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Research Article

Additive Functional Inequalities in Banach Modules

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We investigate the following functional inequality $\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\| \leq \|f(x+z)\|$ in Banach modules over a C^*-algebra and prove the generalized Hyers-Ulam stability of linear mappings in Banach modules over a C^*-algebra in the spirit of the Th. M. Rassias stability approach. Moreover, these results are applied to investigate homomorphisms in complex Banach algebras and prove the generalized Hyers-Ulam stability of homomorphisms in complex Banach algebras.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. The Hyers theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th. M. Rassias approach. Th. M. Rassias [6] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [7], following the same approach as in Th. M. Rassias [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [7] as well as by Th. M. Rassias and Šemrl [8] that one cannot prove a Th. M. Rassias-type theorem when $p = 1$. J. M. Rassias [9] followed the innovative approach of the Th. M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [10–18]).

Gilányi [19] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y). \tag{1.2}$$

See also [20]. Fechner [21] and Gilányi [22] proved the generalized Hyers-Ulam stability of the functional inequality (1.1).

In this paper, we investigate an A-linear mapping associated with the functional inequality

$$\|2f(x) + 2f(y) + 2f(z) - f(x+y) - f(y+z)\| \leq \|f(x+z)\| \tag{1.3}$$

and prove the generalized Hyers-Ulam stability of A-linear mappings in Banach A-modules associated with the functional inequality (1.3). These results are applied to investigate homomorphisms in complex Banach algebras and prove the generalized Hyers-Ulam stability of homomorphisms in complex Banach algebras.

2. Functional inequalities in Banach modules over a C^*-algebra

Throughout this section, let A be a unital C^*-algebra with unitary group $U(A)$ and unit e and B a unital C^*-algebra. Assume that X is a Banach A-module with norm $\|\cdot\|_X$ and that Y is a Banach A-module with norm $\|\cdot\|_Y$.

Lemma 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\|2uf(x) + 2f(y) + 2f(z) - f(ux+y) - f(y+z)\|_Y \leq \|f(ux+z)\|_Y \tag{2.1}$$

for all $x, y, z \in X$ and all $u \in U(A)$. Then f is A-linear.

Proof. Letting $x = y = z = 0$ and $u = e \in U(A)$ in (2.1), we get

$$\|4f(0)\|_Y \leq \|f(0)\|_Y. \tag{2.2}$$

So $f(0) = 0$.

Letting $u = e \in U(A)$, $y = 0$ and $z = -x$ in (2.1), we get

$$\|f(x) + f(-x)\|_Y \leq \|f(0)\|_Y = 0 \tag{2.3}$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x$ and $u = e \in U(A)$ in (2.1), we get

$$\begin{aligned}\|2f(x) + 2f(y) + 2f(-x) - f(x+y) - f(y-x)\|_Y &= \|2f(y) - f(y+x) - f(y-x)\|_Y \\ &\le \|f(0)\|_Y \\ &= 0\end{aligned} \tag{2.4}$$

for all $x, y \in X$. So $f(y+x) + f(y-x) = 2f(y)$ for all $x, y \in X$. Thus

$$f(x+y) = f(x) + f(y) \tag{2.5}$$

for all $x, y \in X$.

Letting $z = -ux$ and $y = 0$ in (2.1), we get

$$\begin{aligned}\|2uf(x) - 2f(ux)\|_Y &= \|2uf(x) + 2f(-uz)\|_Y \\ &\le \|f(0)\|_Y \\ &= 0\end{aligned} \tag{2.6}$$

for all $x \in X$ and all $u \in U(A)$. Thus

$$f(uz) = uf(z) \tag{2.7}$$

for all $u \in U(A)$ and all $z \in X$. Now, let $a \in A (a \neq 0)$ and M an integer greater than $4|a|$. Then $|a/M| < 1/4 < 1 - 2/3 = 1/3$. By [23, Theorem 1], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3(a/M) = u_1 + u_2 + u_3$. So by (2.7)

$$\begin{aligned}f(ax) &= f\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) \\ &= M \cdot f\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) \\ &= \frac{M}{3} f\left(3\frac{a}{M}x\right) \\ &= \frac{M}{3} f(u_1x + u_2x + u_3x) \\ &= \frac{M}{3}\left(f(u_1x) + f(u_2x) + f(u_3x)\right) \\ &= \frac{M}{3}(u_1 + u_2 + u_3) f(x) \\ &= \frac{M}{3} \cdot 3\frac{a}{M} f(x) \\ &= af(x)\end{aligned} \tag{2.8}$$

for all $x \in X$. So $f : X \to Y$ is A-linear, as desired. □

Now, we prove the generalized Hyers-Ulam stability of A-linear mappings in Banach A-modules.

Theorem 2.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|2uf(x) + 2f(y) + 2f(z) - f(ux + y) - f(y + z)\|_Y \le \|f(ux + z)\|_Y + \theta(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r) \tag{2.9}$$

for all $x, y, z \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{3\theta}{2^r - 2}\|x\|_X^r \tag{2.10}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$. So $f(0) = 0$.

Letting $u = e \in U(A)$, $y = x$ and $z = -x$ in (2.9), we get

$$\begin{aligned} \|2f(x) - f(2x)\|_Y &= \|2f(x) + f(-2x)\|_Y \\ &\le 3\theta\|x\|_X^r \end{aligned} \tag{2.11}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y \le \frac{3}{2^r}\theta\|x\|_X^r \tag{2.12}$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_Y \\ &\le \frac{3}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|_X^r \end{aligned} \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.13) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.14}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.10).

It follows from (2.9) that

$$\begin{aligned}
&\|2uL(x) + 2L(y) + 2L(z) - L(ux + y) - L(y + z)\|_Y \\
&\quad = \lim_{n\to\infty} 2^n \left\| 2uf\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) + 2f\left(\frac{z}{2^n}\right) - f\left(\frac{ux+y}{2^n}\right) - f\left(\frac{y+z}{2^n}\right) \right\| \\
&\quad \le \lim_{n\to\infty} 2^n \left\| f\left(\frac{ux+z}{2^n}\right)\right\|_Y + \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}} \left(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r\right) \\
&\quad = \|L(ux + z)\|_Y
\end{aligned} \tag{2.15}$$

for all $x, y, z \in X$ and all $u \in U(A)$. So

$$\|2uL(x) + 2L(y) + 2L(z) - L(ux + y) - L(y + z)\|_Y \le \|L(ux + z)\|_Y \tag{2.16}$$

for all $x, y, z \in X$ and all $u \in U(A)$. By Lemma 2.1, the mapping $L : X \to Y$ is A-linear.

Now, let $T : X \to Y$ be another A-linear mapping satisfying (2.10). Then, we have

$$\begin{aligned}
\|L(x) - T(x)\|_Y &= 2^n \left\| L\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y \\
&\le 2^n \left(\left\| L\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y \right) \\
&\le \frac{6\cdot 2^n}{(2^r - 2)2^{nr}} \theta \|x\|_X^r,
\end{aligned} \tag{2.17}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of L. Thus the mapping $L : X \to Y$ is a unique A-linear mapping satisfying (2.10). □

Theorem 2.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.9). *Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \le \frac{3\theta}{2 - 2^r} \|x\|_X^r \tag{2.18}$$

for all $x \in X$.

Proof. It follows from (2.11) that

$$\left\| f(x) - \frac{1}{2} f(2x)\right\|_Y \le \frac{3}{2}\theta \|x\|_X^r \tag{2.19}$$

for all $x \in X$. Hence

$$\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_Y \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_Y \leq \frac{3}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \theta \|x\|_X^r \tag{2.20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.20) that the sequence $\{(1/2^n) f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(1/2^n) f(2^n x)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \tag{2.21}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.18).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.4. *Let $r > 1/3$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|2uf(x) + 2f(y) + 2f(z) - f(ux + y) - f(y + z)\|_Y \leq \|f(ux + z)\|_Y + \theta \cdot \|x\|_X^r \cdot \|y\|_X^r \cdot \|z\|_X^r \tag{2.22}$$

for all $x, y, z \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\theta}{8^r - 2} \|x\|_X^{3r} \tag{2.23}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$. So $f(0) = 0$.

Letting $u = e \in U(A)$, $y = x$, and $z = -x$ in (2.22), we get

$$\begin{aligned} \|2f(x) - f(2x)\|_Y &= \|2f(x) + f(-2x)\|_Y \\ &\leq \theta \|x\|_X^{3r} \end{aligned} \tag{2.24}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_Y \leq \frac{\theta}{8^r} \|x\|_X^{3r} \tag{2.25}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_Y \\ &\le \frac{\theta}{8^r}\sum_{j=l}^{m-1}\frac{2^j}{8^{rj}}\|x\|_X^{3r}\end{aligned} \tag{2.26}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.26) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.27}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.26), we get (2.23).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.5. *Let $r < 1/3$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.22)*. Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \le \frac{\theta}{2 - 8^r}\|x\|_X^{3r} \tag{2.28}$$

for all $x \in X$.

Proof. It follows from (2.24) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \le \frac{\theta}{2}\|x\|_X^{3r} \tag{2.29}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\|_Y \\ &\le \frac{\theta}{2}\sum_{j=l}^{m-1}\frac{8^{rj}}{2^j}\|x\|_X^{3r}\end{aligned} \tag{2.30}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.30) that the sequence $\{(1/2^n)f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence

$\{(1/2^n)f(2^nx)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \tag{2.31}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.30), we get (2.28).

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Generalized Hyers-Ulam stability of homomorphisms in Banach algebras

Throughout this section, let A and B be complex Banach algebras.

Proposition 3.1. *Let $f : A \to B$ be a multiplicative mapping such that*

$$\|2\mu f(x) + 2f(y) + 2f(z) - f(\mu x + y) - f(y + z)\| \le \|f(\mu x + z)\| \tag{3.1}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then f is an algebra homomorphism.

Proof. Every complex Banach algebra can be considered as a Banach module over $\mathbb{C}$. By Lemma 2.1, the mapping $f : A \to B$ is a $\mathbb{C}$-linear. So the multiplicative mapping $f : A \to B$ is an algebra homomorphism. □

Now, we prove the generalized Hyers-Ulam stability of homomorphisms in complex Banach algebras.

Theorem 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be an odd multiplicative mapping such that*

$$\|2\mu f(x) + 2f(y) + 2f(z) - f(\mu x + y) - f(y + z)\| \le \|f(\mu x + z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{3.2}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{3\theta}{2^r - 2}\|x\|^r \tag{3.3}$$

for all $x \in A$.

Proof. By Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (3.3). The mapping $H : A \to B$ is given by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.4}$$

for all $x \in A$.

Since $f : A \to B$ is multiplicative,

$$\begin{aligned} H(xy) &= \lim_{n\to\infty} 4^n f\left(\frac{xy}{4^n}\right) \\ &= \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)\cdot 2^n f\left(\frac{y}{2^n}\right) \\ &= H(x)H(y) \end{aligned} \tag{3.5}$$

for all $x, y \in A$. Thus the mapping $H : A \to B$ is an algebra homomorphism satisfying (3.3). □

Theorem 3.3. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be an odd multiplicative mapping satisfying* (3.2). *Then there exists a unique algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\| \le \frac{3\theta}{2 - 2^r}\|x\|^r \tag{3.6}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.2. □

Theorem 3.4. *Let $r > 1/3$ and θ be nonnegative real numbers, and let $f : A \to B$ be an odd multiplicative mapping such that*

$$\|2\mu f(x) + 2f(y) + 2f(z) - f(\mu x + y) - f(y + z)\| \le \|f(\mu x + z)\| + \theta\cdot\|x\|^r\cdot\|y\|^r\cdot\|z\|^r \tag{3.7}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{\theta}{8^r - 2}\|x\|^{3r} \tag{3.8}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. □

Theorem 3.5. *Let $r < 1/3$ and θ be positive real numbers, and let $f : A \to B$ be an odd multiplicative mapping satisfying* (3.7). *Then there exists a unique algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\| \le \frac{\theta}{2 - 8^r}\|x\|^{3r} \tag{3.9}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.5 and 3.2. □

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