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POLYHEDRAL REALIZATION OF THE HIGHEST WEIGHT CRYSTALS FOR GENERALIZED KAC-MOODY ALGEBRAS

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ABSTRACT. In this paper, we give a polyhedral realization of the highest weight crystals $B(\lambda)$ associated with the highest weight modules $V(\lambda)$ for the generalized Kac-Moody algebras. As applications, we give explicit descriptions of crystals for the generalized Kac-Moody algebras of ranks 2, 3, and Monster algebras.

INTRODUCTION

The *quantum groups* introduced by Drinfel′d and Jimbo, independently, are deformations of the universal enveloping algebras of Kac-Moody algebras [5, 6]. More precisely, let $\mathfrak{g}$ be a Kac-Moody algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. Then, for each generic parameter q, we associate a Hopf algebra $U_q(\mathfrak{g})$, called the quantum group, whose structure tends to that of $U(\mathfrak{g})$ as q approaches 1. The important feature of quantum groups is that the representation theory of $U_q(\mathfrak{g})$ is the same as that of $U(\mathfrak{g})$. Therefore, to understand the structure of representations over general quantum groups $U_q(\mathfrak{g})$, it is enough to understand that of representations over $U_q(\mathfrak{g})$ for some special parameter q which is easy to treat. The *crystal bases*, introduced by Kashiwara [12, 13], can be viewed as bases at $q = 0$ for the integrable modules over quantum groups. They give a structure of colored oriented graphs, called the *crystal graphs*, reflecting the combinatorial structure of integrable modules. For instance, one of the major goals in representation theory is to find an explicit expression for the characters of representations, and this can be obtained by finding an explicit combinatorial description of crystal bases. So one of the most fundamental problems in crystal basis theory is to construct the crystal basis explicitly. In order to answer this, several kinds of combinatorial objects have been invented [10, 11, 15, 16].

In [14], Kashiwara introduced the embedding of crystals $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}^\infty$, where ι is some infinite sequence from the index set of simple roots. But, in general, it is not easy to find the image $\mathrm{Im}\Psi_\iota$. In [3], Cliff described the image of the Kashiwara embedding for the classical Lie algebras and some reduced expression. For more general types, Zelevinsky and Nakashima obtained the exact image of the

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embedding by a unified method, called the *polyhedral realization* [18]. Moreover, in [17], Nakashima introduced the crystal $R_\lambda = \{r_\lambda\}$, and he showed that the connected component containing $u_\infty \otimes r_\lambda$ is isomorphic to the highest weight crystal $B(\lambda)$, where u_∞ is the highest weight vector in $B(\infty)$. Applying this property to the Kashiwara embedding, he gave the embedding of crystals $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbf{Z}^\infty \otimes R_\lambda$ and described the explicit form of $\mathrm{Im}\Psi_\iota^\lambda$.

The *generalized Kac-Moody algebras* were introduced by Borcherds in his study of Monstrous Moonshine [1, 2]. The structure and representation theories of generalized Kac-Moody algebras are very similar to those of Kac-Moody algebras. The main difference is that the generalized Kac-Moody algebras may have *imaginary simple roots* with nonpositive norms whose multiplicity can be greater than one, and they may have infinitely many simple roots. In [9], Kang introduced the *quantum generalized Kac-Moody algebras* as an analogue of quantum groups, and in [7], Jeong, Kang, and Kashiwara developed the crystal basis theory for quantum generalized Kac-Moody algebras. In [8], Jeong, Kang, Kashiwara and the author introduced the notion of abstract crystals and an analogue Ψ_ι of the Kashiwara embedding of crystals for quantum generalized Kac-Moody algebras. Moreover, in [19], the author gave the generalized version of polyhedral realization of the crystal $B(\infty)$.

In this paper, we give a polyhedral realization of the crystal bases $B(\lambda)$ of the highest weight modules $V(\lambda)$ for quantum generalized Kac-Moody algebras. More precisely, we introduce the crystal R_λ for generalized Kac-Moody algebras and show that the connected component containing $u_\infty \otimes r_\lambda$ is also isomorphic to the highest weight crystal $B(\lambda)$ as in the Kac-Moody case. Moreover, on the basis of the embedding of crystals given in [8] and the crystal R_λ for generalized Kac-Moody algebras, we introduce the crystal structure $\mathbf{Z}^\infty_{\geq 0}[\lambda]$, and we give explicit images of Kashiwara embedding Ψ_ι^λ. As applications, we give explicit descriptions of the crystals over Kac-Moody algebras of ranks 2 and 3. Moreover, for the *Monster Lie algebra* which played an important role in proving the Moonshine conjecture given by Conway and Norton [4], we give the explicit description of $\mathrm{Im}\Psi_\iota^\lambda$. Finally, from this description, we give characters of the highest weight module $V(\lambda)$ over $U_q(\mathfrak{g})$.

1. Crystals for generalized Kac-Moody algebras

Let I be a countable index set. A real matrix $A = (a_{ij})_{i,j\in I}$ is called a *Borcherds-Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (ii) $a_{ij} \leq 0$ if $i \neq j$, (iii) $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$, (iv) $a_{ij} = 0$ if and only if $a_{ji} = 0$. Let $I^{re} = \{i \in I \mid a_{ii} = 2\}$ and $I^{im} = \{i \in I \mid a_{ii} \leq 0\}$. Moreover, we say that an index i in I^{re} (resp. I^{im}) is *real* (resp. *imaginary*).

In this paper, we assume that for all $i, j \in I$, $a_{ij} \in \mathbf{Z}$, $a_{ii} \in 2\mathbf{Z}$, and A is symmetrizable. That is, there is a diagonal matrix $D = \mathrm{diag}(s_i \in \mathbf{Z}_{>0} \mid i \in I)$ such that DA is symmetric. We set a Borcherds-Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ as follows:

$$\begin{aligned}
&A: \text{ a Borcherds-Cartan matrix,}\\
&P^\vee = \Big(\textstyle\bigoplus_{i\in I} \mathbf{Z}h_i\Big) \oplus \Big(\textstyle\bigoplus_{i\in I} \mathbf{Z}d_i\Big): \text{ a free abelian group,}\\
&P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbf{Z}\}: \text{ the weight lattice,}\\
&\Pi^\vee = \{h_i \mid i \in I\} \subset \mathfrak{h}: \text{ the set of simple coroots,}\\
&\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*: \text{ the set of simple roots.}
\end{aligned}$$

Here, the simple roots α_i $(i \in I)$ are defined by

$$\langle h_j, \alpha_i \rangle = a_{ji} \text{ and } \langle d_j, \alpha_i \rangle = \delta_{ji}.$$

We denote by $U_q(\mathfrak{g})$ the *quantum generalized Kac-Moody algebras* associated with the Borcherds-Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$. We also denote by $P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$ the set of *dominant integral weights*.

We recall the definition of abstract crystals introduced in [8] for quantum generalized Kac-Moody algebras.

Definition 1.1 ([8])**.** An *abstract crystal* for $U_q(\mathfrak{g})$ or a $U_q(\mathfrak{g})$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ $(i \in I)$, and $\varepsilon_i, \varphi_i : B \to \mathbf{Z} \cup \{-\infty\}$ $(i \in I)$ such that for all $b \in B$, we have

(i) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}\, b + \alpha_i$ if $i \in I$ and $\tilde{e}_i b \neq 0$,
(ii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}\, b - \alpha_i$ if $i \in I$ and $\tilde{f}_i b \neq 0$,
(iii) for any $i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}\, b \rangle$,
(iv) for any $i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
(v) for any $i \in I$ and $b \in B$ such that $\tilde{e}_i b \neq 0$, we have
  (a) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $i \in I^{re}$,
  (b) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + a_{ii}$ if $i \in I^{im}$,
(vi) for any $i \in I$ and $b \in B$ such that $\tilde{f}_i b \neq 0$, we have
  (a) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $i \in I^{re}$,
  (b) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^{im}$,
(vii) for any $i \in I$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition 1.2 ([8])**.** A *morphism of crystals* or a *crystal morphism* $\psi : B_1 \to B_2$ is a map $\psi : B_1 \to B_2$ such that

(i) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$ for all $b \in B_1$,
(ii) $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $b \in B_1$, $i \in I$,
(iii) if $b \in B_1$ and $i \in I$ satisfy $\tilde{f}_i b \in B_1$, then we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

For a morphism of crystals $\psi : B_1 \to B_2$, ψ is called a *strict morphism* if

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b),\ \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for all } i \in I \text{ and } b \in B_1.$$

Here we understand $\psi(0) = 0$. Moreover, ψ is called an *embedding* if the underlying map $\psi : B_1 \to B_2$ is injective. In this case, we say that B_1 is a *subcrystal* of B_2. If ψ is a strict embedding, we say that B_1 is a *full subcrystal* of B_2.

Example 1.3. (a) The crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ with $\lambda \in P^+$ is a $U_q(\mathfrak{g})$-crystal, where the maps ε_i, φ_i $(i \in I)$ are given by

$$\varepsilon_i(b) = \begin{cases} \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{\mathrm{re}}, \\ 0 & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_i(b) = \begin{cases} \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\} & \text{for } i \in I^{\mathrm{re}}, \\ \langle h_i, \mathrm{wt}(b) \rangle & \text{for } i \in I^{\mathrm{im}}. \end{cases}$$

(b) The crystal basis $B(\infty)$ of $U_q^-(\mathfrak{g})$ is a $U_q(\mathfrak{g})$-crystal, where

$$\varepsilon_i(b) = \begin{cases} \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{\mathrm{re}}, \\ 0 & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle \quad (i \in I).$$

Example 1.4. For $\lambda \in P$, the singletons $T_\lambda = \{t_\lambda\}$ and $R_\lambda = \{r_\lambda\}$ are $U_q(\mathfrak{g})$-crystals with maps defined by

$$\mathrm{wt}(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{for all } i \in I,$$

and

$$\mathrm{wt}(r_\lambda) = \lambda, \quad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \quad \varphi_i(r_\lambda) = 0, \quad \tilde{e}_i r_\lambda = \tilde{f}_i r_\lambda = 0 \quad \text{for all } i \in I.$$

Example 1.5. For each $i \in I$, let $B_i = \{b_i(-n) \,|\, n \geq 0\}$. Then B_i is a crystal with maps defined by

$$\begin{aligned}
&\mathrm{wt}(b_i(-n)) = -n\alpha_i, \\
&\tilde{e}_i b_i(-n) = b_i(-n+1), \quad \tilde{f}_i b_i(-n) = b_i(-n-1), \\
&\tilde{e}_j b_i(-n) = \tilde{f}_j b_i(-n) = 0 \quad \text{if} \quad j \neq i, \\
&\varepsilon_i(b_i(-n)) = n, \quad \varphi_i(b_i(-n)) = -n \quad \text{if} \quad i \in I^{re}, \\
&\varepsilon_i(b_i(-n)) = 0, \quad \varphi_i(b_i(-n)) = -na_{ii} \quad \text{if} \quad i \in I^{im}, \\
&\varepsilon_j(b_i(-n)) = \varphi_j(b_i(-n)) = -\infty \quad \text{if } j \neq i.
\end{aligned}$$

Here, we understand $b_i(-n) = 0$ for $n < 0$. The crystal B_i is called an *elementary crystal*.

We define the tensor product of a pair of crystals as follows: for two crystals B_1 and B_2, their tensor product $B_1 \otimes B_2$ is $\{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$ with the following crystal structure. The maps $\mathrm{wt}, \varepsilon_i, \varphi_i$ are given by

$$\begin{aligned}
\mathrm{wt}(b \otimes b') &= \mathrm{wt}(b) + \mathrm{wt}(b'), \\
\varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \langle h_i, \mathrm{wt}(b) \rangle), \\
\varphi_i(b \otimes b') &= \max(\varphi_i(b) + \langle h_i, \mathrm{wt}(b') \rangle, \varphi_i(b')).
\end{aligned}$$

For $i \in I$, we define

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}$$

For $i \in I^{re}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

and, for $i \in I^{im}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') - a_{ii}, \\ 0 & \text{if } \varepsilon_i(b') < \varphi_i(b) \leq \varepsilon_i(b') - a_{ii}, \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}$$

This tensor product rule is different from the one given in [7]. But when $B_1 = B(\lambda)$ and $B_2 = B(\mu)$ for $\lambda, \mu \in P^+$, the two rules coincide. Note that by the definition above, $B_1 \otimes B_2$ is a crystal. Moreover, the associativity law for the tensor product holds [8].

Example 1.6. Let $R_\lambda = \{r_\lambda\}$ be the crystal given in Example 1.4. Then for a crystal B, $B \otimes R_\lambda$ is a crystal with the maps $\mathrm{wt}, \varepsilon_i, \varphi_i$ given by

$$\begin{aligned}
\mathrm{wt}(b \otimes r_\lambda) &= \mathrm{wt}(b) + \lambda, \\
\varepsilon_i(b \otimes r_\lambda) &= \max(\varepsilon_i(b), -\langle h_i, \lambda + \mathrm{wt}(b)\rangle), \\
\varphi_i(b \otimes r_\lambda) &= \begin{cases} \varphi_i(b) + \langle h_i, \lambda\rangle & \text{for } i \in I^{re}, \\ \max(\varphi_i(b) + \langle h_i, \lambda\rangle, 0) & \text{for } i \in I^{im}, \end{cases} \\
\tilde{e}_i(b \otimes r_\lambda) &= \begin{cases} \tilde{e}_i b \otimes r_\lambda & \text{if } \varphi_i(b) \geq -\langle h_i, \lambda\rangle \text{ and } i \in I^{re}, \\ & \quad \text{or } \varphi_i(b) + \langle h_i, \lambda\rangle + a_{ii} > 0 \text{ and } i \in I^{im}, \\ 0 & \text{otherwise}, \end{cases} \\
\tilde{f}_i(b \otimes r_\lambda) &= \begin{cases} \tilde{f}_i b \otimes r_\lambda & \text{if } \varphi_i(b) > -\langle h_i, \lambda\rangle, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}$$

2. Embedding of the highest weight crystal $B(\lambda)$

Let $*$ be the $\mathbf{Q}(q)$-algebra anti-automorphism of $U_q(\mathfrak{g})$ such that

$$e_i^* = e_i, \;\; f_i^* = f_i \;\text{ and }\; (q^h)^* = q^{-h}.$$

In [8], it is verified that $L(\infty)^* = L(\infty)$ and $B(\infty)^* = B(\infty)$. So let us define for each $i \in I^{re}$, $\varepsilon_i^*(b) = \varepsilon_i(b^*)$ and $\varphi_i^*(b) = \varphi(b^*)$.

Proposition 2.1 ([8]). *For all $i \in I$, there exists a unique strict embedding*

$$\Psi_i : B(\infty) \longrightarrow B(\infty) \otimes B_i \quad \textit{such that} \quad u_\infty \mapsto u_\infty \otimes b_i(0),$$

where u_∞ is the highest weight vector in $B(\infty)$.

Proof. Now, we give a brief sketch of the proof given in [8]. For $b = \tilde{f}_{i_r} \ldots \tilde{f}_{i_1} u_\infty \in B(\infty)$, take $\mu \gg 0$ such that $\widehat{\pi}_\mu(b) = \widehat{\pi}_\mu(\tilde{f}_{i_r} \ldots \tilde{f}_{i_1} u_\infty) = \tilde{f}_{i_r} \ldots \tilde{f}_{i_1} u_\mu \neq 0$. Let $\lambda \in P^+$ such that $\lambda(h_i) = 0$ and $\lambda(h_j) = \mu(h_j) \gg 0$ for all $j \neq i$. Set $\mu(h_i) = l \gg 0$. Then there is a strict embedding $\phi : B(\mu) \to B(\lambda) \otimes B(l\Lambda_i)$ sending u_μ to $u_\lambda \otimes u_{l\Lambda_i}$. Moreover, $\mathrm{Im}(\phi \circ \widehat{\pi}_\mu)$ belongs to $B(\lambda) \otimes \{\tilde{f}_i^m u_{l\Lambda_i} \mid m \geq 0\}$.

Define a map $\psi_i : B(\lambda) \otimes \{\tilde{f}_i^m u_{l\Lambda_i} \mid m \geq 0\} \to B(\infty) \otimes B_i$ by

$$\psi_i(\tilde{f}_{j_s} \ldots \tilde{f}_{j_1} u_\lambda \otimes \tilde{f}_i^m u_{l\Lambda_i}) = \tilde{f}_{j_s} \ldots \tilde{f}_{j_1} u_\infty \otimes b_i(-m).$$

Then ψ_i is injective, and ψ_i commutes with $\tilde{e}_j$ and $\tilde{f}_j$.

Now, we define $\Psi_i(b)$ by

$$\begin{aligned}
\psi_i \circ \phi \circ \widehat{\pi}_\mu(b) &= \psi_i \circ \phi(\tilde{f}_{i_r} \ldots \tilde{f}_{i_1} u_\mu) = \psi_i(\tilde{f}_{i_r} \ldots \tilde{f}_{i_1}(u_\lambda \otimes u_{l\Lambda_i})) \\
&= \psi_i(\tilde{f}_{j_s} \ldots \tilde{f}_{j_1}(u_\lambda \otimes \tilde{f}_i^m u_{l\Lambda_i})) = \tilde{f}_{j_s} \ldots \tilde{f}_{j_1} u_\infty \otimes b_i(-m).
\end{aligned}$$

Then Ψ_i is a strict embedding. Moreover,

$$\tilde{f}_{j_s} \ldots \tilde{f}_{j_1} u_\infty = (\tilde{e}_i^m b^*)^*, \;\text{ and }\; m = \max_k\{k \mid \tilde{e}_i^k b^* \neq 0\}. \tag{2.1}$$

□

Let ι be an infinite sequence, $\iota = (\ldots, i_2, i_1)$ in I, such that every $i \in I$ appears infinitely many times, and let $\mathbf{Z}^{\infty}_{\geq 0,\iota}$ be the crystal introduced in [19]. (Indeed, the crystal $\mathbf{Z}^{\infty}_{\geq 0,\iota}$ is isomorphic to the crystal $B(\iota)$, the (set-theoretical) inductive limit of $B(k) = B_{i_k} \otimes \cdots \otimes B_{i_1}$, given in [8].) By taking the composition of the above crystal embeddings repeatedly, we have

Proposition 2.2 ([8, 19])**.** *There is a strict embedding* $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}^{\infty}_{\geq 0,\iota}$.

Let $\pi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ be the natural projection sending $L(\infty)$ to $L(\lambda)$, and let $\widehat{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda)$ be the induced map sending $B(\infty)$ to $B(\lambda) \cup \{0\}$. Then we have

$$\tilde{f}_i \circ \widehat{\pi}_\lambda = \widehat{\pi}_\lambda \circ \tilde{f}_i,$$
$$\tilde{e}_i \circ \widehat{\pi}_\lambda = \widehat{\pi}_\lambda \circ \tilde{e}_i, \quad \text{if } \widehat{\pi}_\lambda(b) \neq 0,$$
$$\widehat{\pi}_\lambda : B(\infty)\backslash\{\widehat{\pi}_\lambda^{-1}(0)\} \to B(\lambda) \text{ is bijective.}$$

Now, define a map

$$\Phi_\lambda : (B(\infty) \otimes R_\lambda) \cup \{0\} \to B(\lambda) \cup \{0\}$$

by

$$\Phi_\lambda(b \otimes r_\lambda) = \widehat{\pi}_\lambda(b) \quad \text{and} \quad \Phi_\lambda(0) = 0.$$

Let $\widetilde{B}(\lambda) := \{b \otimes r_\lambda \in B(\infty) \otimes R_\lambda | \, \Phi_\lambda(b \otimes r_\lambda) \neq 0\}$. Then we have

Theorem 2.3. (a) *The map* Φ_λ *is a surjective strict morphism of crystals. Moreover, it induces an isomorphism of crystals from* $\widetilde{B}(\lambda)$ *to* $B(\lambda)$.

(b) $\widetilde{B}(\lambda)$ *is the set of* $b \otimes r_\lambda \in B(\infty) \otimes R_\lambda$ *such that*

(i) $\varepsilon_i^*(b) \leq \lambda(h_i)$ $(i \in I^{re})$, (ii) *if* $\lambda(h_i) = 0$, $\tilde{e}_i b^* = 0$.

Proof. (b) is directly derived from the following fact, obtained by (2.1) in the proof of Proposition 2.1:

$$\begin{gathered} \widehat{\pi}_\lambda(b) \neq 0 \text{ if and only if} \\ \varepsilon_i^*(b) \leq \lambda(h_i) \ (i \in I^{re}), \text{ and } \lambda(h_i) = 0 \text{ implies } \tilde{e}_i b^* = 0. \end{gathered} \tag{2.2}$$

For (a), since $\widehat{\pi}_\lambda$ is surjective, it is clear that Φ_λ is also surjective. In order to show that Φ_λ is a strict morphism of crystals, it suffices to show that for $u \in B(\infty) \otimes R_\lambda$,

(1) $\mathrm{wt}(\Phi_\lambda(u)) = \mathrm{wt}(u)$ if $\Phi_\lambda(u) \neq 0$,
(2) $\varepsilon_i(\Phi_\lambda(u)) = \varepsilon_i(u)$ for any $i \in I$ if $\Phi_\lambda(u) \neq 0$,
(3) $\varphi_i(\Phi_\lambda(u)) = \varphi_i(u)$ for any $i \in I$ if $\Phi_\lambda(u) \neq 0$,
(4) $\tilde{e}_i \Phi_\lambda(u) = \Phi_\lambda(\tilde{e}_i u)$ for any i,
(5) $\tilde{f}_i \Phi_\lambda(u) = \Phi_\lambda(\tilde{f}_i u)$ for any i.

If $i \in I^{re}$, (1), (2) and (3) follow from the proof of Theorem 3.1 of [17], and if $i \in I^{im}$, since $\varepsilon_i(b) = 0$ for all $b \in B(\infty)$ or $B(\lambda)$, it is clear. So it suffices to consider (4) and (5). For $u = b \otimes r_\lambda$, suppose that $\Phi_\lambda(u) = \widehat{\pi}_\lambda(b) \neq 0$. When $i \in I^{re}$, we have

$$0 \leq \varphi_i(\widehat{\pi}_\lambda(b)) = \langle h_i, \lambda \rangle + \varphi_i(b),$$

which implies that $\tilde{e}_i(b \otimes r_\lambda) = \tilde{e}_i b \otimes r_\lambda$. Hence

$$\tilde{e}_i \Phi_\lambda(u) = \tilde{e}_i \widehat{\pi}_\lambda(b) = \widehat{\pi}_\lambda(\tilde{e}_i b) = \Phi_\lambda(\tilde{e}_i u).$$

When $i \in I^{im}$, if $\tilde{e}_i(b \otimes r_\lambda) = \tilde{e}_i b \otimes r_\lambda$, there is nothing to prove. If $\tilde{e}_i(b \otimes r_\lambda) = b \otimes \tilde{e}_i r_\lambda = 0$, then $\langle h_i, \lambda + \mathrm{wt}(b)\rangle + a_{ii} \le 0$. Hence $\tilde{e}_i \Phi_\lambda(u) = \tilde{e}_i \widehat{\pi}_\lambda(b) = 0 = \Phi_\lambda(\tilde{e}_i u)$.

Next, consider the case $\Phi_\lambda(u) = \widehat{\pi}_\lambda(b) = 0$. Then it suffices to show that $\Phi_\lambda(\tilde{e}_i(b \otimes r_\lambda)) = 0$. If $\tilde{e}_i(b \otimes r_\lambda) = 0$, there is nothing to prove. If $\tilde{e}_i(b \otimes r_\lambda) \ne 0$, since $\tilde{e}_i r_\lambda = 0$,

$$\tilde{e}_i(b \otimes r_\lambda) = \tilde{e}_i b \otimes r_\lambda,$$

which implies

$$\begin{aligned} &\varphi_i(b) \ge \varepsilon_i(r_\lambda) = -\langle h_i, \lambda\rangle \quad \text{for } i \in I^{re}, \\ &\langle h_i, \mathrm{wt}(b) + \lambda\rangle + a_{ii} > 0 \quad \text{for } i \in I^{im}. \end{aligned} \tag{2.3}$$

Assuming $\Phi_\lambda(\tilde{e}_i b \otimes r_\lambda) = \widehat{\pi}_\lambda(\tilde{e}_i b) \ne 0$, we shall derive a contradiction. Since $\tilde{f}_i$ and $\widehat{\pi}_\lambda$ commute, $\tilde{f}_i \widehat{\pi}_\lambda(\tilde{e}_i b) = \widehat{\pi}_\lambda(\tilde{f}_i \tilde{e}_i b) = \widehat{\pi}_\lambda(b) = 0$. It means that

$$\varphi_i(\widehat{\pi}_\lambda(\tilde{e}_i b)) = 0 \quad (i \in I^{re}) \quad \text{and} \quad \langle h_i, \widehat{\pi}_\lambda(\tilde{e}_i b)\rangle = 0 \quad (i \in I^{im}).$$

When $i \in I^{re}$,

$$\begin{aligned} 0 &= \varphi_i(\widehat{\pi}_\lambda(\tilde{e}_i b)) = \langle h_i, \mathrm{wt}(\widehat{\pi}_\lambda(\tilde{e}_i b))\rangle + \varepsilon_i(\widehat{\pi}_\lambda(\tilde{e}_i b)) \\ &= \langle h_i, \lambda + \mathrm{wt}(b) + \alpha_i\rangle + \varepsilon_i(\tilde{e}_i b) \\ &= \langle h_i, \lambda\rangle + \langle h_i, \mathrm{wt}(b)\rangle + 2 + \varepsilon_i(b) - 1 \\ &= \langle h_i, \lambda\rangle + \varphi_i(b) + 1. \end{aligned}$$

Therefore,

$$\varphi_i(b) = -\langle h_i, \lambda\rangle - 1 = \varepsilon_i(r_\lambda) - 1 < \varepsilon_i(r_\lambda),$$

which contradicts (2.3).

When $i \in I^{im}$,

$$0 = \langle h_i, \widehat{\pi}_\lambda(\tilde{e}_i b)\rangle = \langle h_i, \mathrm{wt}(b) + \lambda + \alpha_i\rangle.$$

This contradicts (2.3).

Now, let us prove (5). We know that $\tilde{f}_i \Phi_\lambda(u) = \tilde{f}_i \widehat{\pi}_\lambda(b) = \widehat{\pi}_\lambda(\tilde{f}_i b)$. So if $\tilde{f}_i u = \tilde{f}_i b \otimes r_\lambda$, then $\widehat{\pi}_\lambda(\tilde{f}_i b) = \Phi_\lambda(\tilde{f}_i u)$. Suppose that $\tilde{f}_i u = b \otimes \tilde{f}_i r_\lambda = 0$. Then we have

$$\begin{aligned} &\varphi_i(b) \le \varepsilon_i(r_\lambda) = -\langle h_i, \lambda\rangle \quad \text{for } i \in I^{re}, \\ &\langle h_i, \mathrm{wt}(b) + \lambda\rangle = 0 \quad \text{for } i \in I^{im}. \end{aligned} \tag{2.4}$$

If $\Phi_\lambda(u) = \widehat{\pi}_\lambda(b) = 0$, there is nothing to prove. Assume that $\Phi_\lambda(u) = \widehat{\pi}_\lambda(b) \ne 0$. Suppose that $\tilde{f}_i \Phi_\lambda(u) = \tilde{f}_i \widehat{\pi}_\lambda(b) \ne 0$. Then

$$\begin{aligned} &0 < \varphi_i(\widehat{\pi}_\lambda(b)) = \langle h_i, \lambda\rangle + \varphi_i(b) \quad \text{for } i \in I^{re}, \\ &\langle h_i, \lambda + \mathrm{wt}(b)\rangle > 0 \quad \text{for } i \in I^{im}, \end{aligned}$$

which contradicts (2.4). Therefore, Φ_λ is a surjective strict morphism of crystals.

Finally, we will show that Φ_λ induces an isomorphism of crystals from $\widetilde{B}(\lambda)$ to $B(\lambda)$. Since $\Phi_\lambda(b \otimes r_\lambda) \ne 0$ is equivalent to $\widehat{\pi}_\lambda(b) \ne 0$, $\phi_\lambda := \Phi_\lambda|_{\widetilde{B}(\lambda)} : \widetilde{B}(\lambda) \to B(\lambda)$ is a 1-1 correspondence. So if we show that $\widetilde{B}(\lambda)$ is stable under $\tilde{e}_i$ and $\tilde{f}_i$, then ϕ_λ is a strict morphism of crystals. In order to prove stability, it is enough to show that $\Phi_\lambda(\tilde{e}_i(b \otimes r_\lambda)) = 0$ (resp. $\Phi_\lambda(\tilde{f}_i(b \otimes r_\lambda)) = 0$) implies $\tilde{e}_i(b \otimes r_\lambda) = 0$ (resp. $\tilde{f}_i(b \otimes r_\lambda) = 0$). First, suppose that $\Phi_\lambda(\tilde{e}_i(b \otimes r_\lambda)) = \tilde{e}_i \widehat{\pi}_\lambda(b) = 0$. Since $\widehat{\pi}_\lambda(b) \ne 0$, $\tilde{e}_i \widehat{\pi}_\lambda(b) = \widehat{\pi}_\lambda(\tilde{e}_i b) = 0$, which implies $\tilde{e}_i b = 0$. Indeed, if $\tilde{e}_i b \ne 0$, we have

$$\max_k\{k \mid \tilde{e}_j^k(\tilde{e}_i b)^* \ne 0\} \le \max_k\{k \mid \tilde{e}_j^k b^* \ne 0\} \quad \text{for any } j \in I.$$

This contradicts $\widehat{\pi}_\lambda(\tilde{e}_i b) = 0$ by (2.2). Therefore, we have $\tilde{e}_i(b \otimes r_\lambda) = 0$. Secondly, suppose that $\Phi_\lambda(\tilde{f}_i(b \otimes r_\lambda)) = \tilde{f}_i \widehat{\pi}_\lambda(b) = 0$. If $i \in I^{re}$, we have $0 = \varphi_i(\widehat{\pi}_\lambda(b)) = \varphi_i(b) + \langle h_i, \lambda \rangle$. Therefore, $\tilde{f}_i(b \otimes r_\lambda) = b \otimes \tilde{f}_i r_\lambda = 0$. If $i \in I^{im}$, we have $0 = \langle h_i, \mathrm{wt}(\widehat{\pi}_\lambda(b)) \rangle = \langle h_i, \lambda + \mathrm{wt}(b) \rangle$. Therefore, $\tilde{f}_i(b \otimes r_\lambda) = 0$. □

Let $\iota = (\ldots, i_k, \ldots, i_1)$ be an infinite sequence such that

$$i_k \neq i_{k+1} \quad \text{and} \quad \#\{k : i_k = i\} = \infty \quad \text{for any } i \in I. \tag{2.5}$$

According to Proposition 2.2 and Theorem 2.3, we derive a crystal structure $\mathbf{Z}^\infty_{\geq 0, \iota}[\lambda]$ on the set of infinite sequences of nonnegative integers

$$\mathbf{Z}^\infty_{\geq 0} := \{(\ldots, x_k, \ldots, x_1) : x_k \in \mathbf{Z}_{\geq 0} \text{ and } x_k = 0 \text{ for } k \gg 0\}$$

associated with ι as follows: Let $\overrightarrow{x} = (\ldots, x_k, \ldots, x_1)$ be an element of $\mathbf{Z}^\infty_{\geq 0}$ corresponding to $\cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \otimes r_\lambda$. For $k \geq 1$, we define

$$\sigma_k(\overrightarrow{x}) = \begin{cases} x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j & \text{if } i_k \in I^{re}, \\ \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j & \text{if } i_k \in I^{im}. \end{cases} \tag{2.6}$$

Let

$$\begin{aligned} &\sigma^{(i)}(\overrightarrow{x}) = \max\nolimits_{k : i_k = i}\{\sigma_k(\overrightarrow{x})\}, \\ &\sigma^{(i)}_0(\overrightarrow{x}) = -\langle h_i, \lambda \rangle + \sum_{j \geq 1} \langle h_i, \alpha_{i_j} \rangle x_j, \\ &n_f = \min\{k : i_k = i,\ \sigma_k(\overrightarrow{x}) = \sigma^{(i)}(\overrightarrow{x})\}, \\ &n_e = \begin{cases} \max\{k : i_k = i,\ \sigma_k(\overrightarrow{x}) = \sigma^{(i)}(\overrightarrow{x})\} & \text{if } i \in I^{re}, \\ n_f & \text{if } i \in I^{im}. \end{cases} \end{aligned}$$

For each $k \geq 1$, we denote by $k^{(+)}$ (resp. $k^{(-)}$) the minimal (resp. maximal) index $j > k$ (resp. $j < k$) such that $i_j = i_k$. We now define

$$\tilde{f}_i \overrightarrow{x} = \begin{cases} (x_k + \delta_{k, n_f} : k \geq 1) & \text{if } i \in I^{re} \text{ and } \sigma^{(i)}(\overrightarrow{x}) > \sigma^{(i)}_0(\overrightarrow{x}), \\ & \quad \text{or } \ i \in I^{im} \text{ and } \sigma^{(i)}_0(\overrightarrow{x}) < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

and $\tilde{e}_i \overrightarrow{x} = (x_k - \delta_{k, n_e} : k \geq 1)$ if $\overrightarrow{x}$ satisfies the following conditions:

$$\begin{aligned} &\text{(i) } i \in I^{re} : \sigma^{(i)}(\overrightarrow{x}) > 0,\ \sigma^{(i)}(\overrightarrow{x}) \geq \sigma^{(i)}_0(\overrightarrow{x}), \\ &\text{(ii) } i \in I^{im} : \sigma^{(i)}_0(\overrightarrow{x}) - a_{ii} < 0, \text{ for } k = n_e \text{ with } k^{(-)} \neq 0, \\ &\qquad\qquad x_k > 1, \text{ or } x_k = 1 \text{ and } \sum_{k^{(-)} < j < k} \langle h_i, \alpha_{i_j} \rangle x_j < 0. \end{aligned} \tag{2.8}$$

Otherwise, $\tilde{e}_i \overrightarrow{x} = 0$. We also define

$$\begin{aligned} \mathrm{wt}(\overrightarrow{x}) &= \lambda - \sum_{j=1}^{\infty} x_j \alpha_{i_j}, \\ \varepsilon_i(\overrightarrow{x}) &= \max(\sigma^{(i)}(\overrightarrow{x}), \sigma^{(i)}_0(\overrightarrow{x})), \\ \varphi_i(\overrightarrow{x}) &= \langle h_i, \mathrm{wt}(\overrightarrow{x}) \rangle + \varepsilon_i(\overrightarrow{x}). \end{aligned}$$

Then it is easy to see that $\mathbf{Z}^{\infty}_{\geq 0}$ is a crystal, and we will denote this crystal by $\mathbf{Z}^{\infty}_{\geq 0,\iota}[\lambda]$. Note that the crystal $\mathbf{Z}^{\infty}_{\geq 0,\iota}[\lambda]$ is isomorphic to $\mathbf{Z}^{\infty}_{\geq 0,\iota} \otimes R_\lambda \cong B(\iota) \otimes R_\lambda$. Therefore, by Theorem 2.3, we have

Proposition 2.4. *Let $B(\lambda)$ be the highest weight crystal with a dominant integral weight. Then there is a strict embedding $\Psi^\lambda_\iota : B(\lambda) \hookrightarrow \mathbf{Z}^{\infty}_{\geq 0,\iota}[\lambda]$.*

3. Polyhedral realization of $B(\lambda)$

Let $\mathbf{Q}^\infty = \{\overrightarrow{x} = (\ldots, x_k, \ldots, x_1) \mid x_k \in \mathbf{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\}$ be an infinite dimensional vector space. Let $\beta^{(\pm)}_k(\overrightarrow{x})$ be linear functions given by

$$\beta^{(+)}_k(\overrightarrow{x}) = \sigma_k(\overrightarrow{x}) - \sigma_{k^{(+)}}(\overrightarrow{x}) \tag{3.1}$$

and

$$\beta^{(-)}_k(\overrightarrow{x}) = \begin{cases} \sigma_{k^{(-)}}(\overrightarrow{x}) - \sigma_k(\overrightarrow{x}) & \text{if } k^{(-)} > 0, \\ \sigma^{(i_k)}_0(\overrightarrow{x}) - \sigma_k(\overrightarrow{x}) & \text{if } k^{(-)} = 0. \end{cases} \tag{3.2}$$

We define an operator $\widehat{S}_k = \widehat{S}_{k,\iota}$ for a linear function $\psi(\overrightarrow{x}) = c + \sum_{k\geq 1} \psi_k x_k$ on $(\mathbf{Q}^\infty)^*$ by

$$\widehat{S_k}(\psi) = \begin{cases} \psi - \psi_k \beta^{(+)}_k & \text{if } \psi_k > 0,\ i_k \in I^{re}, \\ \psi - \psi_k (x_k + \sum_{k<j<k^{(+)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{k^{(+)}}) & \text{if } \psi_k > 0,\ i_k \in I^{im}, \\ \psi - \psi_k \beta^{(-)}_k & \text{if } \psi_k \leq 0. \end{cases}$$

For the fixed sequence ι, we denote by $\iota^{(i)}$ the first number k such that $i_k = i$. For each $i \in I^{re}$, let

$$\lambda^{(i)}(\overrightarrow{x}) = \langle h_i, \lambda\rangle - \sum_{1\leq j<\iota^{(i)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{\iota^{(i)}}.$$

For ι and a dominant integral weight λ,

$$\begin{aligned} \Theta_\iota[\lambda] := \ &\{\widehat{S}_{j_l} \cdots \widehat{S}_{j_1} x_{j_0} : l \geq 0, j_0, \ldots, j_l \geq 1\} \\ &\cup \{\widehat{S}_{j_k} \cdots \widehat{S}_{j_1} \lambda^{(i)}(\overrightarrow{x}) : k \geq 0, j_1, \ldots, j_k \geq 1\}. \end{aligned}$$

Moreover, for a given $s, t \geq 1$ $(t > s)$, let $\Theta^{s\setminus t}_\iota$ be the subset of Θ_ι of linear forms obtained from the coordinate forms x_s by applying transformations S_k with $k \neq t$, i.e.,

$$\Theta^{s\setminus t}_\iota[\lambda] = \{\widehat{S}_{j_l} \cdots \widehat{S}_{j_1} x_s : l \geq 0, s, j_1, \ldots, j_l \geq 1\},$$

where none of $j_1, \ldots, j_l$ is t.

Let $\Gamma_\iota[\lambda]$ be the set of $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geq 0,\iota}[\lambda]$ satisfying the following conditions:

(i) $\psi(\overrightarrow{x}) \geq 0$ for any $\psi \in \Theta_\iota[\lambda]$,
(ii) for each t with $i_t \in I^{im}$, if $x_t \neq 0$, then

$$\begin{aligned} &\sum_{t^{(-)}<j<t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0 && (t^{(-)} \neq 0), \\ -\langle h_{i_t}, \lambda\rangle + &\sum_{1\leq j<t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0 && (t^{(-)} = 0). \end{aligned} \tag{3.3}$$

In addition, if

$$(3.4)\qquad \begin{aligned} \sum_{\substack{t^{(-)}<j<t\\ j\in I^{im}}} \langle h_{i_t},\alpha_{i_j}\rangle x_j = 0 & \qquad (t^{(-)}\neq 0),\\ -\langle h_{i_t},\lambda\rangle + \sum_{\substack{1\le j<t\\ j\in I^{im}}} \langle h_{i_t},\alpha_{i_j}\rangle x_j = 0 & \qquad (t^{(-)}= 0), \end{aligned}$$

then there exists a p ($t^{(-)} < p < t$) with $i_p \in I^{re}$ such that

$$(3.5)\qquad \langle h_{i_t},\alpha_{i_p}\rangle x_p < 0 \text{ and } \psi(\overrightarrow{x}) > 0 \text{ for any } \psi \in \Theta_\iota^{p\setminus t}[\lambda].$$

Theorem 3.1. *Let ι be the sequence of indices satisfying* (2.5)*, and let λ be a dominant integral weight. Suppose that $\overrightarrow{0} = (\dots,0,\dots,0)$ belongs to $\Gamma_\iota[\lambda]$. Let $\Psi_\iota^\lambda : B(\lambda) \hookrightarrow \mathbf{Z}^\infty_{\ge 0,\iota}[\lambda]$ be the embedding. Then we have*

$$\operatorname{Im}\Psi_\iota^\lambda\ \big(\cong \widetilde{B}(\lambda)\big) = \Gamma_\iota[\lambda].$$

Proof. First, let us start by showing that $\Gamma_\iota[\lambda]$ is closed under the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$. Let $\overrightarrow{x}$ be an element of $\Gamma_\iota[\lambda]$. Suppose that $\tilde{f}_i\overrightarrow{x} = (\dots,x_k+1,\dots,x_2,x_1) \neq 0$. Since $\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k \ge \psi_k$ for any $\psi\in\Gamma_\iota[\lambda]$, it suffices to consider the case $\psi_k < 0$. Note that for each k, $\sigma_k(\overrightarrow{x}) > \sigma_{k^{(-)}}(\overrightarrow{x})$ if $k^{(-)} > 0$, and $\sigma_k(\overrightarrow{x}) > \sigma_0^{(i)}(\overrightarrow{x})$ if $k^{(-)} = 0$. (Indeed, when $i_k \in I^{im}$, $\sigma_k(\overrightarrow{x}) = 0$, and $\sigma_{k^{(-)}}(\overrightarrow{x}) < 0$ and $\sigma_0^{(i)}(\overrightarrow{x}) < 0$.) So we have $\beta_k^{(-)}(\overrightarrow{x}) \le -1$, which implies

$$\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k \ge \psi(\overrightarrow{x}) - \psi_k\beta_k^{(-)}(\overrightarrow{x}) = (\widehat{S_k}\psi)(\overrightarrow{x}) \ge 0.$$

Now, suppose that $\tilde{f}_i\overrightarrow{x}$ does not satisfy the condition (3.3). Then we have the following cases:

$$\begin{aligned} &\text{(i) } k^{(-)}\neq 0: && x_k = 0,\ \textstyle\sum_{k^{(-)}<j<k}\langle h_i,\alpha_{i_j}\rangle x_j = 0 \quad (\text{in } \overrightarrow{x})\\ & && \xrightarrow{\tilde{f}_i}\quad x_k = 1,\ \textstyle\sum_{k^{(-)}<j<k}\langle h_i,\alpha_{i_j}\rangle x_j = 0 \quad (\text{in } \tilde{f}_i\overrightarrow{x}),\\ &\text{(ii) } k^{(-)}= 0: && x_k = 0,\ -\langle h_i,\lambda\rangle + \textstyle\sum_{1\le j<k}\langle h_i,\alpha_{i_j}\rangle x_j = 0 \quad (\text{in } \overrightarrow{x})\\ & && \xrightarrow{\tilde{f}_i}\quad x_k = 1,\ -\langle h_i,\lambda\rangle + \textstyle\sum_{1\le j<k}\langle h_i,\alpha_{i_j}\rangle x_j = 0 \quad (\text{in } \tilde{f}_i\overrightarrow{x}). \end{aligned}$$

But, this cannot occur by the definition of the Kashiwara operator $\tilde{f}_i$.

Now, we show that $\tilde{f}_i\overrightarrow{x}$ satisfies the condition (3.5). First, suppose that there exist p and t satisfying (3.5) in $\overrightarrow{x}$. Since $\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k$, it is enough to consider the cases that $\psi_k < 0$. Note that by definition of the set $\Theta_\iota^{p\setminus t}[\lambda]$, $\psi_t > 0$ for all $\psi \in \Theta_\iota^{p\setminus t}[\lambda]$. So it suffices to consider the case that $k \neq t$. If $k\neq t$, then $S_k\psi \in \Theta_\iota^{p\setminus t}$ and so $\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k \ge (S_k\psi)(\overrightarrow{x}) > 0$. Second, suppose that $k = t$, $x_t = 0$, and for any j such that $t^{(-)} < j < t$, $i_j \in I^{re}$, $\langle h_{i_t},\alpha_{i_j}\rangle x_j < 0$, there is a $\psi \in \Theta_\iota^{j\setminus t}[\lambda]$ such that $\psi(\overrightarrow{x}) = 0$ in $\overrightarrow{x}$. Note that since j is the index such that $\langle h_{i_t},\alpha_{i_j}\rangle < 0$, we have $\psi_t > 0$ for all $\psi\in\Theta_\iota^{j\setminus t}[\lambda]$. Therefore, $\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_t \ge \psi_t > 0$ for all $\psi \in \Theta_\iota^{j\setminus t}[\lambda]$. Therefore, $\Gamma_\iota[\lambda]$ is closed under $\tilde{f}_i$.

On the other hand, suppose that $\tilde{e}_i\overrightarrow{x} = (\dots,x_k-1,\dots,x_2,x_1) \neq 0$. Since $\psi(\tilde{e}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) - \psi_k \ge -\psi_k$, it is enough to consider the case $\psi_k > 0$. Note that if $i_k \in I^{re}$, $\sigma_k(\overrightarrow{x}) > \sigma_{k^{(+)}}(\overrightarrow{x})$, and if $i_k\in I^{im}$, then $x_k + \sum_{k<j<k^{(+)}}\langle h_{i_k},\alpha_{i_j}\rangle x_j -$

$x_{k^{(+)}} > 0$. Therefore,

$$\begin{aligned}\psi(\tilde{e}_i\overrightarrow{x}) &= \psi(\overrightarrow{x}) - \psi_k \\ &\geq \begin{cases} \psi(\overrightarrow{x}) - \psi_k\beta_k^{(+)}(\overrightarrow{x}) & (i_k \in I^{re}) \\ \psi(\overrightarrow{x}) - \psi_k(x_k + \sum_{k<j<k^{(+)}}\langle h_{i_k}, \alpha_{i_j}\rangle x_j - x_{k^{(+)}}) & (i_k \in I^{im}) \end{cases} \\ &= (\widehat{S_k}\psi)(\overrightarrow{x}) \geq 0.\end{aligned}$$

Now, suppose that $\tilde{e}_i\overrightarrow{x}$ does not satisfy the condition (ii). By (3.3) and (3.5), it is easy to see that $\tilde{e}_i\overrightarrow{x}$ satisfies (3.3). For instance, if $\tilde{e}_i\overrightarrow{x}$ $(i \in I^{re})$ does not satisfy (3.3) $(t^{(-)} = 0)$, then x_k in $\overrightarrow{x}$ must be 1, and

$$\langle h_{i_t}, \lambda\rangle = \langle h_{i_t}, \alpha_{i_j}\rangle x_j = 0 \text{ for all } j \neq k,\ 1 \leq j < t \quad (\text{in } \overrightarrow{x}).$$

That is, $\overrightarrow{x}$ satisfies the condition (3.4). Then by (3.5), we have $\widehat{S}_k x_k(\overrightarrow{x}) = x_k - \beta_k^{(+)} > 0$, which implies $\beta_k^{(+)} \leq 0$. This contradicts the definition of the Kashiwara operator $\tilde{e}_i$. Moreover, by the same argument given in the proof of Theorem 3.1 in [19], it is proved that $\tilde{e}_i\overrightarrow{x}$ satisfies the condition (3.5). Therefore, Γ_ι is closed under all $\tilde{e}_i$.

To complete the proof, it suffices to show that if $\overrightarrow{x} \in \Gamma_\iota[\lambda]$ satisfies $\tilde{e}_i\overrightarrow{x} = 0$ for any $i \in I$, then $\overrightarrow{x} = \overrightarrow{0}$. Suppose that $\tilde{e}_i\overrightarrow{x} = 0$ for any $i \in I$ and $\overrightarrow{x} \neq \overrightarrow{0}$. Since $\overrightarrow{x} \neq \overrightarrow{0}$, we have $j > 0$ such that $x_j > 0$ and $x_k = 0$ for all $k > j$. First, consider the case $i_j \in I^{re}$. Since $\sigma_j(\overrightarrow{x}) = x_j > 0$, by (2.8), $\sigma^{(i)}(\overrightarrow{x}) < \sigma_0^{(i)}(\overrightarrow{x})$ for $i_j = i$. Hence,

$$0 < \sigma_0^{(i)}(\overrightarrow{x}) - \sigma^{(i)}(\overrightarrow{x}) \leq \sigma_0^{(i)}(\overrightarrow{x}) - \sigma_{\iota^{(i)}}(\overrightarrow{x}) = \beta_{\iota^{(i)}}^{(-)}(\overrightarrow{x}) = -\lambda^{(-)}(\overrightarrow{x}).$$

This is a contradiction.

Second, consider the case that $i_j \in I^{im}$. If $j^{(-)} = 0$, by (2.8), we have

$$\sigma_0^{(i)}(\overrightarrow{x}) - a_{ii} = -\langle h_i, \lambda\rangle + \sum_{l\geq 1}\langle h_i, \alpha_l\rangle x_l - a_{ii} \geq 0,$$

where $i_j = i$. But, by (3.3),

$$\sigma_0^{(i)}(\overrightarrow{x}) - a_{ii} = -\langle h_i, \lambda\rangle + \sum_{1\leq l<j}\langle h_i, \alpha_l\rangle x_l + a_{ii}(x_j - 1) < 0.$$

This is a contradiction. Moreover, if $j^{(-)} \neq 0$, then $\sigma_0^{(i)}(\overrightarrow{x}) - a_{ii} < 0$ and

$$x_j = 1 \quad \text{and} \quad \sum_{j^{(-)}<l<j}\langle h_{i_j}, \alpha_{i_l}\rangle x_l = 0.$$

(Indeed, if $\sigma_0^{(i)}(\overrightarrow{x}) - a_{ii} \geq 0$, we can derive a contradiction by the same argument given in the case that $j^{(-)} = 0$.) But, this contradicts (3.3). □

Now, recall the main result of polyhedral realization of $B(\infty)$ given in [19]. Let $S_k = S_{k,\iota}$ be a piecewise-linear operator on $(\mathbf{Q}^\infty)^*$ such that

$$S_k(\psi) = \begin{cases} \psi - \psi_k\beta_k & \text{if } \psi_k > 0,\ i_k \in I^{re}, \\ \psi - \psi_k(x_k + \sum_{k<j<k^{(+)}}\langle h_i, \alpha_{i_j}\rangle x_j - x_{k^{(+)}}) & \text{if } \psi_k > 0,\ i_k \in I^{im}, \\ \psi - \psi_k\beta_{k^{(-)}} & \text{if } \psi_k \leq 0, \end{cases}$$

where $\beta_k = \beta_k^{(+)}$. Let

$$\Theta_\iota = \{S_{j_l}\cdots S_{j_1}x_{j_0} : l \geq 0, j_0, \ldots, j_l \geq 1\}$$

be the set of linear forms obtained from the coordinate forms x_j by applying transformations S_k. Then under the positivity assumption on ι [18, 19], we have

Theorem 3.2 ([19]). *Let ι be the sequence of indices satisfying* (2.5). *Let $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}^\infty_{\geq 0,\iota}$ be the Kashiwara embedding. Then* $\operatorname{Im}\Psi_\iota$ *is the set Γ_ι of $\overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota}$ such that*

(i) $\psi(\overrightarrow{x}) \geq 0$ *for any $\psi \in \Theta_\iota$,*
(ii) *for each t with $i_t \in I^{im}$, if $x_t \neq 0$ and $t^{(-)} \neq 0$, then*

$$\sum_{t^{(-)}<j<t} \langle h_{i_t}, \alpha_{i_j} \rangle x_j < 0. \tag{3.6}$$

In addition, if $\langle h_{i_t}, \alpha_{i_j} \rangle x_j = 0$ $(t^{(-)} < j < t)$ for all j with $i_j \in I^{im}$, there exists a p $(t^{(-)} < p < t)$ such that $i_p \in I^{re}$,

$$\langle h_{i_t}, \alpha_{i_p} \rangle x_p < 0 \text{ and } \psi(\overrightarrow{x}) > 0 \text{ for any } \psi \in \Theta^{p\setminus t}_\iota. \tag{3.7}$$

Corollary 3.3 ([19]). *Assume that all elements of I are imaginary, that is, $I = I^{im}$. Then the image of the Kashiwara embedding* $\operatorname{Im}\Psi_\iota$ *equals the set of $\overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota}$ satisfying* (3.6) *of Theorem* 3.2.

Corollary 3.4 ([19]). *Let I be an index set such that the cardinality of I^{re} is* 1, *and let ι be a sequence of indices in I satisfying* (2.5). *Then the image* $\operatorname{Im}\Psi_\iota$ *of the crystal embedding is the set Γ_ι of $\overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota}$ satisfying the following conditions:*

(i) $S_j x_j \geq 0$ *for all j with $i_j \in I^{re}$,*
(ii) *for each t with $i_t \in I^{im}$, if $x_t \neq 0$ and $t^{(-)} \neq 0$, then*

$$\sum_{t^{(-)}<j<t} \langle h_{i_t}, \alpha_{i_j} \rangle x_j < 0.$$

In addition, if $\langle h_{i_t}, \alpha_{i_j} \rangle x_j = 0$ $(t^{(-)} < j < t)$ for all $i_j \in I^{im}$, then there exists a p $(t^{(-)} < p < t)$ such that $i_p \in I^{re}$,

$$\langle h_{i_t}, \alpha_{i_p} \rangle x_p < 0 \text{ and } S_p x_p > 0.$$

Now, define a linear form $\xi^{(i)}$ $(i \in I^{re})$ on $\mathbf{Q}^\infty$ by

$$\xi^{(i)}(\overrightarrow{x}) = -\sum_{1 \leq j < \iota^{(i)}} \langle h_i, \alpha_{i_j} \rangle x_j - x^{(i)}_\iota = -\langle h_i, \lambda \rangle + \lambda^{(i)}(\overrightarrow{x}),$$

and set

$$\Theta^{(i)}_\iota = \{S_{j_l} \dots S_{j_1} \xi^{(i)} \mid l \geq 0, j_1, \dots, j_l \geq 1\}.$$

Then the *strict positivity assumption* for ι [17] is as follows:

$$\text{for any } \psi = \textstyle\sum_k \psi_k x_k \in \Big(\Theta_\iota \cup (\bigsqcup_{j\in I} \Theta^j_\iota)\Big) \setminus \{\xi^{(j)} \mid j \in I\},$$

$$\psi_k \geq 0 \quad \text{if} \quad k^{(-)} = 0.$$

Proposition 3.5. *For the sequence ι satisfying the strict positivity assumption, we have*

$$\widehat{S}_{j_l} \dots \widehat{S}_{j_1} x_{j_0} = S_{j_l} \dots S_{j_1} x_{j_0}$$

for any $l \geq 0$, $j_0, \dots, j_l \geq 1$, and

$$\widehat{S}_{j_l} \dots \widehat{S}_{j_1} \lambda^{(i)}(\overrightarrow{x}) = \langle h_i, \lambda \rangle + S_{j_l} \dots S_{j_1} \xi^{(i)}(\overrightarrow{x}) \tag{3.8}$$

for any $l \geq 0$, $j_1, \dots, j_l \geq 1$, if the left hand side of (3.8) *is nonzero.*

Proof. By the same argument as in the proof of Lemma 4.3 in [17], it is proved. So we omit it. □

By Proposition 3.5, we know that under the strict positivity assumption for ι, the set $\Gamma_\iota[\lambda]$ ($\lambda \in P^+$) has $\overrightarrow{0}$. Moreover, we have

Corollary 3.6. *Let ι be a sequence of indices satisfying* (2.5) *and λ be a dominant integral weight. Suppose that $\overrightarrow{0}$ belongs to $\Gamma_\iota[\lambda]$. Then we have*

$$\operatorname{Im}\Psi_\iota^\lambda = \left\{ \overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota} \;\middle|\; \begin{array}{l} \text{(i) } \psi(\overrightarrow{x}) \geq 0 \text{ for any } \psi \in \Theta_\iota, \\ \text{(ii) } \langle h_i, \lambda\rangle + \psi(\overrightarrow{x}) \geq 0 \text{ for any } \psi \in \Theta_\iota^{(i)} \ (i \in I^{re}), \\ \text{(iii) } \overrightarrow{x} \text{ satisfies the condition (ii) of } \Gamma_\iota[\lambda] \end{array} \right\}.$$

Corollary 3.7. *Assume that all elements of I are imaginary, that is, $I = I^{im}$. Then the image of the Kashiwara embedding $\operatorname{Im}\Psi_\iota^\lambda$ equals the set of $\overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota}[\lambda]$ satisfying* (3.3) *of Theorem* 3.1.

Proof. By Corollary 3.6, it suffices to consider the set $\Theta_\iota^{(i)}$. By a simple calculation, it is easy to see that the set $\Theta_\iota^{(i)}$ consists of the linear combinations of the coordinate forms x_j with nonnegative coefficients, which completes the proof. □

By Corollary 3.4 and Corollary 3.6, we have the following simple but important corollary.

Corollary 3.8. *Let I be an index set such that the cardinality of I^{re} is 1, and let ι be a sequence of indices in I satisfying* (2.5). *Then the image $\operatorname{Im}\Psi_\iota^\lambda$ of the crystal embedding is the set $\Gamma_\iota[\lambda]$ of $\overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota}[\lambda]$ satisfying the following conditions:*

(i) *$S_j x_j \geq 0$ for all j with $i_j \in I^{re}$,*

(ii) *$\langle h_i, \lambda\rangle + \psi(\overrightarrow{x}) \geq 0$ for any $\psi \in \Theta_\iota^{(i)}$ ($i \in I^{re}$),*

(iii) *for each t with $i_t \in I^{im}$, if $x_t \neq 0$, then*

$$\sum_{t^{(-)} < j < t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0 \qquad (t^{(-)} \neq 0),$$
$$-\langle h_{i_t}, \lambda\rangle + \sum_{1 \leq j < t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0 \qquad (t^{(-)} = 0).$$

In addition, if

$$\sum_{\substack{t^{(-)} < j < t \\ j \in I^{im}}} \langle h_{i_t}, \alpha_{i_j}\rangle x_j = 0 \qquad (t^{(-)} \neq 0),$$
$$-\langle h_{i_t}, \lambda\rangle + \sum_{\substack{1 \leq j < t \\ j \in I^{im}}} \langle h_{i_t}, \alpha_{i_j}\rangle x_j = 0 \qquad (t^{(-)} = 0),$$

then there exists a p ($t^{(-)} < p < t$) with $i_p \in I^{re}$ such that

$$\langle h_{i_t}, \alpha_{i_p}\rangle x_p < 0 \text{ and } \widehat{S}_p x_p(\overrightarrow{x}) > 0.$$

Set $W(\lambda) = \{\mu \in P \mid B(\lambda)_\mu \neq \emptyset\}$ and denote by $m_{\lambda,\mu}$ the weight multiplicity of μ in $B(\lambda)$. Any $\mu \in W(\lambda)$ is written as $\lambda - \sum_i m_i \alpha_i$ for $m_i \in \mathbf{Z}_{\geq 0}$.

Corollary 3.9. *For $\mu = \lambda - \sum_i m_i \alpha_i \in W(\lambda)$, the weight multiplicity of μ is given by*

$$m_{\lambda,\mu} = \#\left\{ \overrightarrow{x} \in \Gamma_\iota[\lambda] \;\middle|\; m_i = \sum_{i_k = i} x_k \text{ for any } i \in I \right\}.$$

4. Applications: Monster Lie algebras

In this section, we will give an explicit description of the image of the Kashiwara embedding for the generalized Kac-Moody algebras of ranks 2, 3 and Monster Lie algebras. First, consider the rank 2 case. Assume that $I = \{1, 2\}$ and $\iota = (\dots, 2, 1, 2, 1)$. Since we have the results for the Kac-Moody case [17, 18], we also assume that one of $1, 2 \in I$ is imaginary. Thanks to Corollary 3.3, it is enough to consider the case that $1 \in I^{im}$ and $2 \in I^{re}$. Set

$$\langle h_1, \alpha_1 \rangle = -a, \ \ \langle h_1, \alpha_2 \rangle = -b, \ \ \langle h_2, \alpha_1 \rangle = -c, \ \text{ and } \langle h_2, \alpha_2 \rangle = 2,$$

where $a, b, c \in \mathbf{Z}_{\geq 0}$.

Corollary 4.1. *Let λ be a dominant integral weight. Assume that $1 \in I^{im}$ and $2 \in I^{re}$. The image of the Kashiwara embedding $\operatorname{Im} \Psi_\iota^\lambda$ is given by the subset $\Gamma_\iota[\lambda]$ of $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geq 0, \iota}[\lambda]$ as follows:*

(a) *When $b = c = 0$,*

(i) $x_k = 0$ *for* $k \geq 3$,

(ii) $-x_2 + \langle h_2, \lambda \rangle \geq 0$,

(iii) $x_1 \neq 0$ *implies* $\langle h_1, \lambda \rangle > 0$.

(b) *When neither b nor c is 0,*

(i) $c x_1 - x_2 + \langle h_2, \lambda \rangle \geq 0$,

(ii) *for each* $k \geq 1$, $c x_{2k+1} - x_{2k+2} > 0$ *unless* $x_{2k+1} = x_{2k+2} = 0$,

(iii) *for each* $k \geq 1$, *if* $x_{2k+1} \neq 0$, *then* $x_{2k} > 0$,

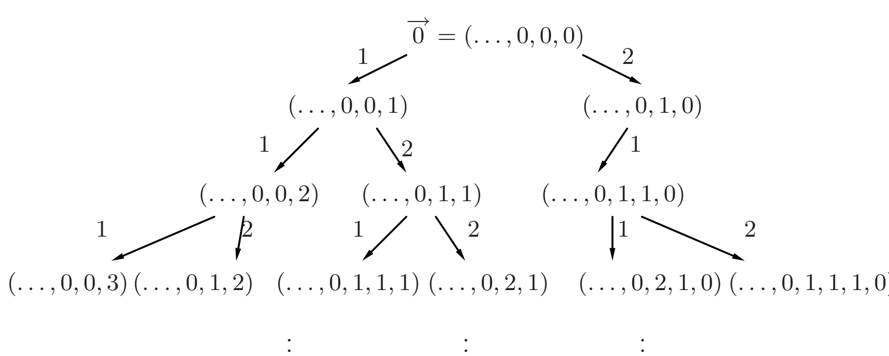
(iv) $x_1 \neq 0$ *implies* $\langle h_1, \lambda \rangle > 0$.

Proof. By Example 3.4 in [19] and Corollary 3.8, it suffices to consider

$$\xi^{(2)} = - \sum_{1 \leq j < 2} \langle h_2, \alpha_{i_j} \rangle x_j - x_2 = c x_1 - x_2 \ \text{ and } \ S_{j_l} \dots S_{j_1} \xi^{(2)}.$$

Since $S_{j_l} \dots S_{j_1} \xi^{(2)}$ is a linear combination of x_i's with nonnegative coefficients, the only meaningful element is $c x_1 - x_2$. Moreover, if x_1 is nonzero, by (3.3), we have $\langle h_1, \lambda \rangle > 0$. □

Example 4.2. If $b = 1$, $c = 2$, and $\langle h_1, \lambda \rangle = \langle h_2, \lambda \rangle = 1$, then the top part of $\operatorname{Im} \Psi_\iota^\lambda \cong B(\lambda)$ is as follows:



Now, let us apply Corollary 3.8 to the generalized Kac-Moody algebras of rank 3. Assume that $I = \{1, 2, 3\}$ and $\iota = (\dots, 1, 3, 2, 1)$. Assume that $1, 2 \in I^{im}$, $3 \in I^{re}$. Let A be a Borcherds-Cartan matrix

$$A = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & 2 \end{pmatrix},$$

where $a, b, c, d, e, f, g, h \in \mathbf{Z}_{\geq 0}$.

Corollary 4.3. *Let λ be a dominant integral weight. The image of the Kashiwara embedding $\operatorname{Im}\Psi_\iota^\lambda$ is given by the subset $\Gamma_\iota[\lambda]$ of $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geq 0,\iota}[\lambda]$ satisfying the following conditions:*

(i) $gx_1 + hx_2 - x_3 + \langle h_3, \lambda\rangle \geq 0$,

(ii) *for each* $k \geq 1$, $gx_{3k+1} + hx_{3k+2} - x_{3k+3} \geq 0$,

(iii) *for each* $k \geq 4$ *with* $i_k = 1, 2$, *if* $x_k \neq 0$, *then*

$$bx_{k-2} + cx_{k-1} > 0 \ \ (i_k = 1), \qquad fx_{k-2} + dx_{k-1} > 0 \ \ (i_k = 2).$$

Moreover, if $\langle h_{i_k}, \alpha_{i_j}\rangle x_j = 0$ *for* $j = k-1, k-2$ *with* $i_j \neq 3$, *then*

$$gx_{j+1} + hx_{j+2} - x_{j+3} > 0.$$

(iv) $x_1 \neq 0$ *and* $x_2 \neq 0$ *imply*

$$\langle h_1, \lambda\rangle > 0 \text{ and } -\langle h_2, \lambda\rangle - dx_1 < 0, \text{ respectively.}$$

Proof. By Theorem 7 in [19], and Corollary 3.6, it suffices to consider the set

$$\Theta_\iota^{(i)} = \{S_{j_l} \dots S_{j_1}\xi^{(i)} \,|\, l \geq 0, j_1, \dots, j_l \geq 1\}.$$

We know that $\xi^{(3)} = gx_1 + hx_2 - x_3$ and

$$S_1(\xi^{(3)}) = (bg + h)x_2 + (cg - 1)x_3 + gx_4,$$
$$S_2(\xi^{(3)}) = gx_1 + (fh - 1)x_3 + dhx_4 + hx_5.$$

Since $S_1(\xi^{(3)})$ and $S_2(\xi^{(3)})$ are linear combinations with nonnegative coefficients of x_i's, it is clear that $S_{j_l} \dots S_{j_1}\xi^{(i)} \geq 0$ $(l \geq 1)$ for all $\overrightarrow{x} \in \Gamma_\iota$. □

Let $I = \{-1 = -1_1\} \cup \{i_t |\, i \in \mathbf{N}, t = 1, \dots, c(i)\}$, where $c(i)$ is the coefficient of the elliptic modular function

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots = \sum_{i=-1}^{\infty} c(i)q^i.$$

We define $A = (a_{pq})_{p,q\in I}$ to be the matrix such that

$$a_{pq} = -(i + j) \quad \text{if } p = i_l,\ q = j_m \text{ for } 1 \leq l \leq c(i),\ 1 \leq m \leq c(j).$$

The associated generalized Kac-Moody algebra $\mathfrak{g}$ is called the *Monster Lie algebra*, and it played a crucial role in Borcherds' proof of the Moonshine conjecture [2]. More precisely, Borcherds derived the *twisted denominator identity* for the Monster Lie algebra with the action of the Monster, from which the replication formulae for the Thompson series follow.

In this paper, we deal with the corresponding quantum group $U_q(\mathfrak{g})$, which we call the *quantum Monster algebra*. Assume that

$$\iota = (\dots, -1, 3_{c(3)}, \dots, 3_1, 2_{c(2)}, \dots, 2_1, 1_{c(1)}, \dots, 1_1, -1,$$
$$2_{c(2)}, \dots, 2_1, 1_{c(1)}, \dots, 1_1, -1, 1_{c(1)}, \dots, 1_1, -1).$$

Let $I_{(-1)}$ be the set of positive integers k such that $i_k = -1$; i.e.,

$$I_{(-1)} = \{1\} \cup \{b(n) = nc(1) + (n-1)c(2) + \cdots + c(n) + n + 1 \mid n \in \mathbf{N}\},$$

and for any $n \geq 1$,

$$\sigma(n) = c(1) + \cdots + c(n).$$

Theorem 4.4. *Let λ be a dominant integral weight. Then the image of the Kashiwara embedding $\operatorname{Im}\Psi_\iota^\lambda$ is given by the subset $\Gamma_\iota[\lambda]$ of $\overrightarrow{x} \in \mathbf{Z}^\infty_{\geq 0,\iota}[\lambda]$ such that*

(i) $0 \leq x_1 \leq \langle h_{-1}, \lambda\rangle$, $x_{c(1)+2} = 0$, *and for each* $n \geq 1$,

$$\sum_{k=1}^{n} k(x_{b(n)+\sigma(k)+1} + \cdots + x_{b(n)+\sigma(k+1)}) - x_{b(n)+\sigma(n+1)+1} \geq 0,$$

(ii) *for each* $k \notin I_{(-1)}$, *if* $x_k \neq 0$ *and* $k^{(-)} \neq 0$, *then*

$$\sum_{k^{(-)}<j<k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j \neq 0.$$

Moreover, if $\langle h_{i_k}, \alpha_{i_j}\rangle x_j = 0$ *for all* $k^{(-)} < j < k$ *with* $j \notin I_{-1}$, *then there exists* $m \geq 1$ *such that* $k^{(-)} < b(m) < k$ *and*

$$\sum_{k=1}^{m} k(x_{b(m)+\sigma(k)+1} + \cdots + x_{b(m)+\sigma(k+1)}) - x_{b(m)+\sigma(m+1)+1} > 0,$$

(iii) *for each* $k \notin I_{-1}$, *if* $x_k \neq 0$ *and* $k^{(-)} = 0$, *then*

$$-\langle h_{i_k}, \lambda\rangle + \textstyle\sum_{1\leq j<k}\langle h_{i_k}, \alpha_{i_j}\rangle x_j < 0.$$

Moreover, if $\langle h_{i_k}, \lambda\rangle = 0$ *and* $\langle h_{i_k}, \alpha_{i_j}\rangle x_j = 0$ *for all* $1 \leq j < k$ *with* $j \notin I_{-1}$, *then there exists* $m \geq 1$ *such that* $1 \leq b(m) < k$ *and*

$$\sum_{k=1}^{m} k(x_{b(m)+\sigma(k)+1} + \cdots + x_{b(m)+\sigma(k+1)}) - x_{b(m)+\sigma(m+1)+1} > 0.$$

Proof. By Theorem 10 in [19] and Corollary 3.6, it is enough to consider the set

$$\Theta_\iota^{(i)} = \{S_{j_l} \dots S_{j_1}\xi^{(i)} \mid l \geq 0, j_1, \dots, j_l \geq 1\}.$$

We know that $\xi^{(-1)} = -x_1$ and $S_{j_l} \dots S_{j_1}\xi^{(i)} = -x_1$ for all l, which completes the proof. □

Finally, by Theorem 4.4, we have the following character of the highest weight module $V(\lambda)$ over $U_q(\mathfrak{g})$.

Corollary 4.5.

$$\operatorname{ch} V(\lambda) = \sum_{\overrightarrow{x}\in\Gamma_\iota[\lambda]} e^{\mathrm{wt}(\overrightarrow{x})} = \sum_{\overrightarrow{x}\in\Gamma_\iota[\lambda]} e^{\lambda - \sum_{j=1}^\infty x_j \alpha_{i_j}}.$$

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