

A SHORT PROOF OF HARA AND NAKAI'S THEOREM

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ABSTRACT. We give a short proof of the following theorem of Hara and Nakai: for a finitely bordered Riemann surface R , one can find an upper bound of the corona constant of R that depends only on the genus and the number of boundary components of R .

THE CORONA PROBLEM AND HARA-NAKAI'S THEOREM

For a given Riemann surface R , let $H^\infty(R)$ denote the uniform algebra of bounded analytic functions on R . To avoid pathological cases, we also assume that $H^\infty(R)$ separates the points in R ; i.e., for any $x_1, x_2 \in R$, $x_1 \neq x_2$, there exists a function $f \in H^\infty(R)$ with $f(x_1) \neq f(x_2)$.

We next consider the maximal ideal space $\mathcal{M}(R)$ of $H^\infty(R)$, and observe that each $M \in \mathcal{M}(R)$ can be identified with $\varphi_M : H^\infty(R) \rightarrow \mathbb{C}$, where φ_M is the complex homomorphism that has M as its kernel. This means that the maximal ideal space $\mathcal{M}(R)$ can be regarded as a subspace of the dual space $(H^\infty(R))^*$ of $H^\infty(R)$. Moreover, it also implies that we can equip $\mathcal{M}(R)$ with the Gelfand topology; thus $\mathcal{M}(R)$ can be thought of as a closed subspace of $(H^\infty(R))^*$ that is contained in the unit sphere. For the details, see for example Chap. V-1 of [10].

We know that each $\xi \in R$ corresponds to the maximal ideal

$$M_\xi = \{f \in H^\infty(R) : f(\xi) = 0\},$$

hence R can be naturally embedded into $\mathcal{M}(R)$ by the inclusion map $\iota : \xi \mapsto M_\xi$. Since we already provided $\mathcal{M}(R)$ with the Gelfand topology, one may ask the following: “is $\iota(R)$ dense in $\mathcal{M}(R)$ with respect to the Gelfand topology?” This is a famous question that is known as the *corona problem*, and we will say that the *corona theorem holds* for R if $\iota(R)$ is dense in $\mathcal{M}(R)$. Otherwise R is said to have *corona* ($= \mathcal{M}(R) \setminus \overline{\iota(R)}$). Note that the complex homomorphism φ_{M_ξ} associated with M_ξ is nothing but the point evaluation map $\lambda_\xi : f \mapsto f(\xi)$.

It is known that the corona theorem holds for R if and only if the following function theoretical statement holds (see Chap. 4 of [9], Chap. VIII of [10], or Chap. 12 of [7]): for given $F_1, \dots, F_n \in H^\infty(R)$ and $\delta \in (0, 1)$ such that

$$(1) \quad \delta \leq \max_{1 \leq j \leq n} |F_j(\zeta)| \leq 1 \quad \text{for all } \zeta \in R,$$

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there exist $G_1, \dots, G_n \in H^\infty(R)$ that satisfy the equation

$$F_1G_1 + F_2G_2 + \cdots + F_nG_n = 1.$$

We refer to F_1, \dots, F_n as *corona data* of index (n, δ) and G_1, \dots, G_n as *corona solutions* associated with the given corona data. The constant

$$C(n, \delta, R) := \sup \inf \max\{\|G_1\|_\infty, \dots, \|G_n\|_\infty\}$$

is called the “corona constant” of R , where the supremum is over all corona data satisfying (1) and the infimum is over all possible corona solutions associated with each set of corona data.

As usual, we interpret the infimum of an empty set as infinity; thus if a Riemann surface R has corona, the corona constant $C(n, \delta, R)$ must be infinite for some index (n, δ) . But what about the converse? If the corona theorem holds for R , is $C(n, \delta, R)$ finite for all indices (n, δ) ? The answer to this question is still unknown for general Riemann surfaces, but the answer is positive at least for finitely bordered Riemann surfaces, as the following theorem shows.

Theorem 1 (Hara and Nakai, [12]). *For a given finitely bordered Riemann surface R , let $g(R)$ denote the genus of R and $b(R)$ denote the number of boundary components of R . Then for each given index (n, δ) and numbers $g \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$, we have*

$$\sup_{R \in \mathfrak{R}(g, b)} C(n, \delta, R) < \infty,$$

where $\mathfrak{R}(g, b)$ is the collection of Riemann surfaces with $g(R) = g$ and $b(R) = b$.

The purpose of our paper is to give a short proof for this theorem. Note that Theorem 1 implies that one can find an upper bound of the corona constant of a finitely bordered Riemann surface that depends only on the index, the genus of R and the number of boundary components of R . The case $g = 0$ was proved by Gamelin in [8], and the case $g = 0$ and $b = 1$ is nothing but the famous Carleson’s corona theorem for the unit disc [6].

There are various planar domains and Riemann surfaces for which the corona theorem holds ([6], [8], [11], [14], [17], [2], [4], [5], and more). On the other hand, a relatively small number of Riemann surfaces are known to have corona. The first such example was constructed by Cole (Chap. 4 of [9]), which was recently reconstructed in a simpler way in [16], and other Riemann surfaces that have corona can be found in [3] and [13].

The corona problem for general planar domains is still open, and the answer is also unknown for a polydisc or a unit ball in \mathbb{C}^n , $n \geq 2$.

PROOF OF THEOREM 1

Our proof is based on the following three theorems and the Carleson’s corona theorem for the unit disc.

Theorem 2 (Mitsuru Nakai, [15]). *Let R and R' be Riemann surfaces and let $f : R' \rightarrow R$ be an m -sheeted branched covering map for some $m < \infty$. Then the corona theorem holds for R' if and only if it holds for R .*

In Theorem 2 Nakai considered only Riemann surfaces, that is, he considered only *connected* surfaces R and R' . However, one can check that the argument is

still valid even when the surfaces are not connected; i.e., in Theorem 2 one can replace R and R' by disjoint unions of Riemann surfaces.

Let \mathbb{D} denote the unit disc. The next argument we need for the proof of Theorem 1 is the following.

Theorem 3 (Ahlfors, [1]). *Suppose R is a finitely bordered Riemann surface with $g(R) = g$ and $b(R) = b$. Then there exists an m -sheeted branched covering map $f : R \rightarrow \mathbb{D}$, called the Ahlfors map, such that $b \leq m \leq 2g + b$.*

The last ingredient of our recipe is the following statement:

Theorem 4. *Let $\{R_j\}$ be a sequence of Riemann surfaces. Then*

$$\sup_j C(n, \delta, R_j) < \infty$$

for every index (n, δ) if and only if the corona theorem holds for $\bigsqcup_j R_j$, the disjoint union of R_j .

Proof. This theorem is essentially Lemma 3.1 of [8]. In fact, in [8] the theorem was stated only for planar domains, but one can easily check that the proof is valid for our case. \square

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Suppose Theorem 1 is not true. Then there exist an index (n_0, δ_0) and a sequence of finitely bordered Riemann surfaces $\{R_j\}$ with $g(R_j) = g$ and $b(R_j) = b$, $j = 1, 2, \dots$, such that

$$(2) \quad C(n_0, \delta_0, R_j) \rightarrow \infty$$

as $j \rightarrow \infty$. Furthermore, by Theorem 3 we can find m_j -sheeted branched covering maps $h_j : R_j \rightarrow D_j := \{z : |z - 3j| < 1\}$ with $b \leq m_j \leq 2g + b$ for all j . However, by passing to a subsequence if necessary, we may assume that all the m_j 's are the same, that is, there exists a constant m such that $m = m_j$ for all j . Now let $D = \bigcup_j D_j$ and $\mathcal{R} = \bigsqcup_j R_j$.

Carleson's corona theorem for the unit disc [6] (Theorem 1 for the case $g = 0$ and $b = 1$) implies that $\sup_j C(n, \delta, D_j) < \infty$ for any index (n, δ) , thus the corona theorem for D follows from Theorem 4. Then by Nakai's theorem (Theorem 2), we see that the corona theorem also holds for \mathcal{R} , because the map $h : \mathcal{R} \rightarrow D$ defined by $h|_{R_j} = h_j$ is an m -sheeted branched covering. According to Theorem 4, however, $\mathcal{R} = \bigsqcup_j R_j$ must have corona, because $\sup_j C(n_0, \delta_0, R_j) = \infty$ by (2). This contradiction completes the proof of Theorem 1. \square

We believe that what makes our proof significantly shorter than the proof of Hara and Nakai is the idea of applying Hara's theorem (Theorem 2) to *disjoint* sets, which is in fact due to Gamelin as Theorem 4 indicates. The rest of the proof is not very far from the original in the sense that both proofs use Ahlfors maps and Nakai's theorem [15] (Theorems 3 and 2 above).

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