$

for all $x, y \in X$. Let $\varphi_q^p(x,y) := (\varphi_q(x,y))^p$. We will use the following lemma in this section.

Lemma 3.1 (see [15]). *Let $0 < p \le 1$ and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \le \sum_{i=1}^{n} x_i{}^p. \tag{3.2}$$

Theorem 3.2. *Let $\varphi_q : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{m\to\infty} 4^m \varphi_q\left(\frac{x}{2^m}, \frac{y}{2^m}\right) = 0 \tag{3.3}$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 4^{pi} \varphi_q{}^p\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{3.4}$$

for all $x \in X$ and all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\Delta f(x,y)\|_Y \le \varphi_q(x,y) \tag{3.5}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{m\to\infty} 4^m \left[f\left(\frac{x}{2^{m-1}}\right) - 16 f\left(\frac{x}{2^m}\right)\right] \tag{3.6}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping satisfying

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{M^{11}}{4} \left[\widetilde{\varphi}_q(x)\right]^{1/p} \tag{3.7}$$

for all $x \in X$*, where*

$$\begin{aligned}
\widetilde{\varphi}_q(x) := \sum_{i=1}^{\infty} 4^{pi} \Bigg\{ \frac{1}{n^{2p}(n^2-1)^p} \Bigg[& \varphi_q^p\left(\frac{x}{2^i}, \frac{(n+2)x}{2^i}\right) + \varphi_q^p\left(\frac{x}{2^i}, \frac{(n-2)x}{2^i}\right) + 4^p \varphi_q^p\left(\frac{x}{2^i}, \frac{(n+1)x}{2^i}\right) \\
& + 4^p \varphi_q^p\left(\frac{x}{2^i}, \frac{(n-1)x}{2^i}\right) + 4^p \varphi_q^p\left(\frac{x}{2^i}, \frac{nx}{2^i}\right) + \varphi_q^p\left(\frac{2x}{2^i}, \frac{2x}{2^i}\right) + 4^p \varphi_q^p\left(\frac{2x}{2^i}, \frac{x}{2^i}\right) \\
& + n^{2p} \varphi_q^p\left(\frac{x}{2^i}, \frac{3x}{2^i}\right) + 2^p \left(3n^2-1\right)^p \varphi_q^p\left(\frac{x}{2^i}, \frac{2x}{2^i}\right) + \left(17n^2-8\right)^p \varphi_q^p\left(\frac{x}{2^i}, \frac{x}{2^i}\right) \\
& + \frac{n^{2p}}{(n^2-1)^p} \left(\varphi_q^p\left(0, \frac{x(n+1)x}{2^i}\right) + \varphi_q^p\left(0, \frac{(n-3)x}{2^i}\right) + 10^p \varphi_q^p\left(0, \frac{(n-1)x}{2^i}\right) \right. \\
& \left. + 4^p \varphi_q^p\left(0, \frac{nx}{2^i}\right) + 4^p \varphi_q^p\left(0, \frac{(n-2)x}{2^i}\right) \right) + \frac{(n^4+1)^p}{(n^2-1)^p} \varphi_q^p\left(0, \frac{2x}{2^i}\right) \\
& + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p} \varphi_q^p\left(0, \frac{x}{2^i}\right) \Bigg] \Bigg\}.
\end{aligned} \tag{3.8}$$

Proof. Setting $x = 0$ in (3.5) and then interchanging x and y, we get

$$\left\| \left(n^2-1\right) f(x) - \left(n^2-1\right) f(-x) \right\| \le \varphi_q(0, x) \tag{3.9}$$

for all $x \in X$. Replacing y by x, $2x$, nx, $(n+1)x$ and $(n-1)x$ in (3.5), respectively, we get

$$\left\| f((n+1)x) + f((n-1)x) - n^2 f(2x) - 2f(nx) + \left(4n^2-2\right) f(x) \right\| \le \varphi_q(x, x), \tag{3.10}$$

$$\begin{aligned}
& \Big\| f((n+2)x) + f((n-2)x) - n^2 f(3x) - n^2 f(-x) - 2f(nx) + 2n^2 f(x) \\
& \quad + 2\left(n^2-1\right) f(2x) \Big\| \le \varphi_q(x, 2x),
\end{aligned} \tag{3.11}$$

$$\left\| f(2nx) - n^2 f((n+1)x) - n^2 f((1-n)x) + 2\left(n^2-2\right) f(nx) + 2n^2 f(x) \right\| \le \varphi_q(x, nx), \tag{3.12}$$

$$\begin{aligned}
& \Big\| f((2n+1)x) + f(-x) - n^2 f((n+2)x) - n^2 f(-nx) - 2f(nx) + 2n^2 f(x) \\
& \quad + 2\left(n^2-1\right) f((n+1)x) \Big\| \le \varphi_q(x, (n+1)x),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
& \Big\| f((2n-1)x) + f(x) - n^2 f((2-n)x) - \left(n^2+2\right) f(nx) + 2n^2 f(x) \\
& \quad + 2\left(n^2-1\right) f((n-1)x) \Big\| \le \varphi_q(x, (n-1)x),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
& \Big\| f(2(n+1)x) + f(-2x) - n^2 f((n+3)x) - n^2 f(-(n+1)x) - 2f(nx) \\
& \quad + 2n^2 f(x) + 2\left(n^2-1\right) f((n+2)x) \Big\| \le \varphi_q(x, (n+2)x),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\Big\| f(2(n-1)x) + f(2x) - n^2 f((n-1)x) - n^2 f(-(n-3)x) - 2f(nx) + 2n^2 f(x)\\&\quad +2\left(n^2-1\right) f((n-2)x)\Big\| \le \varphi_q(x,(n-2)x),\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\Big\| f((n+3)x) + f((n-3)x) - n^2 f(4x) - n^2 f(-2x) - 2f(nx) + 2n^2 f(x)\\&\quad +2\left(n^2-1\right) f(3x)\Big\| \le \varphi_q(x,3x)\end{aligned} \tag{3.17}$$

for all $x \in X$. Combining (3.9) and (3.11)–(3.17), respectively, yields the following inequalities:

$$\begin{aligned}&\left\| f((n+2)x) + f((n-2)x) - n^2 f(3x) - n^2 f(x) - 2f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f(2x)\right\|\\&\quad \le \varphi_q(x,2x) + \frac{n^2}{n^2-1}\varphi_q(0,x),\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\left\| f(2nx) - n^2 f((n+1)x) - n^2 f((n-1)x) + 2\left(n^2-2\right) f(nx) + 2n^2 f(x)\right\|\\&\quad \le \varphi_q(x,nx) + \frac{n^2}{n^2-1}\varphi_q(0,(n-1)x),\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\left\| f((2n+1)x) + f(x) - n^2 f((n+2)x) - n^2 f(nx) - 2f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f((n+1)x)\right\|\\&\quad \le \varphi_q(x,(n+1)x) + \frac{n^2}{n^2-1}\varphi_q(0,nx) + \frac{1}{n^2-1}\varphi_q(0,x),\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left\| f((2n-1)x) + f(x) - n^2 f((n-2)x) - \left(n^2+2\right) f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f((n-1)x)\right\|\\&\quad \le \varphi_q(x,(n-1)x) + \frac{n^2}{n^2-1}\varphi_q(0,(n-2)x),\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\left\| f(2(n+1)x) + f(2x) - n^2 f((n+3)x) - n^2 f((n+1)x) - 2f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f((n+2)x)\right\|\\&\quad \le \varphi_q(x,(n+2)x) + \frac{n^2}{n^2-1}\varphi_q(0,(n+1)x) + \varphi_q(0,2x),\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\left\| f(2(n-1)x) + f(2x) - n^2 f((n-1)x) - n^2 f((n-3)x) - 2f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f((n-2)x)\right\|\\&\quad \le \varphi_q(x,(n-2)x) + \frac{n^2}{n^2-1}\varphi_q(0,(n-3)x),\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\left\| f((n+3)x) + f((n-3)x) - n^2 f(4x) - n^2 f(2x) - 2f(nx) + 2n^2 f(x) + 2\left(n^2-1\right) f(3x)\right\|\\&\quad \le \varphi_q(x,3x) + \frac{n^2}{n^2-1}\varphi_q(0,2x)\end{aligned} \tag{3.24}$$

for all $x \in X$.

Replacing x and y by $2x$ and x in (3.5), respectively, we obtain

$$\left\| f((2n+1)x) + f((2n-1)x) - n^2 f(3x) - 2f(2nx) + 2n^2 f(2x) + \left(n^2-2\right) f(x) \right\| \le \varphi_q(2x, x) \tag{3.25}$$

for all $x \in X$. Putting $2x$ and $2y$ instead of x and y in (3.5), respectively, we have

$$\left\| f(2(n+1)x) + f(2(n-1)x) - n^2 f(4x) - 2f(2nx) + 2\left(2n^2-1\right) f(2x) \right\| \le \varphi_q(2x, 2x) \tag{3.26}$$

for all $x \in X$. It follows from (3.10), (3.18), (3.19), (3.20), (3.21), and (3.25) that

$$\begin{aligned} &\| f(3x) - 6f(2x) + 15f(x) \| \\ &\quad \le \frac{M^5}{n^2(n^2-1)} \Bigg[\varphi_q(x, (n+1)x) + \varphi_q(x, (n-1)x) + \varphi_q(2x, x) + 2\varphi_q(x, nx) + n^2 \varphi_q(x, 2x) \\ &\qquad + \left(4n^2-2\right)\varphi_q(x, x) + \frac{n^2}{n^2-1}(2\varphi_q(0, (n-1)x) + \varphi_q(0, nx) + \varphi_q(0, (n-2)x)) \\ &\qquad + \frac{n^4+1}{n^2-1}\varphi_q(0, x) \Bigg] \end{aligned} \tag{3.27}$$

for all $x \in X$. Also, from (3.10), (3.18), (3.19), (3.22), (3.23), (3.24), and (3.26), we conclude

$$\begin{aligned} &\| f(4x) - 4f(3x) + 4f(2x) + 4f(x) \| \\ &\quad \le \frac{M^6}{n^2(n^2-1)} \Bigg[\varphi_q(x, (n+2)x) + \varphi_q(x, (n-2)x) + \varphi_q(2x, 2x) + 2\varphi_q(x, nx) \\ &\qquad + n^2(\varphi_q(x, 3x) + \varphi_q(x, x)) + 2\left(n^2-1\right)\varphi_q(x, 2x) \\ &\qquad + \frac{n^2}{n^2-1}(2\varphi_q(0, (n-1)x) + \varphi_q(0, (n-3)x) + \varphi_q(0, (n+1)x)) \\ &\qquad + \frac{n^4+1}{n^2-1}\varphi_q(0, 2x) + 2n^2\varphi_q(0, x) \Bigg] \end{aligned} \tag{3.28}$$

for all $x \in X$. Finally, combining (3.27) and (3.28) yields

$$\begin{aligned}
&\|f(4x) - 24f(2x) + 64f(x)\| \\
&\quad\le \frac{M^8}{n^2(n^2-1)}\Bigg[\varphi_q(x,(n+2)x) + \varphi_q(x,(n-2)x) + 4\varphi_q(x,(n+1)x) \\
&\qquad + 4\varphi_q(x,(n-1)x) + 10\varphi_q(x,nx) + \varphi_q(2x,2x) + 4\varphi_q(2x,x) \\
&\qquad + n^2\varphi_q(x,3x) + 2\left(3n^2-1\right)\varphi_q(x,2x) + \left(17n^2-8\right)\varphi_q(x,x) \\
&\qquad + \frac{n^2}{n^2-1}\big(\varphi_q(0,(n+1)x) + \varphi_q(0,(n-3)x) + 10\varphi_q(0,(n-1)x) \\
&\qquad\qquad + 4\varphi_q(0,nx) + 4\varphi_q(0,(n-2)x)\big) + \frac{n^4+1}{n^2-1}\varphi_q(0,2x) \\
&\qquad + \frac{2(3n^4-n^2+2)}{n^2-1}\varphi_q(0,x)\Bigg]
\end{aligned} \tag{3.29}$$

for all $x \in X$. Let

$$\begin{aligned}
\psi_q(x) := \frac{1}{n^2(n^2-1)}\Big[&\varphi_q(x,(n+2)x) + \varphi_q(x,(n-2)x) + 4\varphi_q(x,(n+1)x) \\
&+ 4\varphi_q(x,(n-1)x) + 10\varphi_q(x,nx) + \varphi_q(2x,2x) + 4\varphi_q(2x,x) \\
&+ n^2\varphi_q(x,3x) + 2\left(3n^2-1\right)\varphi_q(x,2x) + \left(17n^2-8\right)\varphi_q(x,x) \\
&+ \frac{n^2}{n^2-1}\big(\varphi_q(0,(n+1)x) + \varphi_q(0,(n-3)x) + 10\varphi_q(0,(n-1)x) + 4\varphi_q(0,nx) \\
&\quad + 4\varphi_q(0,(n-2)x)\big) + \frac{n^4+1}{n^2-1}\varphi_q(0,2x) + \frac{2(3n^4-n^2+2)}{n^2-1}\varphi_q(0,x)\Big].
\end{aligned} \tag{3.30}$$

Then the inequality (3.29) implies that

$$\|f(4x) - 20f(2x) + 64f(x)\| \le M^8\psi_q(x) \tag{3.31}$$

for all $x \in X$.

Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$ for all $x \in X$. From (3.31), we conclude that

$$\|g(2x) - 4g(x)\| \le M^8\psi_q(x) \tag{3.32}$$

for all $x \in X$. If we replace x in (3.32) by $x/2^{m+1}$ and multiply both sides of (3.32) by 4^m, then we get

$$\left\| 4^{m+1} g\left(\frac{x}{2^{m+1}}\right) - 4^m g\left(\frac{x}{2^m}\right) \right\|_Y \leq M^8 4^m \varphi_q\left(\frac{x}{2^{m+1}}\right) \tag{3.33}$$

for all $x \in X$ and all nonnegative integers m. Since Y is a p-Banach space, the inequality (3.33) gives

$$\begin{aligned} \left\| 4^{m+1} g\left(\frac{x}{2^{m+1}}\right) - 4^k g\left(\frac{x}{2^k}\right) \right\|_Y^p &\leq \sum_{i=k}^{m} \left\| 4^{i+1} g\left(\frac{x}{2^{i+1}}\right) - 4^i g\left(\frac{x}{2^i}\right) \right\|_Y^p \\ &\leq M^{8p} \sum_{i=k}^{m} 4^{ip} \varphi_q{}^p\left(\frac{x}{2^{i+1}}\right) \end{aligned} \tag{3.34}$$

for all nonnegative integers m and k with $m \geq k$ and all $x \in X$. Since $0 < p \leq 1$, by Lemma 3.1 and (3.30), we conclude that

$$\begin{aligned} \varphi_q^p(x) \leq \frac{1}{n^{2p}(n^2-1)^p} \Bigg[& \varphi_q^p(x,(n+2)x) + \varphi_q^p(x,(n-2)x) + 4^p \varphi_q^p(x,(n+1)x) \\ & + 4^p \varphi_q^p(x,(n-1)x) + 10^p \varphi_q^p(x,nx) + \varphi_q^p(2x,2x) + 4^p \varphi_q^p(2x,x) \\ & + n^{2p} \varphi_q^p(x,3x) + 2^p \left(3n^2-1\right)^p \varphi_q^p(x,2x) + \left(17n^2-8\right)^p \varphi_q^p(x,x) + \frac{n^{2p}}{(n^2-1)^p} \\ & \times \Big(\varphi_q^p(0,(n+1)x) + \varphi_q^p(0,(n-3)x) + 10^p \varphi_q^p(0,(n-1)x) + 4^p \varphi_q^p(0,nx) \\ & \quad + 4^p \varphi_q^p(0,(n-2)x) \Big) + \frac{(n^4+1)^p}{(n^2-1)^p} \varphi_q^p(0,2x) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p} \varphi_q^p(0,x) \Bigg] \end{aligned} \tag{3.35}$$

for all $x \in X$. Therefore, it follows from (3.4) and (3.35) that

$$\sum_{i=1}^{\infty} 4^{ip} \varphi_q{}^p\left(\frac{x}{2^i}\right) < \infty \tag{3.36}$$

for all $x \in X$. It follows from (3.34) and (3.36) that the sequence $\{4^m g(x/2^m)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^m g(x/2^m)\}$ converges for all $x \in X$. So one can define the mapping $Q : X \to Y$ by

$$Q(x) = \lim_{m \to \infty} 4^m g\left(\frac{x}{2^m}\right) \tag{3.37}$$

for all $x \in X$. Letting $k = 0$ and passing the limit $m \to \infty$ in (3.34), we get

$$\|g(x) - Q(x)\|_Y^p \leq M^{8p} \sum_{i=0}^{\infty} 4^{ip} {\psi_q}^p \left(\frac{x}{2^{i+1}}\right) = \frac{M^{8p}}{4^p} \sum_{i=1}^{\infty} 4^{ip} {\psi_q}^p \left(\frac{x}{2^i}\right) \tag{3.38}$$

for all $x \in X$. Thus (3.7) follows from (3.4) and (3.38).

Now we show that Q is quadratic. It follows from (3.3), (3.33) and (3.37) that

$$\begin{aligned} \|Q(2x) - 4Q(x)\|_Y &= \lim_{m\to\infty} \left\| 4^m g\left(\frac{x}{2^{m-1}}\right) - 4^{m+1} g\left(\frac{x}{2^m}\right) \right\|_Y \\ &= 4 \lim_{m\to\infty} \left\| 4^{m-1} g\left(\frac{x}{4^{m-1}}\right) - 4^m g\left(\frac{x}{2^m}\right) \right\|_Y \\ &\leq M^{11} \lim_{m\to\infty} 4^m \psi_q\left(\frac{x}{2^m}\right) = 0 \end{aligned} \tag{3.39}$$

for all $x \in X$. So

$$Q(2x) = 4Q(x) \tag{3.40}$$

for all $x \in X$. On the other hand, it follows from (3.3), (3.5), (3.6) and (3.37) that

$$\begin{aligned} \|\Delta Q(x,y)\|_Y &= \lim_{m\to\infty} 4^m \left\| \Delta g\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\|_Y \\ &= \lim_{m\to\infty} 4^m \left\| \Delta f\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 16 \Delta f\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\|_Y \\ &\leq M \lim_{m\to\infty} 4^m \left\{ \left\| \Delta f\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) \right\|_Y + 16 \left\| \Delta f\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\|_Y \right\} \\ &\leq M \lim_{m\to\infty} 4^m \left\{ \varphi_q\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) + 16 \varphi_q\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\} = 0 \end{aligned} \tag{3.41}$$

for all $x, y \in X$. Hence the mapping Q satisfies (1.6). By Lemma 2.1, the mapping $Q(2x) - 4Q(x)$ is quadratic. Hence (3.40) implies that the mapping Q is quadratic.

It remains to show that Q is unique. Suppose that there exists another quadratic mapping $Q' : X \to Y$ which satisfies (1.6) and (3.7). Since $Q'(x/2^m) = (1/4^m)Q'(x)$ and $Q(x/2^m) = (1/4^m)Q(x)$ for all $x \in X$, we conclude from (3.7) that

$$\|Q(x) - Q'(x)\|_Y^p = \lim_{m\to\infty} 4^{mp} \left\| g\left(\frac{x}{2^m}\right) - Q'\left(\frac{x}{2^m}\right) \right\|_Y^p \leq \frac{M^{8p}}{4^p} \lim_{m\to\infty} 4^{mp} \widetilde{\psi}_q\left(\frac{x}{2^m}\right) \tag{3.42}$$

for all $x \in X$. On the other hand, since

$$\lim_{m\to\infty} 4^{mp}\sum_{i=1}^{\infty}4^{ip}\varphi_q{}^p\left(\frac{x}{2^{m+i}},\frac{y}{2^{m+i}}\right) = \lim_{m\to\infty}\sum_{i=m+1}^{\infty}4^{ip}\varphi_q{}^p\left(\frac{x}{2^i},\frac{y}{2^i}\right) = 0 \tag{3.43}$$

for all $x \in X$ and all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$, then

$$\lim_{m\to\infty} 4^{mp}\tilde{\varphi}_q\left(\frac{x}{2^m}\right) = 0 \tag{3.44}$$

for all $x \in X$. Using (3.44) and (3.42), we get $Q = Q'$, as desired. □

Theorem 3.3. *Let $\varphi_q : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{m\to\infty}\frac{1}{4^m}\varphi_q(2^m x, 2^m y) = 0 \tag{3.45}$$

for all $x, y \in X$ and

$$\sum_{i=0}^{\infty}\frac{1}{4^{pi}}\varphi_q{}^p\left(2^i x, 2^i y\right) < \infty \tag{3.46}$$

for all $x \in X$ and all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\Delta f(x,y)\|_Y \le \varphi_q(x,y) \tag{3.47}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{m\to\infty}\frac{1}{4^m}\left[f\left(2^{m+1}x\right) - 16f(2^m x)\right] \tag{3.48}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping satisfying

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{M^8}{4}\left[\tilde{\varphi}_q(x)\right]^{1/p} \tag{3.49}$$

for all $x \in X$, where

$$
\begin{aligned}
\widetilde{\varphi}_q(x) := \sum_{i=0}^{\infty} \frac{1}{4^{pi}} \Bigg\{ \frac{1}{n^{2p}(n^2-1)^p} \Big[& \varphi_q^p\left(2^i x, 2^i(n+2)x\right) + \varphi_q^p\left(2^i x, 2^i(n-2)x\right) + 4^p \varphi_q^p\left(2^i x, 2^i(n+1)x\right) \\
& + 4^p \varphi_q^p\left(2^i x, 2^i(n-1)x\right) + 10^p \varphi_q^p\left(2^i x, 2^i nx\right) + \varphi_q^p\left(2^i 2x, 2^i 2x\right) + 4^p \varphi_q^p\left(2^i 2x, 2^i x\right) \\
& + n^{2p} \varphi_q^p\left(2^i x, 2^i 3x\right) + 2^p\left(3n^2-1\right)^p \varphi_q^p\left(2^i x, 2^i 2x\right) + \left(17n^2-8\right)^p \varphi_q^p\left(2^i x, 2^i x\right) \\
& + \frac{n^{2p}}{(n^2-1)^p} \Big(\varphi_q^p\left(0, 2^i(n+1)x\right) + \varphi_q^p\left(0, 2^i(n-3)x\right) \\
& \qquad + 10^p \varphi_q^p\left(0, 2^i(n-1)x\right) + 4^p \varphi_q^p\left(0, 2^i nx\right) + 4^p \varphi_q^p\left(0, 2^i(n-2)x\right) \Big) \\
& + \frac{(n^4+1)^p}{(n^2-1)^p} \varphi_q^p\left(0, 2^i 2x\right) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p} \varphi_q^p\left(0, 2^i x\right) \Big] \Bigg\}.
\end{aligned}
\tag{3.50}
$$

Proof. The proof is similar to the proof of Theorem 3.2. □

Corollary 3.4. *Let θ, r, s be nonnegative real numbers such that $r, s > 2$ or $s < 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$
\|\Delta f(x,y)\|_Y \leq \begin{cases} \theta, & r = s = 0, \\ \theta\|x\|_X^r, & r > 0,\ s = 0, \\ \theta\|y\|_X^s, & r = 0,\ s > 0, \\ \theta\left(\|x\|_X^r + \|y\|_X^s\right), & r,\ s > 0 \end{cases}
\tag{3.51}
$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying

$$
\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)} \begin{cases} \delta_q, & r = s = 0, \\ \alpha_q(x), & r > 0,\ s = 0, \\ \beta_q(x), & r = 0,\ s > 0, \\ \left(\alpha_q^p(x) + \beta_q^p(x)\right)^{1/p}, & r,\ s > 0 \end{cases}
\tag{3.52}
$$

for all $x \in X$*, where*

$$\delta_q = \left\{ \frac{1}{4^p - 1(n^2-1)^p} \left[\left(6n^2-2\right)^p \left(n^2-1\right)^p + \left(17n^2-8\right)^p (n^2-1)^p + \left(6n^4-2n^2+4\right)^p + n^{2p}(2+10^p+2*4^p) + \left(n^4+1\right)^p + n^{2p}\left(n^2-1\right)^p + 3*4^p\left(n^2-1\right)^p + 10^p\left(n^2-1\right)^p + 3\left(n^2-1\right)^p \right] \right\}^{1/p},$$

$$\alpha_q(x) = \left\{ \frac{4^p(2+2^{rp}) + 10^p + (6n^2-2)^p + (17n^2-8)^p + 2^{rp} + n^{2p}}{|4^p - 2^{rp}|} \right\}^{1/p} \|x\|_X^r,$$

$$\beta_q(x) = \left\{ \frac{1}{(n^2-1)^p|4^p-2^{sp}|} \left[2^{sp}\left(6n^2-2\right)^p\left(n^2-1\right)^p + (17n^2-8)^p(n^2-1)^p + \left(6n^4-2n^2+4\right)^p + n^{2p}\left((n+1)^{sp} + (n-3)^{sp} + 10^p(n-1)^{sp} + 4^p n^{sp} + 4^p(n-2)^{sp}\right) + 2^{sp}\left(n^4+1\right)^p + 3^{sp}n^{2p}(n^2-1)^p + 4^p(n^2-1)^p + (n+2)^{sp}\left(n^2-1\right)^p + (n-2)^{sp}\left(n^2-1\right)^p + 4^p(n+1)^{sp}(n^2-1)^p + 4^p(n-1)^{sp}\left(n^2-1\right)^p + 10^p n^{sp}\left(n^2-1\right)^p \right] \right\}^{1/p} \|x\|_X^s. \tag{3.53}$$

Proof. In Theorem 3.2, putting $\varphi_q(x,y) := \theta(\|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$, we get the desired result. □

Corollary 3.5. *Let* $\theta \geq 0$ *and* $r, s > 0$ *be nonnegative real numbers such that* $\lambda := r + s \neq 2$*. Suppose that a maping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|\Delta f(x,y)\|_Y \leq \theta \|x\|_X^r \|y\|_X^s \tag{3.54}$$

for all $x, y \in X$*. Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying*

$$\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)} \left\{ \frac{1}{|4^p - 2^{\lambda p}|} \left[(n+2)^{sp} + (n-2)^{sp} + 4^p(n+1)^{sp} + 4^p(n-1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p}3^{sp} + 2^{sp}\left(6n^2-2\right)^p + \left(17n^2-8\right)^p \right] \right\}^{1/p} \|x\|_X^\lambda \tag{3.55}$$

for all $x \in X$.

Proof. In Theorem 3.2, putting $\varphi_q(x,y) := \theta\|x\|_X^r\|y\|_X^s$ for all $x,y\in X$, we get the desired result. □

Theorem 3.6. *Let $\varphi_t : X\times X \to [0,\infty)$ be a function such that*

$$\lim_{m\to\infty} 16^m\varphi_t\left(\frac{x}{2^m},\frac{y}{2^m}\right)=0 \tag{3.56}$$

for all $x,y\in X$ and

$$\sum_{i=1}^{\infty}16^{pi}\varphi_t{}^p\left(\frac{x}{2^i},\frac{y}{2^i}\right)<\infty \tag{3.57}$$

for all $x\in X$ and all $y\in\{x,2x,3x,nx,(n+1)x,(n-1)x,(n+2)x,(n-2)x,(n-3)x\}$. Suppose that a mapping $f : X\to Y$ with $f(0)=0$ satisfies the inequality

$$\|\Delta f(x,y)\|_Y\le\varphi_t(x,y) \tag{3.58}$$

for all $x,y\in X$. Then the limit

$$T(x) := \lim_{m\to\infty}16^m\left[f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right)\right] \tag{3.59}$$

exists for all $x\in X$ and $T : X\to Y$ is a unique quartic mapping satisfying

$$\|f(2x)-4f(x)-T(x)\|_Y\le\frac{M^8}{16}\left[\widetilde{\varphi}_t(x)\right]^{1/p} \tag{3.60}$$

for all $x\in X$, where

$$\begin{aligned}
\widetilde{\varphi}_t(x) := \sum_{i=1}^{\infty}16^{pi}\Bigg\{\frac{1}{n^{2p}(n^2-1)^p}\Bigg[&\varphi_t^p\left(\frac{x}{2^i},\frac{(n+2)x}{2^i}\right)+\varphi_t^p\left(\frac{x}{2^i},\frac{(n-2)x}{2^i}\right)+4^p\varphi_t^p\left(\frac{x}{2^i},\frac{(n+1)x}{2^i}\right)\\
&+4^p\varphi_t^p\left(\frac{x}{2^i},\frac{(n-1)x}{2^i}\right)+10^p\varphi_t^p\left(\frac{x}{2^i},\frac{nx}{2^i}\right)+\varphi_t^p\left(\frac{2x}{2^i},\frac{2x}{2^i}\right)+4^p\varphi_t^p\left(\frac{2x}{2^i},\frac{x}{2^i}\right)\\
&+n^{2p}\varphi_t^p\left(\frac{x}{2^i},\frac{3x}{2^i}\right)+2^p\left(3n^2-1\right)^p\varphi_t^p\left(\frac{x}{2^i},\frac{2x}{2^i}\right)+\left(17n^2-8\right)^p\varphi_t^p\left(\frac{x}{2^i},\frac{x}{2^i}\right)\\
&+\frac{n^{2p}}{(n^2-1)^p}\Bigg(\varphi_t^p\left(0,\frac{x(n+1)x}{2^i}\right)+\varphi_t^p\left(0,\frac{(n-3)x}{2^i}\right)+10^p\varphi_t^p\left(0,\frac{(n-1)x}{2^i}\right)\\
&+4^p\varphi_t^p\left(0,\frac{nx}{2^i}\right)+4^p\varphi_t^p\left(0,\frac{(n-2)x}{2^i}\right)\Bigg)+\frac{(n^4+1)^p}{(n^2-1)^p}\varphi_t^p\left(0,\frac{2x}{2^i}\right)\\
&+\frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_t^p\left(0,\frac{x}{2^i}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{3.61}$$

Proof. Similar to the proof Theorem 3.2, we have

$$\left\| f(4x) - 20f(2x) + 64f(x) \right\| \le M^8 \psi_t(x) \tag{3.62}$$

for all $x \in X$, where

$$\begin{aligned}
\psi_t(x) = \frac{1}{n^2(n^2-1)} \Bigg[& \varphi_t(x,(n+2)x) + \varphi_t(x,(n-2)x) + 4\varphi_t(x,(n+1)x) \\
& + 4\varphi_t(x,(n-1)x) + 10\varphi_t(x,nx) + \varphi_t(2x,2x) + 4\varphi_t(2x,x) \\
& + n^2\varphi_t(x,3x) + 2\left(3n^2-1\right)\varphi_t(x,2x) + \left(17n^2-8\right)\varphi_t(x,x) \\
& + \frac{n^2}{n^2-1}(\varphi_t(0,(n+1)x) + \varphi_t(0,(n-3)x) + 10\varphi_t(0,(n-1)x) + 4\varphi_t(0,nx) \\
& +4\varphi_t(0,(n-2)x)) + \frac{n^4+1}{n^2-1}\varphi_t(0,2x) + \frac{2(3n^4-n^2+2)}{n^2-1}\varphi_t(0,x) \Bigg].
\end{aligned} \tag{3.63}$$

Let $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 4f(x)$. Then we conclude that

$$\|h(2x) - 16h(x)\| \le M^8 \psi_t(x) \tag{3.64}$$

for all $x \in X$. If we replace x in (3.65) by $x/2^{m+1}$ and multiply both sides of (3.65) by 16^m, then we get

$$\left\| 16^{m+1} h\left(\frac{x}{2^{m+1}}\right) - 16^m h\left(\frac{x}{2^m}\right) \right\|_Y \le M^8 16^m \psi_t\left(\frac{x}{2^{m+1}}\right) \tag{3.65}$$

for all $x \in X$ and all nonnegative integers m. Since Y is a p-Banach space, the inequality (3.66) gives

$$\begin{aligned}
\left\| 16^{m+1} h\left(\frac{x}{2^{m+1}}\right) - 16^k h\left(\frac{x}{2^k}\right) \right\|_Y^p &\le \sum_{i=k}^{m} \left\| 16^{i+1} h\left(\frac{x}{2^{i+1}}\right) - 16^i h\left(\frac{x}{2^i}\right) \right\|_Y^p \\
&\le M^{8p} \sum_{i=k}^{m} 16^{pi} {\psi_t}^p \left(\frac{x}{2^{i+1}}\right)
\end{aligned} \tag{3.66}$$

for all nonnegative integers m and k with $m \geq k$ and all $x \in X$. Since $0 < p \leq 1$, by Lemma 3.1, we conclude from (3.64) that

$$\begin{aligned}
\varphi_t^p(x) \leq \frac{1}{n^{2p}(n^2-1)^p}\Bigg[&\varphi_t^p(x,(n+2)x) + \varphi_t^p(x,(n-2)x) + 4^p\varphi_t^p(x,(n+1)x) \\
&+ 4^p\varphi_t^p(x,(n-1)x) + 10^p\varphi_t^p(x,nx) + \varphi_t^p(2x,2x) + 4^p\varphi_t^p(2x,x) \\
&+ n^{2p}\varphi_t^p(x,3x) + 2^p\left(3n^2-1\right)^p\varphi_t^p(x,2x) + \left(17n^2-8\right)^p\varphi_t^p(x,x) \\
&+ \frac{n^{2p}}{(n^2-1)^p}\Big(\varphi_t^p(0,(n+1)x) + \varphi_t^p(0,(n-3)x) + 10^p\varphi_t^p(0,(n-1)x) \\
&\qquad + 4^p\varphi_t^p(0,nx) + 4^p\varphi_t^p(0,(n-2)x)\Big) + \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_t^p(0,2x) \\
&+ \frac{\left(2\left(3n^4-n^2+2\right)\right)^p}{(n^2-1)^p}\varphi_t^p(0,x)\Bigg]
\end{aligned} \tag{3.67}$$

for all $x \in X$. It follows from (3.57) and (3.67) that

$$\sum_{i=1}^{\infty} 16^{pi}{\varphi_t}^p\left(\frac{x}{2^i}\right) < \infty \tag{3.68}$$

for all $x \in X$. Thus we conclude from (3.67) and (3.69) that the sequence $\{16^m h(x/2^m)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{16^m h(x/2^m)\}$ converges for all $x \in X$. So one can define the mapping $T : X \to Y$ by

$$T(x) = \lim_{m\to\infty} 16^m h\left(\frac{x}{2^m}\right) \tag{3.69}$$

for all $x \in X$. Letting $k = 0$ and passing the limit $m \to \infty$ in (3.67), we get

$$\|h(x) - T(x)\|_Y^p \leq M^{8p}\sum_{i=0}^{\infty} 16^{pi}{\varphi_t}^p\left(\frac{x}{2^{i+1}}\right) = \frac{M^{11p}}{16^p}\sum_{i=1}^{\infty} 16^{pi}{\varphi_t}^p\left(\frac{x}{2^i}\right) \tag{3.70}$$

for all $x \in X$. Thus (3.60) follows from (3.58) and (3.70).

Now we show that T is quartic. From (3.57), (3.66), and (3.70), it follows that

$$\begin{aligned}
\|T(2x) - 16T(x)\|_Y &= \lim_{m\to\infty}\left\|16^m h\left(\frac{x}{2^{m-1}}\right) - 16^{m+1}h\left(\frac{x}{2^m}\right)\right\|_Y \\
&= 16\lim_{m\to\infty}\left\|16^{m-1} h\left(\frac{x}{16^{m-1}}\right) - 16^{m}h\left(\frac{x}{2^m}\right)\right\|_Y \\
&\leq M^8 \lim_{m\to\infty} 16^m \varphi_t\left(\frac{x}{2^m}\right) = 0
\end{aligned} \tag{3.71}$$

for all $x \in X$. So

$$T(2x) = 16T(x) \tag{3.72}$$

for all $x \in X$. On the other hand, by (3.59), (3.69), and (3.70), we have

$$\begin{aligned}\|\Delta T(x,y)\|_Y &= \lim_{m\to\infty} 16^m \left\|\Delta h\left(\frac{x}{2^m},\frac{y}{2^m}\right)\right\|_Y \\ &= \lim_{m\to\infty} 16^m \left\|\Delta f\left(\frac{x}{2^{m-1}},\frac{y}{2^{m-1}}\right) - 4\Delta f\left(\frac{x}{2^m},\frac{y}{2^m}\right)\right\|_Y \\ &\le M \lim_{m\to\infty} 16^m \left\{\left\|\Delta f\left(\frac{x}{2^{m-1}},\frac{y}{2^{m-1}}\right)\right\|_Y + 4\left\|\Delta f\left(\frac{x}{2^m},\frac{y}{2^m}\right)\right\|_Y\right\} \\ &\le M \lim_{m\to\infty} 16^m \left\{\varphi_t\left(\frac{x}{2^{m-1}},\frac{y}{2^{m-1}}\right) + 4\varphi_t\left(\frac{x}{2^m},\frac{y}{2^m}\right)\right\} = 0\end{aligned} \tag{3.73}$$

for all $x, y \in X$. Hence the mapping T satisfies (1.6). By Lemma 2.1, the mapping $T(2x) - 16T(x)$ is quartic. Therefore, (3.75) implies that the mapping T is quartic.

To prove the uniqueness property of T, let $T' : X \to Y$ be another quartic mapping satisfying (3.61). Since

$$\lim_{m\to\infty} 16^{mp} \sum_{i=1}^{\infty} 16^{pi} {\varphi_t}^p \left(\frac{x}{2^{m+i}},\frac{x}{2^{m+i}}\right) = \lim_{m\to\infty} \sum_{i=m+1}^{\infty} 16^{pi} {\varphi_t}^p \left(\frac{x}{2^i},\frac{x}{2^i}\right) = 0 \tag{3.74}$$

for all $x \in X$ and all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$, then

$$\lim_{m\to\infty} 16^{mp} \tilde{\varphi}_t\left(\frac{x}{2^m}\right) = 0 \tag{3.75}$$

for all $x \in X$. It follows from (3.61) and (3.86) that

$$\|T(x) - T'(x)\|_Y = \lim_{m\to\infty} 16^{mp} \left\|h\left(\frac{x}{2^m}\right) - T'\left(\frac{x}{2^m}\right)\right\|_Y^p \le \frac{M^{8p}}{16^p} \lim_{m\to\infty} 16^{mp} \tilde{\varphi}_t\left(\frac{x}{2^m}\right) = 0 \tag{3.76}$$

for all $x \in X$. So $T = T'$, as desired. □

Theorem 3.7. *Let $\varphi_t : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{1}{16^m} \varphi_t(2^m x, 2^m y) = 0 \tag{3.77}$$

for all $x, y \in X$ and

$$\sum_{i=0}^{\infty} \frac{1}{16^{pi}} {\varphi_t}^p \left(2^i x, 2^i y\right) < \infty \tag{3.78}$$

for all $x \in X$ *and all* $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|\Delta f(x,y)\|_Y \leq \varphi_t(x,y) \tag{3.79}$$

for all $x, y \in X$. *Then the limit*

$$T(x) := \lim_{m\to\infty} \frac{1}{16^m}\left[f\left(2^{m+1}x\right) - 4f(2^m x)\right] \tag{3.80}$$

exists for all $x \in X$ *and* $T : X \to Y$ *is a unique quartic mapping satisfying*

$$\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8}{16}\left[\widetilde{\varphi}_t(x)\right]^{1/p} \tag{3.81}$$

for all $x \in X$, *where*

$$\begin{aligned}\widetilde{\varphi}_t(x) := \sum_{i=0}^{\infty}\frac{1}{16^{pi}}\Bigg\{&\frac{1}{n^{2p}(n^2-1)^p}\Big[\varphi_t^p\left(2^i x, 2^i(n+2)x\right) + \varphi_t^p\left(2^i x, 2^i(n-2)x\right) + 4^p\varphi_t^p\left(2^i x, 2^i(n+1)x\right)\\
&+4^p\varphi_t^p\left(2^i x, 2^i(n-1)x\right) + 10^p\varphi_t^p\left(2^i x, 2^i nx\right) + \varphi_t^p\left(2^i 2x, 2^i 2x\right) + 4^p\varphi_t^p\left(2^i 2x, 2^i x\right)\\
&+ n^{2p}\varphi_t^p\left(2^i x, 2^i 3x\right) + 2^p\left(3n^2-1\right)^p\varphi_t^p\left(2^i x, 2^i 2x\right) + \left(17n^2-8\right)^p\varphi_t^p\left(2^i x, 2^i x\right)\\
&+ \frac{n^{2p}}{(n^2-1)^p}\Big(\varphi_t^p\left(0, 2^i(n+1)x\right) + \varphi_t^p\left(0, 2^i(n-3)x\right) + 10^p\varphi_t^p\left(0, 2^i(n-1)x\right)\\
&\qquad +4^p\varphi_t^p\left(0, 2^i nx\right) + 4^p\varphi_t^p\left(0, 2^i(n-2)x\right)\Big) + \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_t^p\left(0, 2^i 2x\right)\\
&+\frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_t^p\left(0, 2^i x\right)\Big]\Bigg\}.\end{aligned} \tag{3.82}$$

Proof. The proof is similar to the proof of Theorem 3.6. □

Corollary 3.8. *Let* θ, r, s *be nonnegative real numbers such that* $r, s > 4$ *or* $0 \leq r, s < 4$. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*(3.51) *for all* $x, y \in X$. *Then there exists a unique quartic mapping* $T : X \to Y$ *satisfying*

$$\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)}\begin{cases}\delta_t, & r = s = 0,\\ \alpha_t(x), & r > 0,\ s = 0,\\ \beta_t(x), & r = 0,\ s > 0,\\ \left(\alpha_t^p(x) + \beta_t^p(x)\right)^{1/p}, & r,\ \ s > 0\end{cases} \tag{3.83}$$

for all $x \in X$, *where*

$$\delta_t = \left\{ \frac{1}{(16^p - 1)(n^2-1)^p} \left[\left(6n^2-2\right)^p \left(n^2-1\right)^p + \left(17n^2-8\right)^p \left(n^2-1\right)^p + \left(6n^4-2n^2+4\right)^p + n^{2p}(2+10^p+2\cdot 4^p) + \left(n^4+1\right)^p + n^{2p}\left(n^2-1\right)^p + 3\cdot 4^p\left(n^2-1\right)^p + 10^p\left(n^2-1\right)^p + 3\left(n^2-1\right)^p \right] \right\}^{1/p},$$

$$\alpha_t(x) = \left\{ \frac{4^p(2+2^{rp}) + 10^p + (6n^2-2)^p + (17n^2-8)^p + 2^{rp} + n^{2p}}{|16^p - 2^{rp}|} \right\}^{1/p} \|x\|_X^r,$$

$$\beta_t(x) = \left\{ \frac{1}{(n^2-1)^p|16^p - 2^{sp}|} \left[2^{sp}\left(6n^2-2\right)^p \left(n^2-1\right)^p + \left(17n^2-8\right)^p\left(n^2-1\right)^p + \left(6n^4-2n^2+4\right)^p + n^{2p}\left((n+1)^{sp} + (n-3)^{sp} + 10^p(n-1)^{sp} + 10^p n^{sp} + 4^p(n-2)^{sp}\right) + 2^{sp}\left(n^4+1\right)^p + 3^{sp}n^{2p}\left(n^2-1\right)^p + 4^p\left(n^2-1\right)^p + (n+2)^{sp}\left(n^2-1\right)^p + (n-2)^{sp}\left(n^2-1\right)^p + 4^p(n+1)^{sp}\left(n^2-1\right)^p + 4^p(n-1)^{sp}\left(n^2-1\right)^p + 4^p n^{sp}\left(n^2-1\right)^p \right] \right\}^{1/p} \|x\|_X^s. \tag{3.84}$$

Proof. In Theorem 3.6, putting $\varphi_t(x,y) := \theta(\|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$, we get the desired result. □

Corollary 3.9. *Let* $\theta \geq 0$ *and* $r, s > 0$ *be nonnegative real numbers such that* $\lambda := r + s \neq 4$. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (3.56) *for all* $x, y \in X$. *Then there exists a unique quartic mapping* $T : X \to Y$ *satisfying*

$$\begin{aligned} &\left\| f(2x) - 4f(x) - T(x) \right\|_Y \\ &\quad \leq \frac{M^8\theta}{n^2(n^2-1)} \left\{ \frac{1}{\left|16^p - 2^{\lambda p}\right|} \left[(n+2)^{sp} + (n-2)^{sp} + 4^p(n+1)^{sp} + 4^p(n-1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p}3^{sp} + 2^{sp}\left(6n^2-2\right)^p + (17n^2-8)^p \right] \right\}^{1/p} \|x\|_X^{\lambda} \end{aligned} \tag{3.85}$$

for all $x \in X$.

Proof. In Theorem 3.6, putting $\varphi_t(x,y) := \theta\|x\|_X^r\|y\|_X^s$ for all $x, y \in X$, we get the desired result. □

Theorem 3.10. *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{m\to\infty} 4^m \varphi\Big(\frac{x}{2^m}, \frac{y}{2^m}\Big) = 0 = \lim_{m\to\infty} \frac{1}{16^m} \varphi(2^m x, 2^m y) \tag{3.86}$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 4^{pi} \varphi^p \left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty,$$
$$\sum_{i=0}^{\infty} \frac{1}{16^{pi}} \varphi^p \left(2^i x, 2^i y\right) < \infty \tag{3.87}$$

for all $x \in X$ and all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\Delta f(x,y)\|_Y \le \varphi(x,y) \tag{3.88}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique quartic mapping $T : X \to Y$ such that

$$\|f(x) - Q(x) - T(x)\|_Y \le \frac{M^9}{192}\Big(4[\widetilde{\varphi}_q(x)]^{1/p} + [\widetilde{\varphi}_t(x)]^{1/p}\Big) \tag{3.89}$$

for all $x \in X$, where $\widetilde{\varphi}_q(x)$ and $\widetilde{\varphi}_t(x)$ are defined in Theorems 3.2 and 3.7, respectively.

Proof. By Theorems 3.2 and 3.7, there exist a quadratic mapping $Q_0 : X \to Y$ and a quartic mapping $T_0 : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q_0(x)\|_Y \le \frac{M^8}{4}[\widetilde{\varphi}_q(x)]^{1/p}, \qquad \|f(2x) - 4f(x) - T_0(x)\|_Y \le \frac{M^8}{16}[\widetilde{\varphi}_t(x)]^{1/p} \tag{3.90}$$

for all $x \in X$. It follows from the last inequalities that

$$\left\|f(x) + \frac{1}{12}Q_0(x) - \frac{1}{12}T_0(x)\right\|_Y \le \frac{M^9}{192}\Big(4[\widetilde{\varphi}_q(x)]^{1/p} + [\widetilde{\varphi}_t(x)]^{1/p}\Big) \tag{3.91}$$

for all $x \in X$. So we obtain (3.92) by letting $Q(x) = -(1/12)Q_0(x)$ and $T(x) = (1/12)T_0(x)$ for all $x \in X$.

To prove the uniqueness property of Q and T, we first show the uniqueness property for Q_0 and T_0 and then we conclude the uniqueness property of Q and T. Let $Q_1, T_1 : X \to Y$

be another quadratic and quartic mappings satisfying (3.92) and let $Q_2 = (1/12)Q_0$, $T_2 = (1/12)T_0$, $Q_3 = Q_2 - Q_1$ and $T_3 = T_2 - T_1$. So

$$\begin{aligned}\|Q_3(x) - T_3(x)\|_Y &\leq M\{\|f(x) - Q_2(x) - T_2(x)\|_Y + \|f(x) - Q_1(x) - T_1(x)\|_Y\} \\ &\leq \frac{M^{10}}{96}\left(4\left[\widetilde{\varphi}_q(x)\right]^{1/p} + \left[\widetilde{\varphi}_t(x)\right]^{1/p}\right)\end{aligned} \tag{3.92}$$

for all $x \in x$. Since

$$\lim_{m\to\infty} 4^{mp}\widetilde{\varphi}_q\left(\frac{x}{2^m}\right) = \lim_{m\to\infty}\frac{1}{16^{mp}}\widetilde{\varphi}_t(2^m x) = 0 \tag{3.93}$$

for all $x \in X$, (3.62) implies that $\lim_{m\to\infty}\|4^m Q_3(x/2^m) + (1/16^m)T_3(2^m x)\|_Y = 0$ for all $x \in X$. Thus $T_3 = Q_3$. But T_3 is only a quartic function and Q_3 is only a quadratic function.

Therefore, we have $T_3 = Q_3 = 0$ and this completes the uniqueness property of Q and T. We can prove the other results similarly. □

Corollary 3.11. *Let θ, r, s be nonnegative real numbers such that $r, s > 4$ or $2 < r, s < 4$ or $0 \leq r, s < 2$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*(3.51) *for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique quartic mapping $T : X \to Y$ such that*

$$\begin{aligned}&\|f(x) - Q(x) - T(x)\|_Y \\ &\quad\leq \frac{M^9\theta}{12n^2(n^2-1)}\begin{cases}\delta_q + \delta_t, & r = s = 0, \\ \alpha_q(x) + \alpha_t(x), & r > 0,\ s = 0, \\ \beta_q(x) + \beta_t(x), & r = 0,\ s > 0, \\ \left(\alpha_q^p(x) + \beta_q^p(x)\right)^{1/p} + \left(\alpha_t^p(x) + \beta_t^p(x)\right)^{1/p}, & r,\ s > 0\end{cases}\end{aligned} \tag{3.94}$$

for all $x \in X$, where $\delta_q, \delta_t, \alpha_q(x), \alpha_t(x), \beta_q(x)$, and $\beta_t(x)$ are defined as in Corollaries 3.4 and 3.8.

Corollary 3.12. *Let $\theta \geq 0$ and $r, s > 0$ be nonnegative real numbers such that $\lambda := r + s \in (0,2) \cup (2,4) \cup (4,\infty)$. Suppose that a mapping $f : X \to Y$ satisfies the inequality* (3.56) *for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique quartic function $T : X \to Y$ such that*

$$\begin{aligned}&\|f(x) - Q(x) - T(x)\|_Y \\ &\quad\leq \frac{M^9\theta}{12n^2(n^2-1)}\left\{\frac{1}{|4^p - 2^{\lambda p}|}\left[(n+2)^{sp} + (n-2)^{sp} + 4^p(n+1)^{sp}\right.\right. \\ &\qquad\qquad + 4^p(n-1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p}3^{sp} \\ &\qquad\qquad \left.\left. + 2^{sp}\left(6n^2-2\right)^p + \left(17n^2-8\right)^p\right]\right\}\|x\|_X^{\lambda}\end{aligned} \tag{3.95}$$

for all $x \in X$.

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