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Research Article

Functional Equations Related to Inner Product Spaces

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Let V, W be real vector spaces. It is shown that an odd mapping $f : V \to W$ satisfies $\sum_{i-1}^{2n} f(x_i - 1/2n\sum_{j=1}^{2n} x_j) = \sum_{i=1}^{2n} f(x_i) - 2nf(1/2n\sum_{i=1}^{2n} x_i)$ for all $x_1, \ldots, x_{2n} \in V$ if and only if the odd mapping $f : V \to W$ is Cauchy additive. Furthermore, we prove the generalized Hyers-Ulam stability of the above functional equation in real Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of Th. M. Rassias' theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function.

The functional equation,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. The generalized Hyers-Ulam stability of the quadratic functional equation has been proved by Czerwik [8], J. M. Rassias [9], Găvruta [10], and others [11]. In [12], Th. M. Rassias proved that the norm defined over a real vector space V is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$\sum_{i=1}^{n} \left\| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n\left\| \frac{1}{n}\sum_{i=1}^{n} x_i \right\|^2 \tag{1.2}$$

holds for all $x_1, \ldots, x_n \in V$. An operator extension of this norm equality is presented in [13]. For more information on the recent results on the stability of quadratic functional equation, see [14]. Inner product spaces, Cauchy equation, Euler-Lagrange-Rassias equations, and Ulam-Găvruta-Rassias stability have been studied by several authors (see [15–27]).

In [28], C. Park, Lee, and Shin proved that if an even mapping $f : V \to W$ satisfies

$$\sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j \right) = \sum_{i=1}^{2n} f(x_i) - 2nf\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i \right), \tag{1.3}$$

then the even mapping $f : V \to W$ is quadratic. Moreover, they proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.3) in real Banach spaces.

Throughout this paper, assume that n is a fixed positive integer, X and Y are real normed vector spaces.

In this paper, we investigate the functional equation

$$\sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j \right) = \sum_{i=1}^{2n} f(x_i) - 2nf\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i \right), \tag{1.4}$$

and prove the generalized Hyers-Ulam stability of the functional equation (1.4) in real Banach spaces.

2. Functional Equations Related to Inner Product Spaces

We investigate the functional equation (1.4).

Lemma 2.1. *Let V and W be real vector spaces. An odd mapping $f : V \to W$ satisfies*

$$\sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j \right) = \sum_{i=1}^{2n} f(x_i) - 2nf\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i \right), \tag{2.1}$$

for all $x_1, \ldots, x_{2n} \in V$ *if and only if the odd mapping* $f : V \to W$ *is Cauchy additive, that is,*

$$f(x+y) = f(x) + f(y), \tag{2.2}$$

for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies (2.1).

Letting $x_1 = \cdots = x_n = x$, $x_{n+1} = \cdots = x_{2n} = y$ in (2.1), we get

$$nf\left(x - \frac{x+y}{2}\right) + nf\left(y - \frac{x+y}{2}\right) = nf(x) + nf(y) - 2nf\left(\frac{x+y}{2}\right), \tag{2.3}$$

for all $x, y \in V$. Since $f : V \to W$ is odd,

$$0 = nf(x) + nf(y) - 2nf\left(\frac{x+y}{2}\right), \tag{2.4}$$

for all $x, y \in V$ and $f(0) = 0$. So

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \tag{2.5}$$

for all $x, y \in V$. Letting $y = 0$ in (2.5), we get $2f(x/2) = f(x)$ for all $x \in V$. Thus

$$f(x+y) = 2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \tag{2.6}$$

for all $x, y \in V$.

It is easy to prove the converse. □

For a given mapping $f : X \to Y$, we define

$$Df(x_1, \ldots, x_{2n}) := \sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j\right) - \sum_{i=1}^{2n} f(x_i) + 2nf\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i\right), \tag{2.7}$$

for all $x_1, \ldots, x_{2n} \in X$.

We are going to prove the generalized Hyers-Ulam stability of the functional equation $Df(x_1, \ldots, x_{2n}) = 0$ in real Banach spaces.

Theorem 2.2. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *for which there exists a function* $\varphi : X^{2n} \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x_1, \ldots, x_{2n}) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j}, \ldots, \frac{x_{2n}}{2^j}\right) < \infty, \tag{2.8}$$

$$\|Df(x_1, \ldots, x_{2n})\| \le \varphi(x_1, \ldots, x_{2n}), \tag{2.9}$$

for all $x_1, \ldots, x_{2n} \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (2.1) *such that*

$$\|f(x) - f(-x) - A(x)\| \le \frac{1}{n}\widetilde{\varphi}\Big(\underbrace{x, \ldots, x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big) + \frac{1}{n}\widetilde{\varphi}\Big(\underbrace{-x, \ldots, -x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big), \tag{2.10}$$

for all $x \in X$.

Proof. Letting $x_1 = \cdots = x_n = x$ and $x_{n+1} = \cdots = x_{2n} = 0$ in (2.9), we get

$$\left\|3nf\left(\frac{x}{2}\right) + nf\left(\frac{-x}{2}\right) - nf(x)\right\| \le \varphi\Big(\underbrace{x, \ldots, x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big), \tag{2.11}$$

for all $x \in X$. Replacing x by $-x$ in (2.11), we get

$$\left\|3nf\left(\frac{-x}{2}\right) + nf\left(\frac{x}{2}\right) - nf(-x)\right\| \le \varphi\Big(\underbrace{-x, \ldots, -x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big), \tag{2.12}$$

for all $x \in X$. Let $g(x) := f(x) - f(-x)$ for all $x \in X$. It follows from (2.11) and (2.12) that

$$\left\|2ng\left(\frac{x}{2}\right) - ng(x)\right\| \le \varphi\Big(\underbrace{x, \ldots, x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big) + \varphi\Big(\underbrace{-x, \ldots, -x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big), \tag{2.13}$$

for all $x \in X$. So

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \le \frac{1}{n}\varphi\Big(\underbrace{x, \ldots, x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big) + \frac{1}{n}\varphi\Big(\underbrace{-x, \ldots, -x}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big), \tag{2.14}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\| \le{}& \sum_{j=l}^{m-1}\frac{2^j}{n}\,\varphi\Big(\underbrace{\frac{x}{2^j}, \ldots, \frac{x}{2^j}}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big) \\ &+ \sum_{j=l}^{m-1}\frac{2^j}{n}\,\varphi\Big(\underbrace{-\frac{x}{2^j}, \ldots, -\frac{x}{2^j}}_{n\text{ times}}, \underbrace{0, \ldots, 0}_{n\text{ times}}\Big),\end{aligned} \tag{2.15}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) and (2.15) that the sequence $\{2^k g(x/2^k)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k g(x/2^k)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k g\left(\frac{x}{2^k}\right), \tag{2.16}$$

for all $x \in X$.

By (2.8) and (2.9),

$$\begin{aligned}\|DA(x_1,\dots,x_{2n})\| &= \lim_{k\to\infty} 2^k \left\|Dg\left(\frac{x_1}{2^k},\dots,\frac{x_{2n}}{2^k}\right)\right\| \\ &\le \lim_{k\to\infty} 2^k \left[\varphi\left(\frac{x_1}{2^k},\dots,\frac{x_{2n}}{2^k}\right) + \varphi\left(-\frac{x_1}{2^k},\dots,-\frac{x_{2n}}{2^k}\right)\right] \\ &= 0,\end{aligned} \tag{2.17}$$

for all $x_1,\dots,x_{2n} \in X$. So $DA(x_1,\dots,x_{2n}) = 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.15), we get (2.10). So there exists a Cauchy additive mapping $A : X \to Y$ satisfying (2.1) and (2.10).

Now, let $L : X \to Y$ be another Cauchy additive mapping satisfying (2.1) and (2.10). Then we have

$$\begin{aligned}\|A(x) - L(x)\| &= 2^q \left\|A\left(\frac{x}{2^q}\right) - L\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \left(\left\|A\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right) + f\left(\frac{-x}{2^q}\right)\right\| + \left\|L\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right) + f\left(\frac{-x}{2^q}\right)\right\|\right) \\ &\le \frac{2\cdot 2^q}{n}\,\tilde{\varphi}\left(\underbrace{\frac{x}{2^q},\dots,\frac{x}{2^q}}_{n \text{ times}},\underbrace{0,\dots,0}_{n \text{ times}}\right) + \frac{2\cdot 2^q}{n}\,\tilde{\varphi}\left(\underbrace{\frac{-x}{2^q},\dots,\frac{-x}{2^q}}_{n \text{ times}},\underbrace{0,\dots,0}_{n \text{ times}}\right),\end{aligned} \tag{2.18}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = L(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 2.3. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|Df(x_1,\dots,x_{2n})\right\| \le \theta \sum_{j=1}^{2n} \|x_j\|^p, \tag{2.19}$$

for all $x_1,\dots,x_{2n} \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (2.1) *such that*

$$\left\|f(x) - f(-x) - A(x)\right\| \le \frac{2^{p+1}\theta}{2^p - 2}\|x\|^p, \tag{2.20}$$

for all $x \in X$.

Proof. Define $\varphi(x_1,\ldots,x_{2n}) = \theta\sum_{j=1}^{2n}\|x_j\|^p$, and apply Theorem 2.2 to get the desired result. $\square$

Corollary 2.4. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying (2.8) and (2.9). Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (2.1) such that*

$$\|2f(x) - A(x)\| \le \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{x,\ldots,x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{-x,\ldots,-x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg), \tag{2.21}$$

or (alternative approximation)

$$\|f(x) - A(x)\| \le \frac{1}{2n}\widetilde{\varphi}\Bigg(\underbrace{x,\ldots,x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg) + \frac{1}{2n}\widetilde{\varphi}\Bigg(\underbrace{-x,\ldots,-x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg), \tag{2.22}$$

for all $x \in X$, where $\widetilde{\varphi}$ is defined in (2.8).

Theorem 2.5. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying (2.9) such that*

$$\widetilde{\varphi}(x_1,\ldots,x_{2n}) := \sum_{j=1}^{\infty}2^{-j}\varphi\Big(2^j x_1,\ldots,2^j x_{2n}\Big) < \infty, \tag{2.23}$$

for all $x_1,\ldots,x_{2n} \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (2.1) such that

$$\|f(x) - f(-x) - A(x)\| \le \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{x,\ldots,x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{-x,\ldots,-x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg), \tag{2.24}$$

for all $x \in X$.

Proof. It follows from (2.13) that

$$\left\|g(x) - \frac{1}{2}g(2x)\right\| \le \frac{1}{2n}\varphi\Bigg(\underbrace{2x,\ldots,2x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg) + \frac{1}{2n}\varphi\Bigg(\underbrace{-2x,\ldots,-2x}_{n\text{ times}},\underbrace{0,\ldots,0}_{n\text{ times}}\Bigg), \tag{2.25}$$

for all $x \in X$. So

$$\left\| \frac{1}{2^l} g\left(2^l x\right) - \frac{1}{2^m} g(2^m x) \right\| \le \sum_{j=l+1}^{m} \frac{1}{2^j n} \varphi\left(\underbrace{2^j x, \ldots, 2^j x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}} \right) + \sum_{j=l+1}^{m} \frac{1}{2^j n} \varphi\left(\underbrace{-2^j x, \ldots, -2^j x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}} \right), \tag{2.26}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.23) and (2.26) that the sequence $\{(1/2^k) g(2^k x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(1/2^k) g(2^k x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k \to \infty} \frac{1}{2^k} g\left(2^k x\right), \tag{2.27}$$

for all $x \in X$.

By (2.9) and (2.23),

$$\begin{aligned} \| DA(x_1, \ldots, x_{2n}) \| &= \lim_{k \to \infty} \frac{1}{2^k} \left\| Dg\left(2^k x_1, \ldots, 2^k x_{2n}\right) \right\| \\ &\le \lim_{k \to \infty} \frac{1}{2^k} \left(\varphi\left(2^k x_1, \ldots, 2^k x_{2n}\right) + \varphi\left(-2^k x_1, \ldots, -2^k x_{2n}\right) \right) \\ &= 0, \end{aligned} \tag{2.28}$$

for all $x_1, \ldots, x_{2n} \in X$. So $DA(x_1, \ldots, x_{2n}) = 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.26), we get (2.24). So there exists a Cauchy additive mapping $A : X \to Y$ satisfying (2.1) and (2.24).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.6. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.19). *Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying* (2.1) *such that*

$$\| f(x) - f(-x) - A(x) \| \le \frac{2^{p+1}\theta}{2 - 2^p} \|x\|^p, \tag{2.29}$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \ldots, x_{2n}) = \theta \sum_{j=1}^{2n} \|x_j\|^p$, and apply Theorem 2.5 to get the desired result. □

Corollary 2.7. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying* (2.9) *and* (2.23). *Then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying* (2.1) *such that*

$$\|2f(x) - A(x)\| \le \frac{1}{n}\tilde{\varphi}\Bigg(\underbrace{x,\dots,x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\tilde{\varphi}\Bigg(\underbrace{-x,\dots,-x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg), \tag{2.30}$$

or (alternative approximation),

$$\|f(x) - A(x)\| \le \frac{1}{2n}\tilde{\varphi}\Bigg(\underbrace{x,\dots,x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{2n}\tilde{\varphi}\Bigg(\underbrace{-x,\dots,-x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg), \tag{2.31}$$

for all $x \in X$, where $\tilde{\varphi}$ is defined in (2.23).

The following was proved in [28].

Remark 2.8 ([28]). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying (2.9) such that

$$\Phi(x_1,\dots,x_{2n}) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{2n}}{2^j}\right) < \infty, \tag{2.32}$$

for all $x_1,\dots,x_{2n} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.1) such that

$$\|f(x) + f(-x) - Q(x)\| \le \frac{1}{n}\Phi\Bigg(\underbrace{x,\dots,x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\Phi\Bigg(\underbrace{-x,\dots,-x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg), \tag{2.33}$$

for all $x \in X$.

Note that

$$\sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{2n}}{2^j}\right) \le \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{2n}}{2^j}\right). \tag{2.34}$$

Combining Theorem 2.2 and Remark 2.8, we obtain the following result.

Theorem 2.9. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying* (2.9) *and* (2.32)*. Then there exist a unique Cauchy additive mapping $A : X \to Y$ satisfying* (2.1) *and a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\|2f(x) - A(x) - Q(x)\| \le \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{x,\dots,x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{-x,\dots,-x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\Phi\Bigg(\underbrace{x,\dots,x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\Phi\Bigg(\underbrace{-x,\dots,-x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg), \tag{2.35}$$

for all $x \in X$, where $\widetilde{\varphi}$ and Φ are defined in (2.8) *and* (2.32)*, respectively.*

Corollary 2.10. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.19)*. Then there exist a unique Cauchy additive mapping $A : X \to Y$ satisfying* (2.1) *and a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\|2f(x) - A(x) - Q(x)\| \le \left(\frac{2^{p+1}}{2^p - 2} + \frac{2^{p+1}}{2^p - 4}\right)\theta\|x\|^p, \tag{2.36}$$

for all $x \in X$.

Proof. Define $\varphi(x_1,\dots,x_{2n}) = \theta\sum_{j=1}^{2n}\|x_j\|^p$, and apply Theorem 2.9 to get the desired result. □

The following was proved in [28].

Remark 2.11 (see [28]). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying (2.9) such that

$$\Phi(x_1,\dots,x_{2n}) := \sum_{j=1}^{\infty} 4^{-j}\varphi\left(2^j x_1,\dots,2^j x_{2n}\right) < \infty, \tag{2.37}$$

for all $x_1,\dots,x_{2n} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.1) such that

$$\|f(x) + f(-x) - Q(x)\| \le \frac{1}{n}\Phi\Bigg(\underbrace{x,\dots,x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg) + \frac{1}{n}\Phi\Bigg(\underbrace{-x,\dots,-x}_{n\text{ times}},\underbrace{0,\dots,0}_{n\text{ times}}\Bigg), \tag{2.38}$$

for all $x \in X$.

Note that

$$\sum_{j=1}^{\infty} 4^{-j} \varphi\left(2^j x_1, \ldots, 2^j x_{2n}\right) \leq \sum_{j=1}^{\infty} 2^{-j} \varphi\left(2^j x_1, \ldots, 2^j x_{2n}\right). \tag{2.39}$$

Combining Theorem 2.5 and Remark 2.11, we obtain the following result.

Theorem 2.12. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying* (2.9) *and* (2.23)*. Then there exist a unique Cauchy additive mapping $A : X \to Y$ satisfying* (2.1) *and a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\begin{aligned} \|2f(x) - A(x) - Q(x)\| \leq{}& \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{x, \ldots, x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}}\Bigg) + \frac{1}{n}\widetilde{\varphi}\Bigg(\underbrace{-x, \ldots, -x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}}\Bigg) \\ &+ \frac{1}{n}\Phi\Bigg(\underbrace{x, \ldots, x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}}\Bigg) + \frac{1}{n}\Phi\Bigg(\underbrace{-x, \ldots, -x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}}\Bigg), \end{aligned} \tag{2.40}$$

for all $x \in X$, where $\widetilde{\varphi}$ and Φ are defined in (2.23) *and* (2.37)*, respectively.*

Corollary 2.13. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.19)*. Then there exist a unique Cauchy additive mapping $A : X \to Y$ satisfying* (2.1) *and a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\|2f(x) - A(x) - Q(x)\| \leq \left(\frac{2^{p+1}}{2 - 2^p} + \frac{2^{p+1}}{4 - 2^p}\right)\theta\|x\|^p, \tag{2.41}$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \ldots, x_{2n}) = \theta\sum_{j=1}^{2n}\|x_j\|^p$, and apply Theorem 2.12 to get the desired result. □

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