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Research Article

Fuzzy Stability of Jensen-Type Quadratic Functional Equations

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We prove the generalized Hyers-Ulam stability of the following quadratic functional equations $2f((x+y)/2)+2f((x-y)/2)=f(x)+f(y)$ and $f(ax+ay)+(ax-ay)=2a^2f(x)+2a^2f(y)$ in fuzzy Banach spaces for a nonzero real number a with $a\neq\pm1/2$.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The work of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

J. M. Rassias [6] proved a similar stability theorem in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\cdot\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q\neq1$ (see also [7, 8] for a number of other new results). The papers of J. M. Rassias [6–8] introduced the Ulam- Găvruţa-Rassias stability of functional equations. See also [9–11].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a*quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [12] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [13] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [14], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

J. M. Rassias [15] introduced and solved the stability problem of Ulam for the Euler-Lagrange-type quadratic functional equation

$$f(rx + sy) + f(sx - ry) = \left(r^2 + s^2\right)[f(x) + f(y)], \tag{1.2}$$

motivated from the following pertinent algebraic equation

$$|ax + by|^2 + |bx - ay|^2 = \left(a^2 + b^2\right)\left(|x|^2 + |y|^2\right). \tag{1.3}$$

The solution of the functional equation (1.2) is called a *Euler-Lagrange-type quadratic mapping*. J. M. Rassias [16, 17] introduced and investigated the relative functional equations. In addition, J. M. Rassias [18] generalized the algebraic equation (1.3) to the following equation

$$mn|ax + by|^2 + |nbx - may|^2 = \left(ma^2 + nb^2\right)\left(n|x|^2 + m|y|^2\right), \tag{1.4}$$

and introduced and investigated the general pertinent Euler-Lagrange quadratic mappings. Analogous quadratic mappings were introduced and investigated in [19, 20].

These Euler-Lagrange mappings are named *Euler-Lagrange-Rassias mappings* and the corresponding Euler-Lagrange equations are called *Euler-Lagrange-Rassias equations*. Before 1992, these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler-Lagrange partial differential equations are known in calculus of variations. Therefore, we think that J. M. Rassias' introduction of Euler-Lagrange mappings and equations in functional equations and inequalities provides an interesting cornerstone in analysis. Already some mathematicians have employed these Euler-Lagrange mappings.

Recently, Jun and Kim [21] solved the stability problem of Ulam for another Euler-Lagrange-Rassias-type quadratic functional equation. Jun and Kim [22] introduced and investigated the following quadratic functional equation of Euler-Lagrange-Rassias type:

$$\sum_{i=1}^{n} r_i Q\left(\sum_{j=1}^{n} r_j (x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) Q\left(\sum_{i=1}^{n} r_i x_i\right) = \left(\sum_{i=1}^{n} r_i\right)^2 \sum_{i=1}^{n} r_i Q(x_i), \tag{1.5}$$

whose solution is said to be a generalized quadratic mapping of Euler-Lagrange-Rassias type.

During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [9, 23–26]).

Katsaras [27] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [28–30]. In particular, Bag and Samanta [31], following Cheng and Mordeson [32], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michálek type [33]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [34].

We use the definition of fuzzy normed spaces given in [31] and [35–38] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the quadratic functional equations

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y), \tag{1.6}$$

$$f(ax+ay)+f(ax-ay)=2a^2f(x)+2a^2f(y) \tag{1.7}$$

in the fuzzy normed vector space setting.

Definition 1.1 (see [31, 35–38]). Let X be a real vector space. A function $N : X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on X if for all $x, y\in X$ and all $s,t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\le 0$;

(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;

(N_3) $N(cx,t)=N(x,t/|c|)$ if $c\neq 0$;

(N_4) $N(x+y,s+t)\ge\min\{N(x,s),N(y,t)\}$;

(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;

(N_6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [35–38].

Definition 1.2 (see [31, 35–38]). Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be *convergent* or *converge* if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty}x_n=x$.

Definition 1.3 (see [31, 35–38]). Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon>0$ and each $t>0$ there exists an $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X\to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0\in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X\to Y$ is continuous at each $x\in X$, then $f : X\to Y$ is said to be *continuous* on X (see [34]).

In this paper, we prove the generalized Hyers-Ulam stability of the quadratic functional equations (1.6) and (1.7) in fuzzy Banach spaces.

Throughout this paper, assume that X is a vector space and that (Y,N) is a fuzzy Banach space. Let a be a nonzero real number with $a\neq(\pm 1/2)$.

2. Fuzzy Stability of Quadratic Functional Equations

We prove the fuzzy stability of the quadratic functional equation (1.6).

Theorem 2.1. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$. Suppose that φ is a mapping from X to a fuzzy normed space (Z, N') such that*

$$N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t+s\right) \geq \min\left\{N'(\varphi(x), t), N'(\varphi(y), s)\right\} \tag{2.1}$$

for all $x, y \in X \setminus \{0\}$ and all positive real numbers t, s. If $\varphi(3x) = \alpha\varphi(x)$ for some positive real number α with $\alpha < 9$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N$-$\lim_{n\to\infty} f(3^n x)/9^n$ and

$$N(Q(x) - f(x), t) \geq M\left(x, \frac{(9-\alpha)t}{18}\right), \tag{2.2}$$

where

$$M(x,t) := \min\left\{N'\left(\varphi(x), \frac{3}{2}t\right), N'\left(\varphi(2x), \frac{3}{2}t\right), N'\left(\varphi(3x), \frac{3}{2}t\right), N'\left(\varphi(0), \frac{3}{2}t\right)\right\}. \tag{2.3}$$

Proof. Putting $y = 3x$ and $s = t$ in (2.1), we get

$$N(2f(2x) + 2f(-x) - f(x) - f(3x), 2t) \geq \min\left\{N'(\varphi(x), t), N'(\varphi(3x), t)\right\} \tag{2.4}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2x$, y by 0, and s by t in (2.1), we obtain

$$N(4f(x) - f(2x), 2t) \geq \min\left\{N'(\varphi(2x), t), N'(\varphi(0), t)\right\}. \tag{2.5}$$

Thus

$$N(9f(x) - f(3x), 6t) \geq \min\left\{N'(\varphi(x), t), N'(\varphi(2x), t), N'(\varphi(3x), t), N'(\varphi(0), t)\right\}, \tag{2.6}$$

and so

$$N\left(f(x) - \frac{f(3x)}{9}, t\right) \geq \min\left\{N'\left(\varphi(x), \frac{3}{2}t\right), N'\left(\varphi(2x), \frac{3}{2}t\right) N'\left(\varphi(3x), \frac{3}{2}t\right), N'\left(\varphi(0), \frac{3}{2}t\right)\right\}. \tag{2.7}$$

Then by the assumption,

$$M(3x, t) = M\left(x, \frac{t}{\alpha}\right). \tag{2.8}$$

Replacing x by $3^n x$ in (2.7) and applying (2.8), we get

$$N\left(\frac{f(3^n x)}{9^n} - \frac{f(3^{n+1}x)}{9^{n+1}}, \frac{\alpha^n t}{9^n}\right) = N\left(f(3^n x) - \frac{f(3^{n+1}x)}{9}, \alpha^n t\right) \geq M(3^n x, \alpha^n t) = M(x,t). \tag{2.9}$$

Thus for each $n > m$ we have

$$\begin{aligned} N\left(\frac{f(3^m x)}{9^m} - \frac{f(3^n x)}{9^n}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{9^k}\right) &= N\left(\sum_{k=m}^{n-1}\left(\frac{f(3^k x)}{9^k} - \frac{f(3^{k+1}x)}{9^{k+1}}\right), \sum_{k=m}^{n-1}\frac{\alpha^k t}{9^k}\right) \\ &\geq \min\left\{\bigcup_{k=m}^{n-1}\left\{N\left(\frac{f(3^k x)}{9^k} - \frac{f(3^{k+1}x)}{9^{k+1}}, \frac{\alpha^k t}{9^k}\right)\right\}\right\} \\ &\geq M(x,t). \end{aligned} \tag{2.10}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M(x,t) = 1$, there is some $t_0 > 0$ such that $M(x,t_0) > 1-\varepsilon$. Since $\sum_{k=0}^{\infty}\alpha^k t_0/9^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1}\alpha^k t_0/9^k < \delta$ for $n > m \geq n_0$. It follows that

$$N\left(\frac{f(3^m x)}{9^m} - \frac{f(3^n x)}{9^n}, \delta\right) \geq N\left(\frac{f(3^m x)}{9^m} - \frac{f(3^n x)}{9^n}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{9^k}\right) \geq M(x,t_0) \geq 1-\varepsilon \tag{2.11}$$

for all $t \geq t_0$. This shows that the sequence $\{f(3^n x)/9^n\}$ is Cauchy in (Y,N). Since (Y,N) is complete, $\{f(3^n x)/9^n\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N - \lim_{t\to\infty} f(3^n x)/9^n$. Moreover, if we put $m = 0$ in (2.10), then we observe that

$$N\left(\frac{f(3^n x)}{9^n} - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{9^k}\right) \geq M(x,t). \tag{2.12}$$

Thus

$$N\left(\frac{f(3^n x)}{9^n} - f(x), t\right) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1}(\alpha/9)^k}\right). \tag{2.13}$$

Next we show that Q is quadratic. Let $x, y \in X$. Then we have

$$\begin{aligned}
&N\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y),t\right)\\
&\quad\geq \min\left\{N\left(2Q\left(\frac{x+y}{2}\right)-\frac{2f(3^n(x+y)/2)}{9^n},\frac{t}{5}\right),\right.\\
&\qquad N\left(2Q\left(\frac{x-y}{2}\right)-\frac{2f(3^n(x-y)/2)}{9^n},\frac{t}{5}\right),\\
&\qquad N\left(\frac{f(3^n x)}{9^n}-Q(x),\frac{t}{5}\right),N\left(\frac{f(3^n y)}{9^n}-Q(y),\frac{t}{5}\right),\\
&\qquad \left. N\left(\frac{2f(3^n(x+y)/2)}{9^n}+\frac{2f(3^n(x-y)/2)}{9^n}-\frac{f(3^n x)}{9^n}-\frac{f(3^n y)}{9^n},\frac{t}{5}\right)\right\}.
\end{aligned} \tag{2.14}$$

The first four terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$ and the fifth term, by (2.1), is greater than or equal to

$$\min\left\{N'\left(\varphi(3^n x),\frac{9^n t}{10}\right),N'\left(\varphi(3^n y),\frac{9^n t}{10}\right)\right\}=\min\left\{N'\left(\varphi(x),\left(\frac{9}{\alpha}\right)^n\frac{t}{10}\right),\left(\varphi(y),\left(\frac{9}{\alpha}\right)^n\frac{t}{10}\right)\right\}, \tag{2.15}$$

which tends to 1 as $n \to \infty$. Hence

$$N\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y),t\right)=1 \tag{2.16}$$

for all $x, y \in X$ and all $t > 0$. This means that Q satisfies the Jensen quadratic functional equation and so it is quadratic.

Next, we approximate the difference between f and Q in a fuzzy sense. For every $x \in X$ and $t > 0$, by (2.13), for large enough n, we have

$$\begin{aligned}
N(Q(x)-f(x),t)&\geq \min\left\{N\left(Q(x)-\frac{f(3^n y)}{9^n},\frac{t}{2}\right),N\left(\frac{f(3^n y)}{9^n}-f(x),\frac{t}{2}\right)\right\}\\
&\geq M\left(x,\frac{t}{2\sum_{k=0}^{\infty}(\alpha/9)^k}\right)\\
&= M\left(x,\frac{(9-\alpha)t}{18}\right).
\end{aligned} \tag{2.17}$$

The uniqueness assertion can be proved by a standard fashion; cf. [36]: Let Q' be another quadratic mapping from X into Y, which satisfies the required inequality. Then for each $x \in X$ and $t > 0$,

$$\begin{aligned} N\left(Q(x) - Q'(x), t\right) &\geq \min\left\{N\left(Q(x) - f(x), \frac{t}{2}\right), N\left(Q'(x) - f(x), \frac{t}{2}\right)\right\} \\ &\geq M\left(x, \frac{(9-\alpha)t}{36}\right). \end{aligned} \tag{2.18}$$

Since Q and Q' are quadratic,

$$\begin{aligned} N\left(Q(x) - Q'(x), t\right) &= N\left(Q(3^n x) - Q'(3^n x), 9^n t\right) \\ &\geq M\left(x, \frac{(9/\alpha)^n (9-\alpha)t}{36}\right). \end{aligned} \tag{2.19}$$

for all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Since $0 < \alpha < 9$, $\lim_{n\to\infty}(9/\alpha)^n = \infty$. Hence the right-hand side of the above inequality tends to 1 as $n \to \infty$. It follows that $Q(x) = Q'(x)$ for all $x \in X$. □

Theorem 2.2. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$. Suppose that φ is a mapping from X to a fuzzy normed space (Z, N') satisfying (2.1). If $\varphi(3x) = \alpha\varphi(x)$ for some real number α with $\alpha > 9$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N\text{-}\lim_{n\to\infty} 9^n f(x/3^n)$ and*

$$N(Q(x) - f(x), t) \geq M\left(x, \frac{(\alpha - 9)t}{2\alpha}\right), \tag{2.20}$$

where

$$M(x,t) := \min\left\{N'\left(\varphi\left(\frac{x}{3}\right), \frac{t}{6}\right), N'\left(\varphi\left(\frac{2x}{3}\right), \frac{t}{6}\right), N'\left(\varphi(x), \frac{t}{6}\right), N'\left(\varphi(0), \frac{t}{6}\right)\right\}. \tag{2.21}$$

Proof. It follows from (2.7) that

$$N\left(f(x) - 9f\left(\frac{x}{3}\right), t\right) \geq \min\left\{N'\left(\varphi\left(\frac{x}{3}\right), \frac{t}{6}\right), N'\left(\varphi\left(\frac{2x}{3}\right), \frac{t}{6}\right), N'\left(\varphi(x), \frac{t}{6}\right), N'\left(\varphi(0), \frac{t}{6}\right)\right\}. \tag{2.22}$$

Then by the assumption,

$$M\left(\frac{x}{3}, t\right) = M(x, \alpha t). \tag{2.23}$$

Replacing x by $x/3^n$ in (2.22) and applying (2.23), we get

$$\begin{aligned} N\left(9^n f\left(\frac{x}{3^n}\right) - 9^{n+1} f\left(\frac{x}{3^{n+1}}\right), \frac{9^n t}{\alpha^n}\right) &= N\left(f\left(\frac{x}{3^n}\right) - 9f\left(\frac{x}{3^{n+1}}\right), \frac{t}{\alpha^n}\right) \\ &\geq M\left(\frac{x}{3^n}, \frac{t}{\alpha^n}\right) \\ &= M(x,t). \end{aligned} \tag{2.24}$$

Thus for each $n > m$ we have

$$\begin{aligned} &N\left(9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), \sum_{k=m}^{n-1} \frac{9^k t}{\alpha^k}\right) \\ &\quad = N\left(\sum_{k=m}^{n-1}\left(9^k f\left(\frac{x}{3^k}\right) - 9^{k+1} f\left(\frac{x}{3^{k+1}}\right)\right), \sum_{k=m}^{n-1} \frac{9^k t}{\alpha^k}\right) \\ &\quad \geq \min\left\{\bigcup_{k=m}^{n-1}\left\{N\left(9^k f\left(\frac{x}{3^k}\right) - 9^{k+1} f\left(\frac{x}{3^{k+1}}\right), \frac{9^k t}{\alpha^k}\right)\right\}\right\} \\ &\quad \geq M(x,t). \end{aligned} \tag{2.25}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M(x,t) = 1$, there is some $t_0 > 0$ such that $M(x,t_0) > 1-\varepsilon$. Since $\sum_{k=0}^{\infty} 9^k t_0/\alpha^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} 9^k t_0/\alpha^k < \delta$ for $n > m \geq n_0$. It follows that

$$\begin{aligned} N\left(9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), \delta\right) &\geq N\left(9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), \sum_{k=m}^{n-1} \frac{9^k t}{\alpha^k}\right) \\ &\geq M(x,t_0) \\ &\geq 1-\varepsilon \end{aligned} \tag{2.26}$$

for all $t \geq t_0$. This shows that the sequence $\{9^n f(x/3^n)\}$ is Cauchy in (Y,N). Since (Y,N) is complete, $\{9^n f(x/3^n)\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{t\to\infty} 9^n f(x/3^n)$. Moreover, if we put $m = 0$ in (2.8), then we observe that

$$N\left(9^n f\left(\frac{x}{3^n}\right) - f(x), \sum_{k=0}^{n-1} \frac{9^k t}{\alpha^k}\right) \geq M(x,t). \tag{2.27}$$

Thus

$$N\left(9^n f\left(\frac{x}{3^n}\right) - f(x), t\right) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1} (9/\alpha)^k}\right). \tag{2.28}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to a fuzzy normed space (Z, N') satisfying (2.1). If $\varphi(2x) = \alpha\varphi(x)$ for some positive real number α with $\alpha < 4$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N$-$\lim_{n\to\infty} f(2^n x)/4^n$ and*

$$N(Q(x) - f(x), t) \geq M\left(x, \frac{(4-\alpha)t}{8}\right) \tag{2.29}$$

where $M(x,t) := \min\{N'(\varphi(2x), 2t), N'(\varphi(0), 2t)\}$.

Proof. Letting $y = 0$ and replacing x by $2x$ and s by t in (2.1), we obtain

$$N(4f(x) - f(2x), 2t) \geq \min\left\{N'(\varphi(2x), t), N'(\varphi(0), t)\right\}. \tag{2.30}$$

Thus

$$N\left(f(x) - \frac{f(2x)}{4}, t\right) \geq \min\left\{N'(\varphi(2x), 2t), N'(\varphi(0), 2t)\right\}. \tag{2.31}$$

Then by the assumption,

$$M(2x, t) = M\left(x, \frac{t}{\alpha}\right). \tag{2.32}$$

Replacing x by $2^n x$ in (2.31) and applying (2.32), we get

$$\begin{aligned} N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) &= N\left(f(2^n x) - \frac{f(4^{n+1}x)}{4}, \alpha^n t\right) \\ &\geq M(2^n x, \alpha^n t) \\ &= M(x, t). \end{aligned} \tag{2.33}$$

Thus for each $n > m$ we have

$$\begin{aligned} &N\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) \\ &\quad = N\left(\sum_{k=m}^{n-1}\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1}x)}{4^{k+1}}\right), \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) \\ &\quad \geq \min\left\{\bigcup_{k=m}^{n-1}\left\{N\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1}x)}{4^{k+1}}, \frac{\alpha^k t}{4^k}\right)\right\}\right\} \\ &\quad \geq M(x, t). \end{aligned} \tag{2.34}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M(x,t) = 1$, there is some $t_0 > 0$ such that $M(x,t_0) > 1-\varepsilon$. Since $\sum_{k=0}^{\infty} \alpha^k t_0/4^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} \alpha^k t_0/4^k < \delta$ for $n > m \geq n_0$. It follows that

$$\begin{aligned} N\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \delta\right) &\geq N\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) \\ &\geq M(x,t_0) \\ &\geq 1-\varepsilon \end{aligned} \tag{2.35}$$

for all $t \geq t_0$. This shows that the sequence $\{f(2^n x)/4^n\}$ is Cauchy in (Y,N). Since (Y,N) is complete, $\{f(2^n x)/4^n\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{t\to\infty} f(2^n x)/4^n$. Moreover, if we put $m = 0$ in (2.34), then we observe that

$$N\left(\frac{f(2^n x)}{4^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^k}\right) \geq M(x,t). \tag{2.36}$$

Thus

$$N\left(\frac{f(2^n x)}{4^n} - f(x), t\right) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/4)^k}\right). \tag{2.37}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to a fuzzy normed space (Z, N') satisfying (2.1). If $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $\alpha > 4$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N\text{-}\lim_{n\to\infty} 4^n f(x/2^n)$ and*

$$N(Q(x) - f(x), t) \geq M\left(x, \frac{(\alpha-4)t}{2\alpha}\right), \tag{2.38}$$

where $M(x,t) := \min\{N'(\varphi(x), t/2), N'(\varphi(0), t/2)\}$.

Proof. It follows from (2.31) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \min\left\{N'\left(\varphi(x), \frac{t}{2}\right), N'\left(\varphi(0), \frac{t}{2}\right)\right\}. \tag{2.39}$$

Then by the assumption,

$$M\left(\frac{x}{2}, t\right) = M(x, \alpha t). \tag{2.40}$$

Replacing x by $x/2^n$ in (2.39) and applying (2.40), we get

$$\begin{aligned} N\left(4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right), \frac{4^n t}{\alpha^n}\right) &= N\left(f\left(\frac{x}{2^n}\right) - 4f\left(\frac{x}{2^{n+1}}\right), \frac{t}{\alpha^n}\right) \\ &\geq M\left(\frac{x}{2^n}, \frac{t}{\alpha^n}\right) \\ &= M(x,t). \end{aligned} \tag{2.41}$$

Thus for each $n > m$ we have

$$\begin{aligned} &N\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), \sum_{k=m}^{n-1} \frac{4^k t}{\alpha^k}\right) \\ &\quad = N\left(\sum_{k=m}^{n-1}\left(4^k f\left(\frac{x}{2^k}\right) - 4^{k+1} f\left(\frac{x}{2^{k+1}}\right)\right), \sum_{k=m}^{n-1} \frac{4^k t}{\alpha^k}\right) \\ &\quad \geq \min\left\{\bigcup_{k=m}^{n-1}\left\{N\left(4^k f\left(\frac{x}{2^k}\right) - 4^{k+1} f\left(\frac{x}{2^{k+1}}\right), \frac{4^k t}{\alpha^k}\right)\right\}\right\} \\ &\quad \geq M(x,t). \end{aligned} \tag{2.42}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M(x,t) = 1$, there is some $t_0 > 0$ such that $M(x,t_0) > 1-\varepsilon$. Since $\sum_{k=0}^{\infty} 4^k t_0/\alpha^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} 4^k t_0/\alpha^k < \delta$ for $n > m \geq n_0$. It follows that

$$\begin{aligned} N\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), \delta\right) &\geq N\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), \sum_{k=m}^{n-1} \frac{4^k t}{\alpha^k}\right) \\ &\geq M(x,t_0) \\ &\geq 1-\varepsilon \end{aligned} \tag{2.43}$$

for all $t \geq t_0$. This shows that the sequence $\{4^n f(x/2^n)\}$ is Cauchy in (Y,N). Since (Y,N) is complete, $\{4^n f(x/2^n)\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{t\to\infty} 4^n f(x/2^n)$. Moreover, if we put $m = 0$ in (2.42), then we observe that

$$N\left(4^n f\left(\frac{x}{2^n}\right) - f(x), \sum_{k=0}^{n-1} \frac{4^k t}{\alpha^k}\right) \geq M(x,t). \tag{2.44}$$

Thus

$$N\left(4^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1} (4/\alpha)^k}\right). \tag{2.45}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Now we prove the fuzzy stability of the quadratic functional equation (1.7) for the case $a \neq (\pm 1/2)$.

Theorem 2.5. *Let $|2a| > 1$ and $f : X \to Y$ a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to a fuzzy normed space (Z, N') such that*

$$N\left(f(ax+ay) + f(ax-ay) - 2a^2 f(x) - 2a^2 f(y), t+s\right) \geq \min\left\{N'(\varphi(x), t), N'(\varphi(y), s)\right\} \tag{2.46}$$

for all $x, y \in X \setminus \{0\}$ and all positive real numbers t, s. If $\varphi(2ax) = \alpha\varphi(x)$ for some positive real number α with $0 < \alpha < 4a^2$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N\text{-}\lim_{n\to\infty} f((2a)^n x)/(2a)^{2n}$ and

$$N(Q(x) - f(x), t) \geq N'\left(\varphi(x), \frac{(4a^2-\alpha)t}{4}\right) \tag{2.47}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ and $s = t$ in (2.46), we get

$$N\left(f(2ax) - 4a^2 f(x), 2t\right) \geq N'(\varphi(x), t) \tag{2.48}$$

for all $x \in X$ and all $t > 0$. Thus

$$N\left(f(x) - \frac{f(2ax)}{4a^2}, \frac{t}{2a^2}\right) \geq N'(\varphi(x), t) \tag{2.49}$$

and so

$$N\left(f(x) - \frac{f(2ax)}{4a^2}, t\right) \geq N'\left(\varphi(x), 2a^2 t\right). \tag{2.50}$$

Replacing x by $(2a)^n x$ in (2.50), we get

$$\begin{aligned} N\left(\frac{f((2a)^n x)}{(2a)^{2n}} - \frac{f\left((2a)^{n+1}x\right)}{(2a)^{2n+2}}, \frac{\alpha^n t}{(2a)^{2n}}\right) &= N\left(f((2a)^n x) - \frac{f\left((2a)^{n+1}x\right)}{4a^2}, \alpha^n t\right) \\ &\geq N'\left(\varphi(x), 2a^2 t\right). \end{aligned} \tag{2.51}$$

Thus for each $n > m$ we have

$$\begin{aligned} &N\left(\frac{f((2a)^m x)}{(2a)^{2m}} - \frac{f((2a)^n x)}{(2a)^{2n}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}}\right) \\ &\quad = N\left(\sum_{k=m}^{n-1}\left(\frac{f\left((2a)^k x\right)}{(2a)^{2k}} - \frac{f\left((2a)^{k+1} x\right)}{(2a)^{2k+2}}\right), \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}}\right) \\ &\quad \geq \min\left\{\bigcup_{k=m}^{n-1}\left\{N\left(\frac{f\left((2a)^k x\right)}{(2a)^{2k}} - \frac{f\left((2a)^{k+1} x\right)}{(2a)^{2k+2}}, \frac{\alpha^k t}{(2a)^{2k}}\right)\right\}\right\} \\ &\quad \geq N'\left(\varphi(x), 2a^2 t\right). \end{aligned} \tag{2.52}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} N'(\varphi(x), 2a^2 t) = 1$, there is some $t_0 > 0$ such that $N'(\varphi(x), 2a^2 t_0) > 1 - \varepsilon$. Since $\sum_{k=0}^{\infty} \alpha^k t_0/(2a)^{2k} < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} \alpha^k t_0/(2a)^{2k} < \delta$ for $n > m \geq n_0$. It follows that

$$\begin{aligned} N\left(\frac{f((2a)^m x)}{(2a)^{2m}} - \frac{f((2a)^n x)}{(2a)^{2n}}, \delta\right) &\geq N\left(\frac{f((2a)^m x)}{(2a)^{2m}} - \frac{f((2a)^n x)}{(2a)^{2n}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}}\right) \\ &\geq N'\left(\varphi(x), 2a^2 t_0\right) \\ &\geq 1 - \varepsilon \end{aligned} \tag{2.53}$$

for all $t \geq t_0$. This shows that the sequence $\{f((2a)^n x)/(2a)^{2n}\}$ is Cauchy in (Y, N). Since (Y, N) is complete, $\{f((2a)^n x)/(2a)^{2n}\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{t\to\infty} f((2a)^n x)/(2a)^{2n}$. Moreover, if we put $m = 0$ in (2.52), then we observe that

$$N\left(\frac{f((2a)^n x)}{(2a)^{2n}} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{(2a)^{2k}}\right) \geq N'\left(\varphi(x), 2a^2 t\right). \tag{2.54}$$

Thus

$$N\left(\frac{f((2a)^n x)}{(2a)^{2n}} - f(x), t\right) \geq N'\left(\varphi(x), \frac{2a^2 t}{\sum_{k=0}^{n-1} \left(\alpha/(2a)^2\right)^k}\right). \tag{2.55}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.6. *Let $|2a| < 1$ and $f : X \to Y$ a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to a fuzzy normed space (Z, N') satisfying* (2.46). *If $\varphi(2ax) = \alpha\varphi(x)$ for some real number*

α with $\alpha > 4a^2$, then there is a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = N\text{-}\lim_{n\to\infty}(2a)^{2n} f(x/(2a)^n)$ and

$$N(Q(x) - f(x), t) \geq M\left(x, \frac{(\alpha - 4a^2)t}{4}\right) \tag{2.56}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (2.50) that

$$N\left(f(x) - (2a)^2 f\left(\frac{x}{2a}\right), 2t\right) \geq N'\left(\varphi\left(\frac{x}{2a}\right), t\right) \tag{2.57}$$

for all $x \in X$ and all $t > 0$. Thus

$$N\left(f(x) - 4a^2 f\left(\frac{x}{2a}\right), t\right) \geq N'\left(\varphi\left(\frac{x}{2a}\right), \frac{t}{2}\right) = N'\left(\varphi(x), \frac{\alpha}{2}t\right). \tag{2.58}$$

Replacing x by $x/(2a)^n$ in (2.58), we get

$$\begin{aligned} &N\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - (2a)^{2n+2} f\left(\frac{x}{(2a)^{n+1}}\right), \frac{(2a)^{2n}t}{\alpha^n}\right) \\ &\quad = N\left(f\left(\frac{x}{(2a)^n}\right) - 4a^2 f\left(\frac{x}{(2a)^{n+1}}\right), \alpha^n t\right) \\ &\quad \geq N'\left(\varphi(x), \frac{\alpha}{2}t\right). \end{aligned} \tag{2.59}$$

Thus for each $n > m$ we have

$$\begin{aligned} &N\left((2a)^{2m} f\left(\frac{x}{(2a)^m}\right) - (2a)^{2n} f\left(\frac{x}{(2a)^n}\right), \sum_{k=m}^{n-1}\frac{(2a)^{2k}t}{\alpha^k}\right) \\ &\quad = N\left(\sum_{k=m}^{n-1}\left((2a)^{2k} f\left(\frac{x}{(2a)^k}\right) - (2a)^{2k+2} f\left(\frac{x}{(2a)^{k+1}}\right)\right), \sum_{k=m}^{n-1}\frac{(2a)^{2k}t}{\alpha^k}\right) \\ &\quad \geq \min\left\{\bigcup_{k=m}^{n-1}\left\{N\left((2a)^{2k} f\left(\frac{x}{(2a)^k}\right) - (2a)^{2k+2} f\left(\frac{x}{(2a)^{k+1}}\right), \frac{(2a)^{2k}t}{\alpha^k}\right)\right\}\right\} \\ &\quad \geq N'\left(\varphi(x), \frac{\alpha}{2}t\right). \end{aligned} \tag{2.60}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} N'(\varphi(x), (\alpha/2)t) = 1$, there is some $t_0 > 0$ such that $N'(\varphi(x), (\alpha/2)t_0) > 1 - \varepsilon$. Since $\sum_{k=0}^{\infty} (2a)^{2k} t_0/\alpha^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} (2a)^{2k} t_0/\alpha^k < \delta$ for $n > m \geq n_0$. It follows that

$$\begin{aligned} &N\left((2a)^{2m} f\left(\frac{x}{(2a)^m}\right) - (2a)^{2n} f\left(\frac{x}{(2a)^n}\right), \delta\right) \\ &\quad \geq N\left((2a)^{2m} f\left(\frac{x}{(2a)^m}\right) - (2a)^{2n} f\left(\frac{x}{(2a)^n}\right), \sum_{k=m}^{n-1} \frac{(2a)^{2k} t}{\alpha^k}\right) \\ &\quad \geq N'\left(\varphi(x), \frac{\alpha}{2} t_0\right) \\ &\quad \geq 1 - \varepsilon \end{aligned} \tag{2.61}$$

for all $t \geq t_0$. This shows that the sequence $\{(2a)^{2n} f(x/(2a)^n)\}$ is Cauchy in (Y, N). Since (Y, N) is complete, $\{(2a)^{2n} f(x/(2a)^n)\}$ converges to some $Q(x) \in Y$. Thus we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{t\to\infty} (2a)^{2n} f(x/(2a)^n)$. Moreover, if we put $m = 0$ in (2.60), then we observe that

$$N\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - f(x), \sum_{k=0}^{n-1} \frac{(2a)^{2k} t}{\alpha^k}\right) \geq N'\left(\varphi(x), \frac{\alpha}{2} t\right). \tag{2.62}$$

Thus

$$N\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - f(x), t\right) \geq N'\left(\varphi(x), \frac{\alpha t}{2 \sum_{k=0}^{n-1} \left((2a)^2/\alpha\right)^k}\right). \tag{2.63}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

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References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[6] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.

[7] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[8] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.

[9] P. Găvruţa, "An answer to a question of John M. Rassias concerning the stability of Cauchy equation," in *Advances in Equations and Inequalities*, Hadronic Mathematics Series, pp. 67–71, Hadronic Press, Palm Harbor, Fla, USA, 1999.

[10] K. Ravi and M. Arunkumar, "On the Ulam-Găvruţa-Rassias stability of the orthogonally Euler-Lagrange type functional equation," *International Journal of Applied Mathematics & Statistics*, vol. 7, no. Fe07, pp. 143–156, 2007.

[11] M. A. SiBaha, B. Bouikhalene, and E. Elqorachi, "Ulam-Găvruţa-Rassias stability of a linear functional equation," *International Journal of Applied Mathematics & Statistics*, vol. 7, no. Fe07, pp. 157–166, 2007.

[12] F. Skof, "Proprietà locali e approssimazione di operatori," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, no. 1, pp. 113–129, 1983.

[13] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.

[14] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, no. 1, pp. 59–64, 1992.

[15] J. M. Rassias, "On the stability of the Euler-Lagrange functional equation," *Chinese Journal of Mathematics*, vol. 20, no. 2, pp. 185–190, 1992.

[16] J. M. Rassias, "On the stability of the non-linear Euler-Lagrange functional equation in real normed linear spaces," *Journal of Mathematical and Physical Sciences*, vol. 28, no. 5, pp. 231–235, 1994.

[17] J. M. Rassias, "On the stability of the general Euler-Lagrange functional equation," *Demonstratio Mathematica*, vol. 29, no. 4, pp. 755–766, 1996.

[18] J. M. Rassias, "Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 2, pp. 613–639, 1998.

[19] J. M. Rassias, "On the stability of the multi-dimensional Euler-Lagrange functional equation," *The Journal of the Indian Mathematical Society*, vol. 66, no. 1–4, pp. 1–9, 1999.

[20] M. J. Rassias and J. M. Rassias, "On the Ulam stability for Euler-Lagrange type quadratic functional equations," *The Australian Journal of Mathematical Analysis and Applications*, vol. 2, no. 1, article 11, pp. 1–10, 2005.

[21] K.-W. Jun and H.-M. Kim, "Ulam stability problem for quadratic mappings of Euler-Lagrange," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 7, pp. 1093–1104, 2005.

[22] K.-W. Jun and H.-M. Kim, "On the generalized A-quadratic mappings associated with the variance of a discrete-type distribution," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 6, pp. 975–987, 2005.

[23] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[24] G. H. Kim, "On the stability of the quadratic mapping in normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 25, no. 4, pp. 217–229, 2001.

[25] Th. M. Rassias, "On the stability of the quadratic functional equation and its applications," *Studia Universitatis Babeş-Bolyai. Mathematica*, vol. 43, no. 3, pp. 89–124, 1998.

[26] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[27] A. K. Katsaras, "Fuzzy topological vector spaces—II," *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 143–154, 1984.

[28] C. Felbin, "Finite-dimensional fuzzy normed linear space," *Fuzzy Sets and Systems*, vol. 48, no. 2, pp. 239–248, 1992.

[29] S. V. Krishna and K. K. M. Sarma, "Separation of fuzzy normed linear spaces," *Fuzzy Sets and Systems*, vol. 63, no. 2, pp. 207–217, 1994.

[30] J.-Z. Xiao and X.-H. Zhu, "Fuzzy normed space of operators and its completeness," *Fuzzy Sets and Systems*, vol. 133, no. 3, pp. 389–399, 2003.

[31] T. Bag and S. K. Samanta, "Finite dimensional fuzzy normed linear spaces," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–705, 2003.

[32] S. C. Cheng and J. N. Mordeson, "Fuzzy linear operators and fuzzy normed linear spaces," *Bulletin of the Calcutta Mathematical Society*, vol. 86, no. 5, pp. 429–436, 1994.

[33] I. Kramosil and J. Michálek, "Fuzzy metrics and statistical metric spaces," *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975.

[34] T. Bag and S. K. Samanta, "Fuzzy bounded linear operators," *Fuzzy Sets and Systems*, vol. 151, no. 3, pp. 513–547, 2005.

[35] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, "Fuzzy stability of the Jensen functional equation," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 730–738, 2008.

[36] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy versions of Hyers-Ulam-Rassias theorem," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 720–729, 2008.

[37] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy approximately cubic mappings," *Information Sciences*, vol. 178, no. 19, pp. 3791–3798, 2008.

[38] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy almost quadratic functions," *Results in Mathematics*, vol. 52, no. 1-2, pp. 161–177, 2008.