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*Research Article*

# A Fixed Point Approach to the Stability of a Quadratic Functional Equation in $C^*$-Algebras

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We use a fixed point method to investigate the stability problem of the quadratic functional equation $f(x+y)+f(x-y)=2f(\sqrt{xx^*+yy^*})$ in $C^*$-algebras.



## 1. Introduction and Preliminaries

In 1940, the following question concerning the stability of group homomorphisms was proposed by Ulam [1]: *Under what conditions does there exist a group homomorphism near an approximately group homomorphism?* In 1941, Hyers [2] considered the case of approximately additive functions $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon \tag{1.1}$$

for all $x, y \in E$. Aoki [3] and Th. M. Rassias [4] provided a generalization of the Hyers' theorem for additive mappings and for linear mappings, respectively, by allowing the Cauchy difference to be unbounded (see also [5]).

**Theorem 1.1** (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p) \tag{1.2}$$

*for all* $x, y \in E$, *where* $\epsilon$ *and* $p$ *are constants with* $\epsilon > 0$ *and* $p < 1$. *Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.3}$$

*exists for all* $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.4}$$

*for all* $x \in E$. *If* $p < 0$ *then inequality* (1.2) *holds for* $x, y \neq 0$ *and* (1.4) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* $L$ *is* $\mathbb{R}$*-linear.*

The result of the Th. M. Rassias theorem has been generalized by Găvruţa [6] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [7–20]). We also refer the readers to the books [21–25]. A *quadratic functional equation* is a functional equation of the following form:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.5}$$

In particular, every solution of the quadratic equation (1.5) is said to be a *quadratic mapping.* It is well known that a mapping $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping $B$ such that $f(x) = B(x, x)$ for all $x$ (see [16, 21, 26, 27]. The biadditive mapping $B$ is given by

$$B(x, y) = \frac{1}{4}[f(x+y) - f(x-y)]. \tag{1.6}$$

The Hyers-Ulam stability problem for the quadratic functional equation (1.5) was studied by Skof [28] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if we replace $E_1$ by an Abelian group. Czerwik [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.5). Grabiec [11] has generalized these results mentioned above. Jun and Lee [14] proved the generalized Hyers-Ulam stability of a Pexiderized quadratic functional equation.

Let $E$ be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on $E$ if $d$ satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

**Theorem 1.2** (see [29]). *Let $(E, d)$ be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.7}$$

*for all nonnegative integers $n$ or there exists a non-negative integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* $y^*$ *of* $J$;

(3) $y^*$ *is the unique fixed point of* $J$ *in the set* $Y = \{y \in E : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper $A$ will be a $C^*$-algebra. We denote by $\sqrt{a}$ the unique positive element $b \in A$ such that $b^2 = a$ for each positive element $a \in A$. Also, we denote by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ the set of real, complex, and rational numbers, respectively. In this paper, we use a fixed point method (see [7, 15, 17]) to investigate the stability problem of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f\left(\sqrt{xx^* + yy^*}\right) \tag{1.8}$$

in $C^*$-algebras. A systematic study of fixed point theorems in nonlinear analysis is due to Hyers et al. [30] and Isac and Rassias [13].

## 2. Solutions of (1.8)

**Theorem 2.1.** *Let $X$ be a linear space. If a mapping $f : A \to X$ satisfies $f(0) = 0$ and the functional equation* (1.8), *then $f$ is quadratic.*

*Proof.* Letting $u = x + y$ and $v = x - y$ in (1.8), respectively, we get

$$f(u) + f(v) = 2f\left(\sqrt{\frac{uu^* + vv^*}{2}}\right) \tag{2.1}$$

for all $u, v \in A$. It follows from (1.8) and (2.1) that

$$f(u) + f(v) = f\left(\frac{u+v}{\sqrt{2}}\right) + f\left(\frac{u-v}{\sqrt{2}}\right) \tag{2.2}$$

for all $u, v \in A$. Letting $v = 0$ in (2.2), we get

$$2f\left(\frac{u}{\sqrt{2}}\right) = f(u) \tag{2.3}$$

for all $u \in A$. Thus (2.2) implies that

$$f(u+v) + f(u-v) = 2f(u) + 2f(v) \tag{2.4}$$

for all $u, v \in A$. Hence $f$ is quadratic. □

*Remark 2.2.* A quadratic mapping does not satisfy (1.8) in general. Let $f : A \to A$ be the mapping defined by $f(x) = x^2$ for all $x \in A$. It is clear that $f$ is quadratic and that $f$ does not satisfy (1.8).

**Corollary 2.3.** *Let X be a linear space. If a mapping $f : A \to X$ satisfies the functional equation* (1.8)*, then there exists a symmetric biadditive mapping $B : A \times A \to X$ such that $f(x) = B(x, x)$ for all $x \in A$.*

## 3. Generalized Hyers-Ulam Stability of (1.8) in $C^*$-Algebras

In this section, we use a fixed point method (see [7, 15, 17]) to investigate the stability problem of the functional equation (1.8) in $C^*$-algebras.

For convenience, we use the following abbreviation for a given mapping $f : A \to X$ :

$$Df(x,y) := f(x+y) + f(x-y) - 2f\left(\sqrt{xx^* + yy^*}\right) \tag{3.1}$$

for all $x, y \in A$, where X is a linear space.

**Theorem 3.1.** *Let X be a linear space and let $f : A \to X$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.2}$$

*for all $x, y \in A$. If there exists a constant $0 < L < 1$ such that*

$$\varphi\left(\sqrt{2}x, \sqrt{2}y\right) \le 2L\varphi(x,y) \tag{3.3}$$

*for all $x, y \in A$, then there exists a unique quadratic mapping $Q : A \to X$ such that*

$$\|f(x) - Q(x)\| \le \frac{1}{2-2L}\phi(x) \tag{3.4}$$

*for all $x \in A$, where*

$$\phi(x) := \varphi(x, 0) + \varphi\left(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right). \tag{3.5}$$

*Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then $Q$ is $\mathbb{R}$-quadratic, that is, $Q(tx) = t^2Q(x)$ for all $x \in A$ and all $t \in \mathbb{R}$.*

*Proof.* Replacing $x$ and $y$ by $(x+y)/2$ and $(x-y)/2$ in (3.2), respectively, we get

$$\left\| f(x) + f(y) - 2f\left(\sqrt{\frac{xx^* + yy^*}{2}}\right) \right\| \le \varphi\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \tag{3.6}$$

for all $x, y \in A$. Replacing $x$ and $y$ by $x/\sqrt{2}$ and $y/\sqrt{2}$ in (3.2), respectively, we get

$$\left\| f\left(\frac{x+y}{\sqrt{2}}\right) + f\left(\frac{x-y}{\sqrt{2}}\right) - 2f\left(\sqrt{\frac{xx^* + yy^*}{2}}\right) \right\| \le \varphi\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \tag{3.7}$$

for all $x, y \in A$. It follows from (3.6) and (3.7) that

$$\left\| f\left(\frac{x+y}{\sqrt{2}}\right) + f\left(\frac{x-y}{\sqrt{2}}\right) - f(x) - f(y) \right\| \le \varphi\left(\frac{x+y}{2}, \frac{x-y}{2}\right) + \varphi\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \tag{3.8}$$

for all $x, y \in A$. Letting $y = x$ in (3.8), we get

$$\left\| f\left(\sqrt{2}x\right) - 2f(x) \right\| \le \varphi(x, 0) + \varphi\left(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right) \tag{3.9}$$

for all $x \in A$. By (3.3) we have $\phi(\sqrt{2}x) \le 2L\phi(x)$ for all $x \in A$. Let $E$ be the set of all mappings $g : A \to X$ with $g(0) = 0$. We can define a generalized metric on $E$ as follows:

$$d(g, h) := \inf\{C \in [0, \infty] : \|g(x) - h(x)\| \le C\phi(x)\ \forall x \in A\}. \tag{3.10}$$

$(E, d)$ is a generalized complete metric space [7].

Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(x) = \frac{1}{2}g\left(\sqrt{2}x\right) \quad \forall g \in E \text{ and all } x \in A. \tag{3.11}$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$. From the definition of $d$, we have

$$\|g(x) - h(x)\| \le C\phi(x) \tag{3.12}$$

for all $x \in A$. Hence

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \frac{1}{2}\left\|g\left(\sqrt{2}x\right) - h\left(\sqrt{2}x\right)\right\| \le \frac{1}{2}C\phi\left(\sqrt{2}x\right) \le CL\phi(x) \tag{3.13}$$

for all $x \in A$. So

$$d(\Lambda g, \Lambda h) \le Ld(g, h) \tag{3.14}$$

for any $g, h \in E$. It follows from (3.9) that $d(\Lambda f, f) \le 1/2$. According to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point $Q$ of $\Lambda$, that is,

$$Q : A \to X, \qquad Q(x) = \lim_{k\to\infty}\left(\Lambda^k f\right)(x) = \lim_{k\to\infty}\frac{1}{2^k} f\left(2^{k/2}x\right), \tag{3.15}$$

and $Q(\sqrt{2}x) = 2Q(x)$ for all $x \in A$. Also,

$$d(Q, f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{1}{2-2L}, \tag{3.16}$$

and $Q$ is the unique fixed point of $\Lambda$ in the set $E^* = \{g \in E : d(f, g) < \infty\}$. Thus the inequality (3.4) holds true for all $x \in A$. It follows from the definition of $Q$, (3.2), and (3.3) that

$$\|DQ(x,y)\| = \lim_{k\to\infty}\frac{1}{2^k}\|Df\left(2^{k/2}x, 2^{k/2}y\right)\| \le \lim_{k\to\infty}\frac{1}{2^k}\varphi\left(2^{k/2}x, 2^{k/2}y\right) = 0 \tag{3.17}$$

for all $x, y \in A$. By Theorem 2.1, the function $Q : A \to X$ is quadratic.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then by the same reasoning as in the proof of [4] $Q$ is $\mathbb{R}$-quadratic. □

**Corollary 3.2.** *Let $0 < r < 2$ and $\theta, \delta$ be non-negative real numbers and let $f : A \to X$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x,y)\| \le \delta + \theta(\|x\|^r + \|y\|^r) \tag{3.18}$$

*for all $x, y \in A$. Then there exists a unique quadratic mapping $Q : A \to X$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\delta}{2 - 2^{r/2}} + \frac{2 + 2^{r/2}}{2^{r/2}(2 - 2^{r/2})}\theta\|x\|^r \tag{3.19}$$

*for all $x \in A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then $Q$ is $\mathbb{R}$-quadratic.*

The following theorem is an alternative result of Theorem 3.1 and we will omit the proof.

**Theorem 3.3.** *Let $f : A \to X$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \to [0, \infty)$ satisfying (3.2) for all $x, y \in A$. If there exists a constant $0 < L < 1$ such that*

$$2\varphi(x,y) \le L\varphi\left(\sqrt{2}x, \sqrt{2}y\right) \tag{3.20}$$

*for all $x, y \in A$, then there exists a unique quadratic mapping $Q : A \to X$ such that*

$$\|f(x) - Q(x)\| \le \frac{L}{2-2L}\phi(x) \tag{3.21}$$

*for all $x \in A$, where $\phi(x)$ is defined as in Theorem 3.1. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then $Q$ is $\mathbb{R}$-quadratic.*

**Corollary 3.4.** *Let $r > 2$ and $\theta$ be non-negative real numbers and let $f : A \to X$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x,y)\| \leq \theta\left(\|x\|^r + \|y\|^r\right) \tag{3.22}$$

*for all $x, y \in A$. Then there exists a unique quadratic mapping $Q : A \to X$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2 + 2^{r/2}}{2^{r/2}\left(2^{r/2} - 2\right)}\theta\|x\|^r \tag{3.23}$$

*for all $x \in A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then $Q$ is $\mathbb{R}$-quadratic.*

For the case $r = 2$ we use the Gajda's example [31] to give the following counterexample (see also [9]).

*Example 3.5.* Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) := \begin{cases} |x|^2, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases} \tag{3.24}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{4^n}\phi(2^n x). \tag{3.25}$$

It is clear that $f$ is continuous and bounded by $4/3$ on $\mathbb{C}$. We prove that

$$\left| f(x+y) + f(x-y) - 2f\left(\sqrt{|x|^2 + |y|^2}\right)\right| \leq \frac{64}{3}\left(|x|^2 + |y|^2\right) \tag{3.26}$$

for all $x, y \in \mathbb{C}$. To see this, if $|x|^2 + |y|^2 = 0$ or $|x|^2 + |y|^2 \geq 1/4$, then

$$\left| f(x+y) + f(x-y) - 2f\left(\sqrt{|x|^2 + |y|^2}\right)\right| \leq \frac{16}{3} \leq \frac{64}{3}\left(|x|^2 + |y|^2\right). \tag{3.27}$$

Now suppose that $0 < |x|^2 + |y|^2 < 1/4$. Then there exists a positive integer $k$ such that

$$\frac{1}{4^{k+1}} \leq |x|^2 + |y|^2 < \frac{1}{4^k}. \tag{3.28}$$

Thus

$$2^{k-1}|x \pm y|,\ 2^k\sqrt{|x|^2+|y|^2} \in (-1,1). \tag{3.29}$$

Hence

$$2^m|x \pm y|,\ 2^m\sqrt{|x|^2+|y|^2} \in (-1,1) \tag{3.30}$$

for all $m = 0,1,\dots,k-1$. It follows from the definition of $f$ and (3.28) that

$$\begin{aligned} &\left|f(x+y)+f(x-y)-2f\left(\sqrt{|x|^2+|y|^2}\right)\right| \\ &\quad= \left|\sum_{n=k}^{\infty}\frac{1}{4^n}\left[\phi(2^n(x+y))+\phi(2^n(x-y))-2\phi\left(2^n\sqrt{|x|^2+|y|^2}\right)\right]\right| \\ &\quad\le 4\sum_{n=k}^{\infty}\frac{1}{4^n} = \frac{64}{3\times 4^{k+1}} \le \frac{64}{3}\left(|x|^2+|y|^2\right). \end{aligned} \tag{3.31}$$

Thus $f$ satisfies (3.26). Let $Q : \mathbb{C} \to \mathbb{C}$ be a quadratic function such that

$$|f(x)-Q(x)| \le \beta|x|^2 \tag{3.32}$$

for all $x \in \mathbb{C}$, where $\beta$ is a positive constant. Then there exists a constant $c \in \mathbb{C}$ such that $Q(x) = cx^2$ for all $x \in \mathbb{Q}$. So we have

$$|f(x)| \le (\beta+|c|)|x|^2 \tag{3.33}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If $x_0 \in (0, 2^{-m}) \cap \mathbb{Q}$, then $2^n x_0 \in (0,1)$ for all $n = 0,1,\dots,m-1$. So

$$f(x_0) \ge \sum_{n=0}^{m-1}\frac{1}{4^n}\phi(2^n x_0) = m|x_0|^2 > (\beta+|c|)|x_0|^2 \tag{3.34}$$

which contradicts (3.33).

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