Density Dependence of Transport Coefficients from Holographic Hydrodynamics

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We study the transport coefficients of Quark-Gluon-Plasma in finite temperature and finite baryon density. We consider AdS/QCD of charged AdS black hole background with bulk-filling branes identifying the $U(1)$ charge as the baryon number. Using Reissner-Nordström-AdS background, Green functions are explicitly obtained. We calculate the diffusion constant, the shear viscosity and the thermal conductivity, and plot their density and temperature dependences. Hydrodynamic relations between those are shown to hold exactly. The diffusion constant and the shear viscosity are decreasing as a function of density for fixed total energy. For fixed temperature, the fluid becomes less diffusible and more viscous for larger baryon density.

§1. Introduction

After the discovery of consistency on the ratio of the viscosity to the entropy density (η/s)[1)] in AdS/CFT correspondence[2)–4)] and RHIC (Relativistic Heavy Ion Collider) experiment, much attention has been drawn to the calculational scheme provided by string theory. Even some attempt has been made to map the entire process of RHIC experiment in terms of the gravity dual.[5)] The way to include a chemical potential in the theory was figured out in the context of D4D8$\overline{\text{D8}}$ setup.[6),7)] Phases of these theories were discussed in D3/D7 setup and new phases were reported where instability due to the strong attraction is a feature.[8)–10)]

Although QCD and $\mathcal{N} = 4$ SYM are different, it is expected that some of the properties are shared by the two theories. It is an interesting question to ask how much one can learn by studying the various versions of AdS/CFT correspondence. The relevance is based on the universality of low energy physics. In this respect, the hydrodynamic limit is interesting since such limit can be shared by many theories in spite of the differences in UV limit.

The calculation scheme of transport coefficients is to use Kubo formula, which gives a relation to the low energy limit of Wightman Green functions. In AdS/CFT correspondence, one calculates the retarded Green function which is related to the Wightman function by fluctuation-dissipation theorem. Such scheme has been de-

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veloped in a series of papers.[11)–15)]

For charged case, the calculations are more involved and corresponding works have been done partially by various groups.[16)–19)] For generic STU black hole,[20),21)] the hydrodynamic calculation was performed for tensor type perturbations in the paper 16). In the paper 17), apart from the tensor type perturbations in generic STU black hole, vector type perturbations for $(1,0,0)$ charged case in STU black hole have been analyzed. From the tensor type perturbations, the ratio (η/s) was found to be $1/(4\pi)$ analytically in these works.[16),17)] In the paper 18), Reissner-Nordström-Anti-deSitter (RN-AdS) background was considered and it was shown numerically that the ratio (η/s) was $1/(4\pi)$ with very good accuracy by using the vector type perturbations. Later, it was also proven that the ratio was universal in more general setup.[19)]

In this paper, we perform hydrodynamic calculation analytically for RN-AdS black hole, which corresponds to the $(1,1,1)$ charged STU black hole. We emphasize that RN-AdS black hole and STU black hole with $(1,0,0)$ charge could have different properties due to their different phase diagrams.[22)] In the metric perturbations, both of the vector and tensor type perturbations are considered. Master equations for the decoupled modes are worked out, so that we can obtain Green functions explicitly. From the vector type perturbations, vector modes of Maxwell field (as well as the vector modes of metric) have diffusion pole, contrary to the chargeless case. The diffusion constant is calculated explicitly. In the tensor type perturbations, the ratio $(\eta/s) = 1/(4\pi)$ is also confirmed to be exactly same as the previous result in the papers 16)–18). We observe that the diffusion constant and the shear viscosity decrease as we increase the charge with fixed total system energy (or equivalently the black hole mass), while the shear viscosity increases for fixed temperature. The thermal conductivity is also calculated from the tensor type perturbations.

The charge in RN-AdS black hole is usually regarded as R-charge of SUSY.[26)] We here consider another interpretation in the following way: One can introduce quarks and mesons by considering the bulk-filling branes in AdS$_5$ space. The overall $U(1)$ of the flavor branes is identified as the baryon charge. The $U(1)$ charge in this model minimally couples to the bulk gravity since the bulk and the world volume of brane are identified. RN-AdS metric can be considered as the consequence of the back reaction of the AdS black hole to the charge. Therefore the $U(1)$ charge in RN-AdS can be identical to the baryon charge. As a result, we can calculate the transport coefficients in the presence of the baryon density. (See also 27).) The density dependence of the transport coefficients can be plotted in this paper.

One can give an explanation of hydrodynamic mode in meson physics. In our interpretation, the Maxwell fields are the fluctuations of bulk-filling branes, therefore they should be interpreted as master fields of the mesons. Then hydrodynamic modes are lowest lying massless meson spectrum. In terms of brane embedding picture, this massless-ness is due to the touching of the brane on the black hole horizon. Near the horizon, the tension of the brane is zero due to the metric factor and it can lead to the massless fluctuation. Then the massless spectrum cannot go far from the horizon in radial direction. In this picture, hydrodynamic nature is closely related to the near horizon behavior of the branes.

This paper is organized as follows: In §2, we introduce RN-AdS black hole and review correlation function calculation at finite temperature in AdS/CFT correspondence. In §3, a formulation on the metric and the gauge perturbations in RN-AdS background is given. We then calculate the correlators in hydrodynamic regime and obtain the diffusion pole in §4. In §5, the shear viscosity is calculated via Kubo formula. We also show that the result is consistent with the hydrodynamic relation of diffusion constant and the viscosity. The thermal conductivity is also calculated in this section. Conclusions and discussions are given in §6. Three appendices are given to provide the details of the calculations.

§2. Basic setup

2.1. *Minkowskian correlators in AdS/CFT correspondence*

Before introducing RN-AdS black hole, we briefly summarize Minkowskian correlators in AdS/CFT correspondence. We follow the prescription proposed in 11). Let us consider fluctuations of fields which satisfy equations of motion at the linearized order. We work on the five-dimensional background,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{uu}(du)^2, \tag{2.1}$$

where x^μ and u are the four-dimensional and the radial coordinates, respectively. We refer the boundary as $u = 0$ and the horizon as $u = 1$. A solution of the equation of motion may be given,

$$\phi(u, x) = \int \frac{d^4k}{(2\pi)^4} \; e^{ikx} f_k(u)\phi_0(k), \tag{2.2}$$

where $f_k(u)$ is normalized such that $f_k(0) = 1$ at the boundary. An on-shell action might be reduced to surface terms by using the equation of motion,

$$S[\phi_0] = \int \frac{d^4k}{(2\pi)^4} \phi_0(-k)\mathcal{G}(k, u)\phi_0(k)\bigg|_{u=0}^{u=1}. \tag{2.3}$$

Here, the function $\mathcal{G}(k, u)$ can be written in terms of $f_{\pm k}(u)$ and $\partial_u f_{\pm k}(u)$, for example, for a scalar field,

$$\mathcal{G}(k, u) = K\sqrt{-g}g^{uu}f_{-k}(u)\partial_u f_k(u),$$

with some constant K. The direct generalization of AdS/CFT correspondence, or Gubser-Klebanov-Polyakov/Witten relation,[3),4)] to Minkowski spacetime gives the relation,

$$\left\langle e^{i\int \phi_0 \mathcal{O}} \right\rangle = e^{iS[\phi_0]},$$

where the operator $\mathcal{O}$ is defined in the boundary field theory. From this relation, one may obtain a Green function by taking second derivative of the action with respect to the boundary value of the field,

$$G(k) = -\mathcal{G}(k, u)\bigg|_{u=0}^{u=1} - \mathcal{G}(-k, u)\bigg|_{u=0}^{u=1}.$$

However, this quantity is real and cannot be a retarded Green function. This can be seen as follows. The imaginary part of $\mathcal{G}(k, u)$ is proportional to a conserved flux. Then, its contributions at the boundary $u = 0$ and at the horizon $u = 1$ cancel completely. Even if one neglects the contribution from the horizon, $G(k)$ is still real. The reality condition of the equation of motion implies $\mathcal{G}(-k, u) = \mathcal{G}^*(k, u)$, and the imaginary part of $G(k)$ vanishes again. Therefore we should impose the "retarded" condition to the Green function.

Son and Starinets proposed that the retarded (advanced) Green function is given by

$$G^{\mathrm{R}}(k) = 2\mathcal{G}(k, u)\Big|_{u=0}, \qquad (2\cdot4)$$

with incoming (outgoing) boundary condition at the horizon. Generally, the contribution at the horizon is oscillating and averaged out to zero. In order to avoid this, we have to consider incoming or outgoing boundary condition. Taking away the contribution at the horizon, we obtain $\mathcal{G}(k, u)$ with a non-zero imaginary part. Physics at the horizon affects the Green function only through the boundary condition. In general, there are several fields in the model. We write the Green function as $G_{ij}(k)$, where indices i and j distinguish these fields. The surface terms are always associated with equations of motion. We choose the former index to indicate the field whose equation of motion is associated with the Green function.

In this paper, we work in RN-AdS background and consider its perturbations so that essential ingredients are perturbed metric field and $U(1)$ gauge field. Here we define the precise form of the retarded Green function which we discuss later:

$$G_{\mu\nu\ \rho\sigma}(k) = -i \int d^4x \ e^{-ikx} \theta(t) \langle [T_{\mu\nu}(x),\ T_{\rho\sigma}(0)] \rangle,$$

$$G_{\mu\nu\ \rho}(k) = -i \int d^4x \ e^{-ikx} \theta(t) \langle [T_{\mu\nu}(x),\ J_{\rho}(0)] \rangle,$$

$$G_{\mu\ \nu}(k) = -i \int d^4x \ e^{-ikx} \theta(t) \langle [J_{\mu}(x),\ J_{\nu}(0)] \rangle, \qquad (2\cdot5)$$

where the operators $T_{\mu\nu}(x)$ and $J_{\mu}(x)$ are energy-momentum tensor and $U(1)$ current which couple to the metric and the gauge fields, respectively.

2.2. *Reissner-Nordström-AdS background*

In this paper, we consider N_c D3-branes and N_f D7-branes, and treat the D3-branes as a gravitational background. The D7-branes are wrapping on S^3 of S^5, and we neglect this S^3 dependence. We do not consider the perpendicular fluctuations of D7-branes, and the effective action then becomes that for five-dimensional gauge theory. If the D7-branes touch the D3-branes, the D7-branes fill the AdS$_5$ completely. The induced metric on the D7-brane is identical to the AdS bulk metric. This model corresponds to $\mathcal{N} = 4$ SYM with massless quarks. If we introduce the baryon charge at the boundary theory, its chemical potential is identified as the tail of the $U(1)$ gauge potential on the flavor brane.[6),7)]   (See also 8)–10), 28), 29) for later development.) We consider the phenomenological model taking only AdS$_5$ part and

neglecting S^5 part. Then there is no way to distinguish the bulk field and the brane field. The baryon charge and the R-charge have the same description in terms of the $U(1)$ gauge field living in the AdS$_5$ space. A charged black hole (RN-AdS black hole) is then induced by its back reaction. This corresponds to the $\mathcal{N} = 4$ SYM in finite temperature with finite baryon density. It is the case that we consider in this paper.

We now describe the gravity side of this system which is given by ten-dimensional Einstein gravity and the effective action of D7-branes. Due to the effect of D3-branes, the gravitational background becomes AdS spacetime. Since we treat the system classically, we can reduce it to five dimensions. The baryon current corresponds to $U(1)$ gauge field[*] on D7-branes. The effective action of this gauge field is given by Dirac-Born-Infeld action[**]

$$S_{\text{D7}} = -\frac{1}{4e^2} \int d^5 x \sqrt{-g} \, \text{Tr} \left(\mathcal{F}_{mn} \mathcal{F}^{mn} \right), \qquad (2\cdot6)$$

where the gauge coupling constant e is given by

$$\frac{l}{e^2} = \frac{N_c N_f}{(2\pi)^2}, \qquad (2\cdot7)$$

with l the radius of the AdS spacetime. We pick up an overall $U(1)$ part of this gauge field to consider the baryon current at the boundary. Together with the gravitation part, we arrive at the following action which is our starting point:

$$S[g_{mn}, \mathcal{A}_m] = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-g} \left(R - 2\Lambda \right) - \frac{1}{4e^2} \int d^5 x \sqrt{-g} \mathcal{F}_{mn} \mathcal{F}^{mn}, \qquad (2\cdot8)$$

where we denote the gravitation constant and the cosmological constant as $\kappa^2 = 8\pi G_5$ and Λ, respectively. The $U(1)$ gauge field strength is given by $\mathcal{F}_{mn}(x) = \partial_m \mathcal{A}_n(x) - \partial_n \mathcal{A}_m(x)$. The gravitation constant is related to the gauge theory quantities by

$$\frac{l^3}{\kappa^2} = \frac{N_c^2}{4\pi^2}. \qquad (2\cdot9)$$

Suppose we have baryon charge Q. Due to this charge, the AdS background should be modified to RN-AdS.[***],[27] This should be identified to the source of $U(1)$ charge on the brane hence on the bulk. Then we can relate it to the parameter in RN black hole solution by considering the full solution to the equation of motion,

$$R_{mn} - \frac{1}{2} g_{mn} R + g_{mn} \Lambda = \kappa^2 T_{mn}, \qquad (2\cdot10)$$

where energy-momentum tensor $T_{mn}(x)$ is given by

$$T_{mn} = \frac{1}{e^2} \left(\mathcal{F}_{mk} \mathcal{F}_{nl} g^{kl} - \frac{1}{4} g_{mn} \mathcal{F}_{kl} \mathcal{F}^{kl} \right). \qquad (2\cdot11)$$

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[*] This gauge field decouples and does not appear in QCD side.

[**] The indices m and n run through five-dimensional spacetime while μ and ν would be reserved for four-dimensional Minkowski spacetime. Their spatial coordinates are labeled by i and j.

[***] RN-AdS black hole corresponds to $(1, 1, 1)$ charged case in STU black hole which is a solution of SUGRA.

An equation of motion for the gauge field $\mathcal{A}_m(x)$ gives Maxwell equation,

$$\nabla_m \mathcal{F}^{mn} = \frac{1}{\sqrt{-g}} \partial_m \left(\sqrt{-g} g^{mk} g^{nl} \left(\partial_k \mathcal{A}_l - \partial_l \mathcal{A}_k \right) \right) = 0. \tag{2·12}$$

Here we assumed that there is no electromagnetic source outside the black hole. One can confirm that the following metric and gauge potential satisfy the equations of motion (2·10) and (2·12),

$$ds^2 = \frac{r^2}{l^2} \left(-f(r)(dt)^2 + \sum_{i=1}^{3} (dx^i)^2 \right) + \frac{l^2}{r^2 f(r)} (dr)^2, \tag{2·13a}$$

$$\mathcal{A}_t = -\frac{Q}{r^2} + \mu, \tag{2·13b}$$

with

$$f(r) = 1 - \frac{ml^2}{r^4} + \frac{q^2 l^2}{r^6}, \qquad \Lambda = -\frac{6}{l^2},$$

if and only if q is related to the Q by

$$e^2 = \frac{2Q^2}{3q^2} \kappa^2. \tag{2·14}$$

It should be noted that a ratio of the gauge coupling constant e^2 to the gravitation constant κ^2 is

$$\frac{e^2}{\kappa^2} = \frac{N_c}{N_f} l^{-2}. \tag{2·15}$$

Since the gauge potential $\mathcal{A}_t(x)$ must vanish at the horizon, the charge Q and the chemical potential μ are related. The parameters m and q are the mass and charge of AdS space, respectively. This is nothing but Reissner-Nordström-Anti-deSitter (RN-AdS) background in which we are interested throughout this paper.

The horizons of RN-AdS black hole are located at the zero for $f(r)$,[*]

$$f(r) = 1 - \frac{ml^2}{r^4} + \frac{q^2 l^2}{r^6} = \frac{1}{r^6} \left(r^2 - r_+^2 \right) \left(r^2 - r_-^2 \right) \left(r^2 - r_0^2 \right), \tag{2·16}$$

where their explicit forms of the horizon radiuses are given by

$$r_+^2 = \left(\frac{m}{3q^2} \left(1 + 2\cos\left(\frac{\theta}{3} + \frac{4}{3}\pi \right) \right) \right)^{-1}, \tag{2·17a}$$

$$r_-^2 = \left(\frac{m}{3q^2} \left(1 + 2\cos\left(\frac{\theta}{3} \right) \right) \right)^{-1}, \tag{2·17b}$$

$$r_0^2 = \left(\frac{m}{3q^2} \left(1 + 2\cos\left(\frac{\theta}{3} + \frac{2}{3}\pi \right) \right) \right)^{-1}, \tag{2·17c}$$

---

[*] In order to define the horizon, the charge q must satisfy a relation $q^4 \leq 4m^3 l^2/27$.

with

$$\theta = \arctan\left(\frac{3\sqrt{3}q^2\sqrt{4m^3l^2 - 27q^4}}{2m^3l^2 - 27q^4}\right),$$

and satisfy a relation $r_+^2 + r_-^2 = -r_0^2$. The positions expressed by r_+ and r_- correspond to the outer and the inner horizon, respectively. It will be useful to notice that the charge q can be expressed in terms of θ and m by

$$q^4 = \frac{4m^3l^2}{27}\sin^2\left(\frac{\theta}{2}\right).$$

The outer horizon takes a value in

$$\sqrt{\frac{m}{3}}\,l \le r_+^2 \le \sqrt{m}\,l,$$

where the upper bound and the lower bound correspond to the case for $q = 0$ and the extremal case, respectively.

We shall give various thermodynamic quantities of RN-AdS black hole.[26),27)] The temperature is defined from the conical singularity free condition around the horizon r_+,

$$T = \frac{r_+^2 f'(r_+)}{4\pi l^2} = \frac{r_+}{\pi l^2}\left(1 - \frac{1}{2}\frac{q^2l^2}{r_+^6}\right) \equiv \frac{1}{2\pi b}\left(1 - \frac{a}{2}\right), \quad (> 0) \tag{2.18}$$

where we defined the parameters a and b by

$$a \equiv \frac{q^2l^2}{r_+^6}, \qquad b \equiv \frac{l^2}{2r_+}. \tag{2.19}$$

In the limit $q \to 0$, these parameters go to

$$a \to 0, \qquad b \to \frac{l^{3/2}}{2m^{1/4}},$$

and the temperature becomes

$$T \to T_0 = \frac{m^{1/4}}{\pi l^{3/2}}. \tag{2.20}$$

The entropy density s, the energy density ϵ, the pressure p, the chemical potential μ and the density of physical charge ρ can be also computed as

$$s = \frac{2\pi r_+^3}{\kappa^2 l^3} = \frac{\pi l^3}{4\kappa^2 b^3}, \tag{2.21}$$

$$\epsilon = \frac{3m}{2\kappa^2 l^3} = \frac{3l^3}{32\kappa^2 b^4}\left(1 + a\right), \tag{2.22}$$

$$p = \frac{\epsilon}{3}, \tag{2.23}$$

$$\mu = \frac{Q}{r_+^2}, \tag{2.24}$$

$$\rho = \frac{2Q}{e^2 l^3}. \tag{2.25}$$

§3. Perturbations in RN-AdS background

In RN-AdS background, we study small perturbations of the metric $g_{mn}(x)$ and the gauge field $\mathcal{A}_m(x)$,

$$
\begin{aligned}
g_{mn} &\equiv g_{mn}^{(0)} + h_{mn}, \\
\mathcal{A}_m &\equiv A_m^{(0)} + A_m,
\end{aligned}
\tag{3.1}
$$

where the background metric $g_{mn}^{(0)}(x)$ and the background gauge field $A_m^{(0)}(x)$ are given in (2·13a) and (2·13b), respectively. In the metric perturbation, one can define a inverse metric as

$$
g^{mn} = g^{(0)mn} - h^{mn} + \mathcal{O}(h^2),
$$

and raise and lower indices by using the background metric $g_{mn}^{(0)}(x)$ and $g^{(0)mn}(x)$.

Now we shall consider a linearized theory of the symmetric tensor field $h_{mn}(x)$ and the vector field $A_m(x)$ propagating in RN-AdS background. In the first order of $h_{mn}(x)$ and $A_m(x)$, the Einstein equation (2·10) can be written as

$$
R_{mn}^{(1)} - \frac{1}{2} g_{mn}^{(0)} R^{(1)} - \frac{1}{2} h_{mn} R^{(0)} + h_{mn} \Lambda = \kappa^2 T_{mn}^{(1)}.
\tag{3.2}
$$

In the expression above, the scalar curvature $R^{(0)}(x)$ is constructed by using the background metric $g_{mn}^{(0)}(x)$ and the following tensors are newly defined:

$$
\begin{aligned}
R_{mn}^{(1)} &= \frac{1}{2}\Big(\nabla_k \nabla_m h_n{}^k + \nabla_k \nabla_n h_m{}^k - \nabla_k \nabla^k h_{mn} - \nabla_m \nabla_n h \Big), \\
R^{(1)} &= g^{(0)kl} R_{kl}^{(1)} - h^{kl} R_{kl}^{(0)} \\
&= \nabla_k \nabla_l h^{kl} - \nabla_k \nabla^k h - h^{kl} R_{kl}^{(0)}, \\
T_{mn}^{(1)} &= \frac{1}{e^2}\Big(-F_{mk}^{(0)} F_{nl}^{(0)} h^{kl} + \frac{1}{2} g_{mn}^{(0)} F_{kp}^{(0)} F^{(0)}{}_l{}^p h^{kl} - \frac{1}{4} h_{mn} F_{kl}^{(0)} F^{(0)kl} \\
&\qquad + F_{mk}^{(0)} F_n{}^k + F_{nk}^{(0)} F_m{}^k - \frac{1}{2} g_{mn}^{(0)} F_{kl}^{(0)} F^{kl} \Big),
\end{aligned}
$$

where the Ricci tensor $R_{mn}^{(0)}(x)$, the covariant derivative and the field strength $F_{mn}^{(0)}(x)$ are defined through the background metric $g_{mn}^{(0)}(x)$ and the gauge field $A_m^{(0)}(x)$. We denote a trace part of the metric and a field strength for the perturbative parts as $h(x) \equiv h_{mn} g^{(0)mn}(x)$ and $F_{mn}(x) \equiv \partial_m A_n(x) - \partial_n A_m(x)$, respectively. On the other hand, the Maxwell equation (2·12) becomes

$$
\begin{aligned}
0 &= \nabla_m \left(F^{mn} - F^{(0)m}{}_k h^{nk} + F^{(0)n}{}_k h^{mk} + \frac{1}{2} F^{(0)mn} h \right) \\
&= \frac{1}{\sqrt{-g^{(0)}}} \partial_m \bigg\{ \sqrt{-g^{(0)}} \bigg(g^{(0)mk} g^{(0)nl} (\partial_k A_l - \partial_l A_k) \\
&\qquad\qquad - F^{(0)m}{}_k h^{nk} + F^{(0)n}{}_k h^{mk} + \frac{1}{2} F^{(0)mn} h \bigg) \bigg\}.
\end{aligned}
\tag{3.3}
$$

The above equations of motion (3·2) and (3·3) can be derived from the following action:

$$
\begin{aligned}
S[h_{mn}, A_m] = & -\frac{1}{4\kappa^2}\int d^5x\sqrt{-g^{(0)}}\Bigg\{ \nabla_m h^{mn}\nabla_n h - \nabla_m h^{nk}\nabla_n h^m{}_k \\
& +\frac{1}{2}\nabla_m h^{kl}\nabla^m h_{kl} - \frac{1}{2}\nabla_m h\nabla^m h \\
& +\left(\frac{1}{2}R^{(0)} - \Lambda - \frac{\kappa^2}{4e^2}F^{(0)}_{kl}F^{(0)kl}\right)\left(\frac{1}{2}h^2 - h_{mn}h^{mn}\right) \\
& +\frac{\kappa^2}{e^2}F^{(0)}_{mn}F^{(0)}_{kl}h^{mk}h^{nl}\Bigg\} \\
& -\frac{1}{4e^2}\int d^5x\sqrt{-g^{(0)}}\Bigg\{ F_{mn}F^{mn} \\
& -2\left(F^{(0)}_{mk}F_n{}^k h^{mn} + F^{(0)}_{nk}F_m{}^k h^{mn} - \frac{1}{2}F^{(0)}_{mn}F^{mn}h\right)\Bigg\}.
\end{aligned}
$$
(3·4)

By using the equations of motion, an on-shell action is reduced to surface term

$$
\begin{aligned}
S[h^{cl}_{mn}, A^{cl}_m] = & -\frac{1}{8\kappa^2}\int d^5x\,\partial_m\Bigg\{ \sqrt{-g^{(0)}}\left(h^{mn}\nabla_n h + h\nabla_n h^{mn} - 2h^{nk}\nabla_n h^m{}_k \right. \\
& \left. +h^{kl}\nabla^m h_{kl} - h\nabla^m h\right)\Bigg\} \\
& -\frac{1}{2e^2}\int d^5x\,\partial_m\Bigg\{ \sqrt{-g^{(0)}}A_n\left(F^{mn} \right. \\
& \left. -F^{(0)m}{}_k h^{nk} + F^{(0)n}{}_k h^{mk} + \frac{1}{2}F^{(0)mn}h\right)\Bigg\}.
\end{aligned}
$$
(3·5)

We shall work in the $h_{rm}(x) = 0$ and $A_r(x) = 0$ gauges and use the Fourier decomposition

$$
h_{\mu\nu}(t,z,r) = \int\frac{d^4k}{(2\pi)^4}\,e^{-i\omega t + ikz}h_{\mu\nu}(k,r),
$$

$$
A_\mu(t,z,r) = \int\frac{d^4k}{(2\pi)^4}\,e^{-i\omega t + ikz}A_\mu(k,r),
$$

where we choose the momenta which are along the z-direction. In this case, one can categorize the metric perturbations to the following three types by using the spin under the $O(2)$ rotation in (x,y)-plane:[12]

- vector type: $h_{xt} \neq 0$, $h_{xz} \neq 0$, (others) = 0
  $$\left(\text{equivalently}, \ h_{yt} \neq 0, \ h_{yz} \neq 0, \ (\text{others}) = 0\right)$$

- tensor type: $h_{xy} \neq 0$, $h_{xx} = -h_{yy} \neq 0$, (others) = 0

- scalar type: $h_{tz} \neq 0$, $h_{tt} \neq 0$, $h_{xx} = h_{yy} \neq 0$, and $h_{zz} \neq 0$, (others) = 0

We consider the first two types in this paper. The scalar type perturbation would be studied elsewhere.

3.1. *Vector type perturbation*

In this subsection, we study the vector type perturbation in RN-AdS background. From explicit calculation, one can show that only x-component of the gauge field $A_x(x)$ could participate in the linealized perturbative equations of motion. Thus independent variables are

$$h_{xt}(x) \neq 0, \quad h_{xz}(x) \neq 0, \quad A_x(x) \neq 0, \quad (\text{others}) = 0.$$

We start by introducing new field valiables, $h_t^x(r) = g^{(0)xx}h_{xt}(r) = (l^2/r^2)h_{xt}(r)$ and $h_z^x(r) = g^{(0)xx}h_{xz}(r) = (l^2/r^2)h_{xz}(r)$. Nontrivial equations in the Einstein equation (3·2) appear from (t,x), (r,x) and (x,z) components, respectively:

$$0 = h_t^{x\prime\prime} + \frac{5}{r}h_t^{x\prime} - \frac{l^4}{r^4 f}\left(\omega k h_z^x + k^2 h_t^x\right) + \frac{6q^2 l^2}{Q r^5}A_x', \tag{3·6a}$$

$$0 = kf h_z^{x\prime} + \omega h_t^{x\prime} + \frac{6q^2 l^2 \omega}{Q r^5}A_x, \tag{3·6b}$$

$$0 = h_z^{x\prime\prime} + \frac{(r^5 f)'}{r^5 f}h_z^{x\prime} + \frac{l^4}{r^4 f^2}\left(\omega k h_t^x + \omega^2 h_z^x\right), \tag{3·6c}$$

where the prime implies the derivative with respect to r. In the set of equations, the equations (3·6a) and (3·6b) imply (3·6c). On the other hand, in the Maxwell equation (3·3), the x-component gives a nontrivial contribution,

$$0 = A_x'' + \frac{(r^3 f)'}{r^3 f}A_x' + \frac{l^4}{r^4 f^2}\left(\omega^2 - k^2 f\right)A_x + \frac{2Q}{r^3 f}h_t^{x\prime}. \tag{3·7}$$

Taking the limit in which the charge q goes to zero, the metric and the gauge perturbations are completely decoupled.

We now look for solutions of our set of equations. First of all, from the equations (3·6a) and (3·6b), we can obtain a second order differential equation for $h_t^{x\prime}(r)$ and $A_x(r)$,

$$0 = h_t^{x\prime\prime\prime} + \frac{(r^9 f)'}{r^9 f}h_t^{x\prime\prime} + \frac{1}{r^4 f}\left(5(r^3 f)' + \frac{l^4}{f}\left(\omega^2 - k^2 f\right)\right)h_t^{x\prime}$$
$$+ \frac{6q^2 l^2}{Q}\left(\frac{A_x''}{r^5} + \frac{(r^{-1} f)'}{r^4 f}A_x' + \frac{l^4 \omega^2}{r^9 f^2}A_x\right). \tag{3·8}$$

Together with the equation of motion (3·7), we treat $h_t^{x\prime}(r)$ and $A_x(r)$ as independent variables. Having the solutions for these, one can get one for $h_z^{x\prime}(r)$ by using the equation (3·6b). In order to solve these equations, we find it is useful to introduce linear combinations of the variables

$$\Phi_\pm \equiv -\frac{8b^4}{l^8} r^5 h_t^{x\prime} + \left(-\frac{3al^4}{4Qb^2} + \frac{C_\pm}{Q} r^2 \right) A_x, \qquad (3\cdot9)$$

with constants

$$C_\pm = (1+a) \pm \sqrt{(1+a)^2 + 3ab^2 k^2},$$

so that we can obtain second order ordinary differential equations in terms of these new variables. In fact, the equations of motion (3·7) and (3·8) could be rearranged as

$$0 = \Phi_\pm{''} + \frac{(r^{-1}f)'}{r^{-1}f}\Phi_\pm{'} + \frac{l^4}{r^4 f^2}\Big(\omega^2 - k^2 f\Big)\Phi_\pm - \frac{l^8 C_\pm}{4b^4 r^6 f}\Phi_\pm. \qquad (3\cdot10)$$

In the chargeless limit, the two equations of motion (3·10) for $\Phi_+(r)$ and $\Phi_-(r)$ give decoupled ones for $A_x(r)$ and $h_t^{x\prime}(r)$, respectively.

We will consider these equations of motion in low frequency limit so-called hydrodynamic regime. In the hydrodynamic regime we could obtain the diffusion pole and the thermal conductivity from retarded Green functions.

3.2. *Tensor type perturbation*

Next we shall focus on the tensor type perturbation. By considering the spin or by calculating directly, the metric perturbation is decoupled from the gauge perturbation. Thus independent variables are

$$h_{xy}(x) \neq 0, \qquad h_{xx}(x) = -h_{yy}(x), \qquad \text{(others)} = 0.$$

A nontrivial equation of motion in (3·2) is coming from (x, y) component. As we did in the vector type perturbation, it might be convenient to introduce new variable $h_y^x(r) = g^{(0)xx}h_{xy}(r) = (l^2/r^2)h_{xy}(r)$. We then get the following equation of motion:

$$0 = h_y^{x\prime\prime} + \frac{(r^5 f)'}{r^5 f}h_y^{x\prime} + \frac{l^4}{r^4 f^2}\Big(\omega^2 - k^2 f\Big)h_y^x. \qquad (3\cdot11)$$

An another equation of motion for $h_{xx}(r) = -h_{yy}(r)$ is as the same form of (3·11). We use this equation of motion to study the shear viscosity in the hydrodynamic approximation.

Equations (3·10) and (3·11) can be rewritten as Schrödinger-like equations through suitable field redefinitions. Their potentials were derived first by Kodama and Ishibashi.[30]

§4. Diffusion pole in hydrodynamic regime

In the hydrodynamic regime, it is standard to introduce new dimensionless coordinate $u = r_+^2/r^2$ which is normalized by the outer horizon. In this coordinate

system, the horizon and the boundary are located at $u = 1$ and $u = 0$, respectively. Defining the new variable $B(u) \equiv \dfrac{A_x(u)}{\mu} = \dfrac{l^4}{4Qb^2}A_x(u)$ where μ is the chemical potential given by (2·24), our basic equations (3·6a)–(3·6c) and (3·7) are rewritten in this new coordinate system:

$$0 = h_t^{x\prime\prime} - \frac{1}{u}h_t^{x\prime} - \frac{b^2}{uf}\Big(\omega k h_z^x + k^2 h_t^x\Big) - 3auB', \tag{4·1a}$$

$$0 = kf h_z^{x\prime} + \omega h_t^{x\prime} - 3a\omega uB, \tag{4·1b}$$

$$0 = h_z^{x\prime\prime} + \frac{(u^{-1}f)'}{u^{-1}f}h_z^{x\prime} + \frac{b^2}{uf^2}\Big(\omega^2 h_z^x + \omega k h_t^x\Big), \tag{4·1c}$$

$$0 = B'' + \frac{f'}{f}B' + \frac{b^2}{uf^2}\Big(\omega^2 - k^2 f\Big)B - \frac{1}{f}h_t^{x\prime}, \tag{4·1d}$$

with

$$f(u) = (1-u)(1+u-au^2).$$

Here the prime now means the derivative with respect to u. Equation (3·10) may be also written down as

$$0 = \Phi_\pm{}'' + \frac{(u^2 f)'}{u^2 f}\Phi_\pm{}' + \frac{b^2}{uf^2}\Big(\omega^2 - k^2 f\Big)\Phi_\pm - \frac{C_\pm}{f}\Phi_\pm, \tag{4·2a}$$

for

$$\Phi_\pm = \frac{1}{u}h_t^{x\prime} - 3aB + \frac{C_\pm}{u}B. \tag{4·2b}$$

Getting the solution for $\Phi_\pm(u)$, one can access solutions for $h_t^{x\prime}(u)$ and $B(u)$,

$$h_t^{x\prime} = u\Phi_- + \frac{3a}{C_+ - C_-}u^2\Big(\Phi_+ - \Phi_-\Big) - \frac{C_-}{C_+ - C_-}u\Big(\Phi_+ - \Phi_-\Big), \tag{4·3a}$$

$$B = \frac{1}{C_+ - C_-}u\Big(\Phi_+ - \Phi_-\Big). \tag{4·3b}$$

The constants $C_\pm$ could be expanded in this regime,

$$\begin{aligned} C_+ &= 2(1+a) + \frac{3ab^2}{2(1+a)}k^2 + \mathcal{O}(k^4), \\ C_- &= -\frac{3ab^2}{2(1+a)}k^2 + \mathcal{O}(k^4). \end{aligned} \tag{4·4}$$

First, let us consider the equation for $\Phi_-(u)$. Following the usual way to solve differential equations, we impose a solution as $\Phi_-(u) = (1-u)^\nu F_-(u)$ where $F_-(u)$ is a regular function at the horizon $u = 1$. Substituting this form into the equation of motion, one can fix the parameter ν as $\nu = \pm i\omega/(4\pi T)$ where T is the temperature defined by Eq. (2·18). We here choose

$$\nu = -i\frac{\omega}{4\pi T},$$

as the incoming wave condition.

Now we are in the position to solve the equation of motion in the hydrodynamic regime. We start by introducing the following series expansion with respect to small ω and k:

$$F_-(u) = F_0(u) + \omega F_1(u) + k^2 G_1(u) + \mathcal{O}(\omega^2,\ \omega k^2), \tag{4·5}$$

where $F_0(u)$, $F_1(u)$ and $G_1(u)$ are determined by imposing suitable boundary conditions. In order to do the perturbative analysis, it might be convenient to rewrite the equation (4·2a) for $\Phi_-(u)$ as,

$$
\begin{aligned}
0 = &\left(u^2(1-u)(1+u-au^2)F_-{}' \right)' \\
&+ i\omega \frac{2b}{2-a} u^2 \left(1+u-au^2\right) F_-' + i\omega \frac{b}{2-a} u\left(2+3u-4au^2\right) F_- \\
&+ \omega^2 \frac{b^2}{(2-a)^2} \frac{u}{1+u-au^2} \\
&\qquad \times \left((2-a)^2 + (1-a)(3-a)u + (1-4a+a^2)u^2 - a(2-a)u^3 + a^2 u^4 \right) F_- \\
&- k^2 b^2 u \left(1 - \frac{3a}{2(1+a)} u \right) F_-.
\end{aligned}
\tag{4·6}
$$

The solution can be then obtained recursively.[*] The result is as follows:

$$F_0(u) = C, \quad \text{(const)} \tag{4·7a}$$

$$
\begin{aligned}
F_1(u) = iCb\Bigg\{ &\frac{1+2a-2a^2}{2\sqrt{1+4a}(2-a)} \left(\log\left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 - \dfrac{1-2a}{\sqrt{1+4a}}} \right) - \log\left(\frac{1 + \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2a}{\sqrt{1+4a}}} \right) \right) \\
&+ 1 - \frac{1}{u} + \frac{1}{2(2-a)} \log\left(\frac{1+u-au^2}{2-a} \right) \Bigg\},
\end{aligned}
\tag{4·7b}
$$

$$G_1(u) = \frac{Cb^2}{2(1+a)}\left(-1 + \frac{1}{u} \right). \tag{4·7c}$$

All of the solutions should be regular at the horizon $u = 1$ and the functions $F_1(u)$ and $G_1(u)$ should be vanished there. The constant of integration C will be estimated later.

Next, we shall study the equation for $\Phi_+(u)$. It might be useful to introduce new variable $\widetilde{\Phi}_+(u)$,

$$\Phi_+ \equiv \left(-\frac{3a}{2(1+a)} + \frac{1}{u} \right)\widetilde{\Phi}_+. \tag{4·8}$$

---

[*] The derivation of the solutions is given in Appendix A.

In terms of new variable, the equation of motion (4·2a) for $\Phi_+(u)$ becomes

$$0 = \widetilde{\Phi}''_+ + \frac{\left(\left(1 - \frac{3a}{2(1+a)}u\right)^2 f\right)'}{\left(1 - \frac{3a}{2(1+a)}u\right)^2 f}\widetilde{\Phi}'_+ + \frac{b^2}{uf^2}\left(\omega^2 - k^2 f\left(1 + \frac{3a}{2(1+a)}u\right)\right)\widetilde{\Phi}_+. \quad (4\cdot9)$$

Assuming again $\widetilde{\Phi}_+(u) = (1-u)^\nu \widetilde{F}(u)$ where $\widetilde{F}(u)$ is a regular function at $u = 1$, the singularity might be extracted. The equation of motion (4·9) becomes

$$\begin{aligned}
0 = &\left(\left(1-u\right)\left(1+u-au^2\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2\widetilde{F}'\right)' \\
&+2i\omega\frac{b}{2-a}\left(1+u-au^2\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2\widetilde{F}' \\
&+i\omega\frac{b}{2-a}\left(\left(1+u-au^2\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2\right)'\widetilde{F} \\
&+\frac{\omega^2 b^2}{(2-a)^2}\frac{\left(1 - \frac{3a}{2(1+a)}u\right)^2}{u(1+u-au^2)} \\
&\qquad\times\left((2-a)^2 + (1-a)(3-a)u + (1-4a+a^2)u^2 - a(2-a)u^3 + au^4\right)\widetilde{F} \\
&-\frac{k^2 b^2}{u}\left(1 + \frac{3a}{2(1+a)}u\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2\widetilde{F}, \quad (4\cdot10)
\end{aligned}$$

where we used the incoming wave condition $\nu = -i\omega/(4\pi T)$ as same as before.

We impose a perturbative solution as

$$\widetilde{F}(u) = \widetilde{F}_0(u) + \omega\widetilde{F}_1(u) + k^2\widetilde{G}_1(u) + \mathcal{O}(\omega^2, \omega k^2), \quad (4\cdot11)$$

and then we obtain the following result:[*]

$$\begin{aligned}
\widetilde{F}_0(u) &= \widetilde{C}, \quad (\text{const}) \quad &(4\cdot12\text{a}) \\
\widetilde{F}_1(u) &\equiv \widetilde{C}\widetilde{H}(u) \\
&= i\frac{\widetilde{C}b}{2-a}\left\{\frac{27a^2}{1+4a}\left(\frac{1-u}{2+2a-3au}\right)\right. \\
&\qquad+\frac{1-10a-2a^2}{2(1+4a)^{3/2}}\left(\log\left(\frac{1 - \frac{1-2au}{\sqrt{1+4a}}}{1 - \frac{1-2a}{\sqrt{1+4a}}}\right) - \log\left(\frac{1 + \frac{1-2au}{\sqrt{1+4a}}}{1 + \frac{1-2a}{\sqrt{1+4a}}}\right)\right) \\
&\qquad\left.+\frac{1}{2}\log\left(\frac{1+u-au^2}{2-a}\right)\right\}, \quad &(4\cdot12\text{b})
\end{aligned}$$

---

[*] The detail is given in Appendix B.

$$\widetilde{G}_1(u) \equiv \widetilde{C}\widetilde{J}(u)$$

$$= \widetilde{C}b^2 \Bigg\{ -\frac{9a^2(14 + 31a + 8a^2)}{2(1+a)(1+4a)(2-a)^2}\left(\frac{1-u}{2+2a-3au}\right)$$

$$+ \frac{(1+a)\Big(3a(2-a)(5+2a) - 2(1+a)(1-10a-2a^2)\log(3a)\Big)}{(2-a)^3(1+4a)^{3/2}}$$

$$\times \Bigg(\log\left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 - \dfrac{1-2a}{\sqrt{1+4a}}}\right) - \log\left(\frac{1 + \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2a}{\sqrt{1+4a}}}\right) \Bigg)$$

$$- \frac{4(1+a)^2}{(2-a)^3}\log u \log\Big(1-u\Big)$$

$$- \frac{(1+a)\Big(9a(2-a) + 2(1+a)(1+4a)\log(3a)\Big)}{(2-a)^3(1+4a)}\log\left(\frac{1+u-au^2}{2-a}\right)$$

$$- \frac{54a^2(1+a)}{(2-a)^2(1+4a)}\left(\frac{u\log u}{2+2a-3au}\right)$$

$$- \frac{4(1+a)^2}{(2-a)^3}\left(\mathrm{Li}_2(u) - \frac{\pi^2}{6}\right)$$

$$+ \frac{2(1+a)^2}{(1+4a)^{3/2}(2-a)^3}$$

$$\times \Bigg(\Big(1 - 10a - 2a^2 + (1+4a)^{3/2}\Big)$$

$$\times \Bigg(\log u \log\left(1 - \frac{2au}{1-\sqrt{1+4a}}\right) + \log(3a)\log\left(\frac{1 - \dfrac{2au}{1-\sqrt{1+4a}}}{1 - \dfrac{2a}{1-\sqrt{1+4a}}}\right)$$

$$+ \mathrm{Li}_2\left(\frac{2au}{1-\sqrt{1+4a}}\right) - \mathrm{Li}_2\left(\frac{2a}{1-\sqrt{1+4a}}\right) \Bigg)$$

$$- \Big(1 - 10a - 2a^2 - (1+4a)^{3/2}\Big)$$

$$\times \Bigg(\log u \log\left(1 - \frac{2au}{1+\sqrt{1+4a}}\right) + \log(3a)\log\left(\frac{1 - \dfrac{2au}{1+\sqrt{1+4a}}}{1 - \dfrac{2a}{1+\sqrt{1+4a}}}\right)$$

$$+ \mathrm{Li}_2\left(\frac{2au}{1+\sqrt{1+4a}}\right) - \mathrm{Li}_2\left(\frac{2a}{1+\sqrt{1+4a}}\right) \Bigg) \Bigg) \Bigg\}, \qquad (4\cdot12\mathrm{c})$$

where $\mathrm{Li}_2(u)$ is the polylogalithm.[*] It should be mentioned that the defined functions $\widetilde{H}(u)$ and $\widetilde{J}(u)$ in $\widetilde{F}_1(u)$ and $\widetilde{G}_1(u)$ are finite at the boundary $u = 0$.

Let us consider the integration constants C and $\widetilde{C}$. These could be estimated in terms of boundary values of the fields

$$\lim_{u\to 0} h_t^x(u) = (h_t^x)^0, \quad \lim_{u\to 0} h_z^x(u) = (h_z^x)^0, \quad \lim_{u\to 0} B(u) = (B)^0.$$

Taking a derivative of $\Phi_\pm(u)$ and using the equation of motion (4·1a), we can get relations

$$u^2\Phi_\pm' - C_\pm u B' = \frac{b^2}{f}\left(\omega k h_z^x + k^2 h_t^x\right) - C_\pm B.$$

We evaluate the equations above at the boundary,

$$\lim_{u\to 0}\left(u^2\Phi_\pm' - C_\pm u B'\right) = b^2\left(\omega k (h_z^x)^0 + k^2 (h_t^x)^0\right) - C_\pm (B)^0, \qquad (4\cdot13)$$

so that we may fix the constants C and $\widetilde{C}$ from $\mp$ parts, respectively,

$$C = \frac{b\left(\omega k (h_z^x)^0 + k^2 (h_t^x)^0\right) + \dfrac{3ab}{2(1+a)}k^2 (B)^0}{i\omega - \dfrac{b}{2(1+a)}k^2}, \qquad (4\cdot14\mathrm{a})$$

$$\widetilde{C} = \frac{-b^2\left(\omega k (h_z^x)^0 + k^2 (h_t^x)^0\right) + \left(2(1+a) + \dfrac{3ab^2}{2(1+a)}k^2\right)(B)^0}{1 + \omega \widetilde{H}(0) + k^2 \widetilde{J}(0)}, \qquad (4\cdot14\mathrm{b})$$

where we used the obtained solutions $\Phi_\pm(u)$ and the relation (4·3b) for $B'(u)$. It should be noted that the boundary value of $uB'(u)$ is vanished. In the equation (4·14a), one can see the existence of the hydrodynamic pole in the complex ω-plane.

Now we proceed to calculate the Minkowskian correlators. For the vector type perturbation, the on-shell action (3·5) becomes

$$S[h_t^x, h_z^x, B] = \frac{l^3}{32\kappa^2 b^4} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{u}h_t^x(-k,u)h_t^{x\prime}(k,u) - \frac{1}{u^2}h_t^x(-k,u)h_t^x(k,u) \right.$$
$$- \frac{f(u)}{u}h_z^x(-k,u)h_z^{x\prime}(k,u) + \frac{f(u)}{u^2}h_z^x(-k,u)h_z^x(k,u)$$
$$\left. -3af(u)B(-k,u)\left(B'(k,u) - \frac{1}{f(u)}h_t^x(k,u)\right) \right\}\Bigg|_{u=0}^{u=1}. \qquad (4\cdot15)$$

---

[*] The polylogalithm appears from

$$\left(\mathrm{Li}_2(u)\right)' = -\frac{\log(1-u)}{u}.$$

Some values are given as, $\mathrm{Li}_2(-1) = -\pi^2/12$, $\mathrm{Li}_2(0) = 0$ and $\mathrm{Li}_2(1) = \pi^2/6$.

Using the obtained solutions, we can lead the following relations between the radial derivative of the fields and their boundary values near the boundary $u = \varepsilon$:

$$
h_t^{x\prime}(\varepsilon) = -b^2 \Big(\omega k (h_z^x)^0 + k^2 (h_t^x)^0 \Big)
$$
$$
+ \frac{\varepsilon}{i\omega - \dfrac{b}{2(1+a)}k^2} \Bigg\{ b\Big(\omega k (h_z^x)^0 + k^2 (h_t^x)^0 \Big) + 3ia\omega(B)^0
$$
$$
+ \mathcal{O}(\omega^2 k, \omega k^2) \Bigg\} + \mathcal{O}(\varepsilon^2), \tag{4.16a}
$$

$$
h_z^{x\prime}(\varepsilon) = b^2 \Big(\omega^2 (h_z^x)^0 + \omega k (h_t^x)^0 \Big)
$$
$$
- \frac{\varepsilon}{i\omega - \dfrac{b}{2(1+a)}k^2} \Bigg\{ b\Big(\omega^2 (h_z^x)^0 + \omega k (h_t^x)^0 \Big) + \frac{3ab}{2(1+a)}\omega k (B)^0
$$
$$
+ \mathcal{O}(\omega^2 k, \omega k^2) \Bigg\} + \mathcal{O}(\varepsilon^2), \tag{4.16b}
$$

$$
B'(\varepsilon) = - \frac{1}{i\omega - \dfrac{b}{2(1+a)}k^2} \Bigg\{ \frac{b}{2(1+a)} \Big(\omega k (h_z^x)^0 + k^2 (h_t^x)^0 \Big) + i\frac{3a}{2(1+a)}\omega(B)^0
$$
$$
- i\Big(i\omega - \frac{b}{2(1+a)}k^2 \Big) \frac{(2-a)^2 b}{4(1+a)^2}\omega(B)^0 + \mathcal{O}(\omega^2 k^2, k^4) \Bigg\}
$$
$$
+ \Big(b^2 k^2 (B)^0 + \mathcal{O}(\omega k^2) \Big) \log \varepsilon + \mathcal{O}(\varepsilon). \tag{4.16c}
$$

By using the relation (2·4) and the definition (2·5), we can read off the correlators in the hydrodynamic approximation,

$$
G_{xt\ xt}(\omega, k) = \frac{l^3}{16\kappa^2 b^3}\left(\frac{k^2}{i\omega - Dk^2} \right), \tag{4.17a}
$$

$$
G_{xt\ xz}(\omega, k) = G_{xz\ xt}(\omega, k) = -\frac{l^3}{16\kappa^2 b^3}\left(\frac{\omega k}{i\omega - Dk^2} \right), \tag{4.17b}
$$

$$
G_{xz\ xz}(\omega, k) = \frac{l^3}{16\kappa^2 b^3}\left(\frac{\omega^2}{i\omega - Dk^2} \right), \tag{4.17c}
$$

$$
G_{xt\ x}(\omega, k) = G_{x\ xt}(\omega, k) = -\frac{2Q}{e^2 l^3}\left(\frac{i\omega}{i\omega - Dk^2} \right), \tag{4.17d}
$$

$$
G_{xz\ x}(\omega, k) = G_{x\ xz}(\omega, k) = \frac{Qb}{(1+a)e^2 l^3}\left(\frac{\omega k}{i\omega - Dk^2} \right), \tag{4.17e}
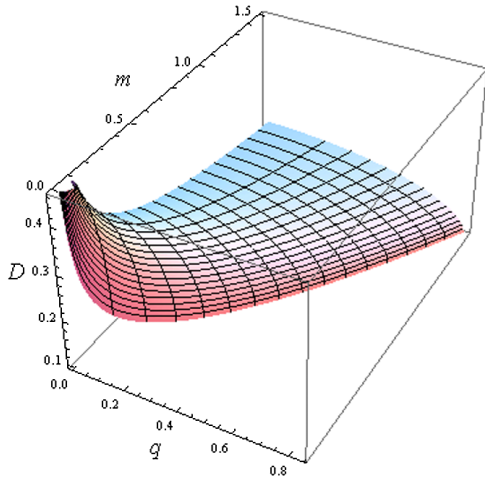$$

Fig. 1. D vs q and m ($l = 1$).
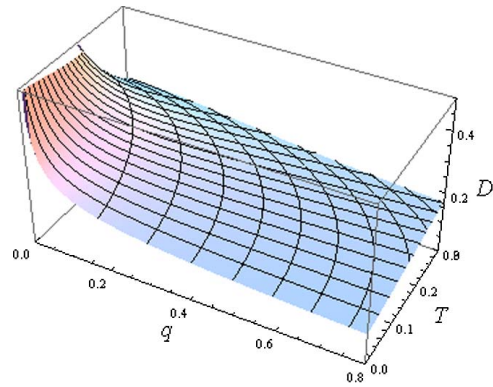

Fig. 2. D vs q and T ($l = 1$).

$$G_{x\ x}(\omega, k) = \frac{3al}{4(1+a)b^2 e^2} \left(\frac{i\omega}{i\omega - Dk^2} \right) - \frac{(2-a)^2 l}{8(1+a)^2 be^2} i\omega, \qquad (4\cdot17\text{f})$$

where we subtracted the contact terms. In the final expression above we rescaled the gauge field $(B)^0$ to the original one $(A_x)^0 = \dfrac{4Qb^2}{l^4}(B)^0$ and raised and lowered the indices by using the flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$ in the four-dimensional boundary theory. Taking the limit in which the charge q goes to zero, the results coincide with the known ones in 12). In this limit, the correlators (4·17d) and (4·17e) vanish, while the correlator (4·17f) has no diffusion pole and the subleading term reproduces the consistent result. The same interesting structure was found in the single $(1, 0, 0)$ R-charged black hole.[17] The constant D is the diffusion constant

$$D = \frac{b}{2(1+a)} = \frac{1}{4} \left(\frac{m^{5/3}}{3q^2} \left(1 + 2\cos\left(\frac{\theta}{3} + \frac{4}{3}\pi \right) \right) \right)^{-\frac{3}{2}}, \qquad (4\cdot18)$$

with

$$\theta = \arctan\left(\frac{3\sqrt{3}q^2 \sqrt{4m^3 l^2 - 27q^4}}{2m^3 l^2 - 27q^4} \right).$$

All of the correlators in the vector type perturbation exhibit a diffusion pole. The behavior of the diffusion constant is drawn as a function of the charge q and the mass m in Fig. 1 and as a function of the charge q and the temperature T in Fig. 2. In the chargeless limit, the diffusion constant becomes

$$D \to D_0 = \frac{1}{4\pi T_0},$$

where the temperature T_0 is given in (2·20).

§5. Shear viscosity in hydrodynamic regime

In this section, we solve the equation of motion (3·11) in the hydrodynamic regime and obtain the shear viscosity. We could also see the hydrodynamic relation and the formulation of the thermal conductivity.

After changing the coordinate r to $u = r_+^2/r^2$, Eq. (3·11) can be rewritten as

$$0 = h_y^{x\prime\prime} + \frac{(u^{-1}f)'}{u^{-1}f} h_y^{x\prime} + \frac{b^2}{uf^2}\Big(\omega^2 - k^2 f\Big) h_y^x, \tag{5·1}$$

with

$$f(u) = (1-u)(1+u-au^2),$$

where the prime means the derivative with respect to u. Removing the singularity around $u = 1$, the equation becomes

$$0 = \Big(\frac{1}{u}(1-u)(1+u-au^2)F'\Big)'$$
$$+ i\omega \frac{2b}{(2-a)} \frac{1}{u}(1-u)(1+u-au^2)F' - i\omega \frac{b}{(2-a)} \frac{1}{u^2}(1+au^2)F$$
$$+ \omega^2 \frac{b^2}{(2-a)^2 u^2(1+u-au^2)}$$
$$\times \Big((a-2)^2 + (a-3)(a-1)u + (a^2-4a+1)u^2 + a(a-2)u^3 + a^2u^4\Big)F$$
$$- k^2 \frac{b^2}{u^2}F, \tag{5·2}$$

where we imposed the incoming wave condition

$$h_y^x(u) = (1-u)^{-i\omega/(4\pi T)}F(u). \tag{5·3}$$

Perturbative solutions for $F(u)$,

$$F(u) = F_0(u) + \omega F_1(u) + k^2 G_1(u) + \mathcal{O}(\omega^2, \, \omega k^2), \tag{5·4}$$

can be obtained as[*]

$$F_0(u) = C, \quad \text{(const)} \tag{5·5a}$$
$$F_1(u) \equiv CH(u)$$
$$= i\frac{Cb}{2(2-a)}\Bigg\{ -\frac{3}{\sqrt{1+4a}}\Bigg(\log\Bigg(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 - \dfrac{1-2a}{\sqrt{1+4a}}}\Bigg) - \log\Bigg(\frac{1 + \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2a}{\sqrt{1+4a}}}\Bigg)\Bigg)$$
$$+ \log\Bigg(\frac{1+u-au^2}{2-a}\Bigg)\Bigg\}, \tag{5·5b}$$

---

[*] The detail is given in Appendix C.

$$G_1(u) \equiv CJ(u)$$

$$= -\frac{Cb^2}{\sqrt{1+4a}} \left\{ \log \left(\frac{1 - \frac{1-2au}{\sqrt{1+4a}}}{1 - \frac{1-2a}{\sqrt{1+4a}}} \right) - \log \left(\frac{1 + \frac{1-2au}{\sqrt{1+4a}}}{1 + \frac{1-2a}{\sqrt{1+4a}}} \right) \right\}. \qquad (5\cdot5\text{c})$$

Since the function $h_y^x(u)$ goes to $(h_y^x)^0$ at the boundary $u = 0$, the constant C can be fixed as

$$C = \frac{(h_y^x)^0}{1 + \omega H(0) + k^2 J(0)}. \qquad (5\cdot6)$$

Taking the limit $q \to 0$, the solution recovers the result in 12). The solution of $h_x^x(u)$ is the same form as $h_y^x(u)$.

Let us evaluate the Minkowskian correlators. The relevant part of the metric perturbation in the on-shell action (3·5) becomes

$$S[h_y^x, h_x^x, h_y^x] = -\frac{l^3}{32\kappa^2 b^4} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{f(u)}{u} h_y^x(-k, u) h_y^{x\prime}(k, u) \right.$$
$$-\frac{f(u)}{u^2} h_y^x(-k, u) h_y^x(k, u)$$
$$+\frac{f(u)}{u} h_x^x(-k, u) h_x^{x\prime}(k, u)$$
$$\left. -\frac{f(u)}{u^2} h_x^x(-k, u) h_x^x(k, u) \right\} \Bigg|_{u=0}^{u=1}. \qquad (5\cdot7)$$

Near the boundary $u = \varepsilon$, using the perturbative solution for $h_y^x(u)$, we can obtain

$$h_y^{x\prime}(\varepsilon) = \varepsilon b \left(i\omega + bk^2 \right) (h_y^x)^0 - b^2 k^2 (h_y^x)^0 + \mathcal{O}(\omega^2, \omega k^2). \qquad (5\cdot8)$$

The same relation for $h_x^x(u)$ might be satisfied. Therefore we can read off the correlation functions from the on-shell action (5·7),

$$G_{xy\ xy}(\omega, k) = G_{xx\ xx}(\omega, k) = G_{yy\ yy}(\omega, k)$$
$$= -\frac{l^3}{16\kappa^2 b^3} \left(i\omega + bk^2 \right), \qquad (5\cdot9)$$

where we subtract contact terms.

The result above can be used to estimate the shear viscosity η via Kubo formula,

$$\eta = -\lim_{\omega \to 0} \frac{\text{Im}(G(\omega, 0))}{\omega} = \frac{l^3}{16\kappa^2 b^3}. \qquad (5\cdot10)$$

Therefore we can confirm the following relation[16)–18)] between the shear viscosity η and the entropy density s which is given in Eq. (2·21):

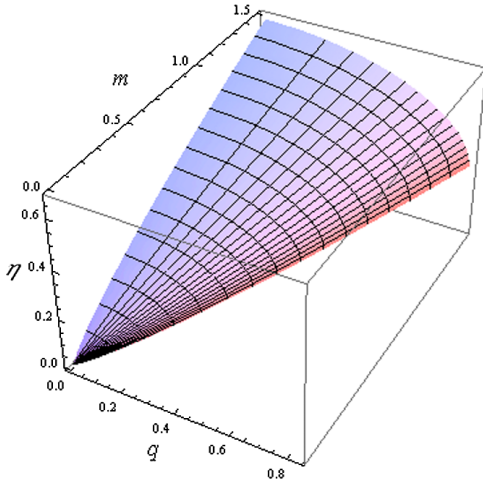$$\frac{\eta}{s} = \frac{1}{4\pi}. \qquad (5\cdot11)$$

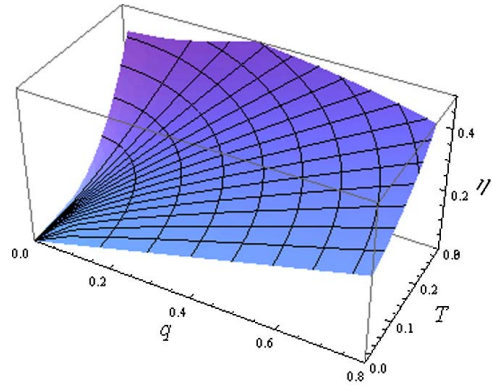Fig. 3. η vs q and m ($\kappa = l = 1$).

Fig. 4. η vs q and T ($\kappa = l = 1$).

The behavior of the shear viscosity is drawn as a function of the charge q and the mass m in Fig. 3 and as a function of the charge q and the temperature T in Fig. 4.

In hydrodynamics, the following relation is held:

$$D = \frac{\eta}{\epsilon + p}, \tag{5·12}$$

where ϵ and p are the energy density and the pressure defined in (2·22) and (2·23), respectively. Using the obtained diffusion constant (4·18), the shear viscosity could be calculated. We can confirm the result coincides with (5·10) which was obtained from Kubo formula.

The thermal conductivity κ_T can be also computed from the Green function by using Kubo formula,[17]

$$\kappa_T = -\frac{(\epsilon + p)^2}{\rho^2 T} \lim_{\omega \to 0} \frac{\text{Im}(G(\omega, 0))}{\omega}, \tag{5·13}$$

where the density of physical charge ρ is given by (2·25). Here we can use the retarded Green function $G_{x\ x}(\omega, 0)$ given by (4·17f) as $G(\omega, 0)$. Thus we obtain

$$\kappa_T = 2\pi^2 \left(\frac{e^2 l^2}{\kappa^2}\right) \frac{\eta T}{\mu^2} = 2\pi^2 \frac{N_c}{N_f} \frac{\eta T}{\mu^2}. \tag{5·14}$$

The behavior of the thermal conductivity κ_T is drawn as a function of the charge q and the mass in Fig. 5 and as a function of the charge q and the temperature T in Fig. 6.

§6. Conclusions and discussions

In this paper we considered holographic QCD in the presence of the baryon density by introducing the bulk-filling branes. We used RN-AdS black hole geometry
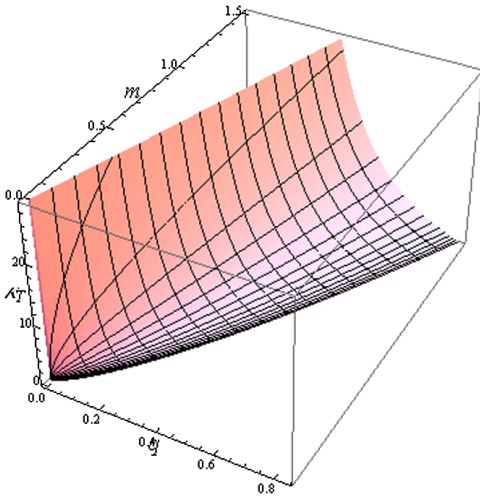
Fig. 5. κ_T vs q and m ($\kappa = l = 1$).
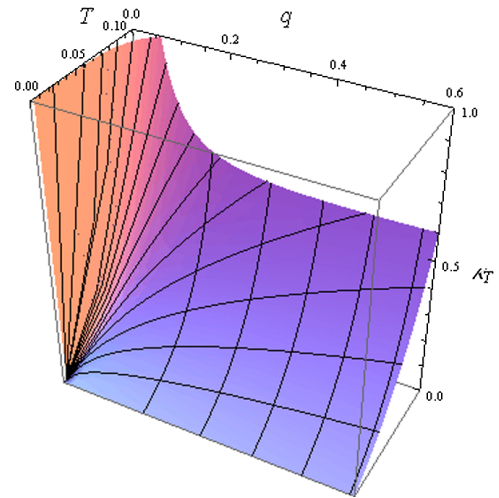
Fig. 6. κ_T vs q and T ($\kappa = l = 1$).

as the gravity dual of such system. The vector and tensor type perturbations have been worked out and the Green functions were explicitly calculated. We have seen the diffusion pole structure in vector type perturbation. It is worth mentioning that the correlator for Maxwell fields in the vector mode $G_{x\;x}(\omega, k)$ has the diffusion pole unlike the charge free case. The transport coefficients have been calculated in holographic hydrodynamics and their temperature and density dependence was demonstrated.

The diffusion constant decreases as charge increases for fixed temperature. Physically, this implies that the fluid is less diffusible for large baryon density. By calculating the shear viscosity analytically, we showed that the shear viscosity η and the entropy density s satisfy the universal ratio $(\eta/s) = 1/(4\pi)$ which has been originally suggested in 1). For fixed temperature, the fluid becomes thicker as the charge increases. We have also seen that the diffusion constant and the shear viscosity satisfy the suitable relation for hydrodynamics. The calculation of the thermal conductivity shows that it satisfies (an analogue of) the Wiedemann-Franz low.

It is very interesting to study the pole structure of scalar type as well as vector type of gravitational perturbations. Also it is important to carry out higher order calculations. Such result will be useful to get the higher order transport coefficients in the presence of the conserved current. We will report on these issues in the forthcoming publications.

In our interpretation, the fluctuations of bulk-filling branes are regarded as master fields of the mesons. Near the horizon, the tension of the brane is zero due to the metric factor and it can lead to the long range fluctuation. This becomes the hydrodynamic mode. One important question is about the meaning of hydrodynamic mode in terms of meson physics.

Further question in this direction is how we can understand the dispersion relations of vector modes of Maxwell fields in terms of the particle spectrum with dissipation. This vector mode cannot propagate in neutral medium while it can in

charged medium. In addition, the tensor mode does not propagate in the medium. It is interesting to consider their interpretations in terms of meson physics. More thought on these points is to be pursued in the future.

Acknowledgements

Appendix A
—— *Perturbative Solutions for Φ_-* ——

From the equation of motion (4·6), one can read off one for $F_0(u)$,

$$0 = \left(u^2 (1-u)(1+u-au^2) F_0' \right)'. \tag{A·1}$$

A general solution is given by

$$F_0(u) = C_0 + D_0 \left\{ -\frac{1}{u} + \frac{1+2a-2a^2}{2\sqrt{1+4a}\,(2-a)} \log \left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2au}{\sqrt{1+4a}}} \right) \right.$$
$$\left. -\frac{1}{2-a} \log \left(1-u\right) + \frac{1}{2(2-a)} \log \left(1+u-au^2\right) \right\}. \tag{A·2}$$

Constants of integration C_0 and D_0 should be determined to be a regular function at the horizon. So we here choose $D_0 = 0$ and get

$$F_0(u) = C_0 = C. \text{ (const)} \tag{A·3}$$

By using this solution, one can get an equation for $F_1(u)$ from (4·6),

$$0 = \left(u^2 (1-u)(1+u-au^2) F_1' \right)' + i\frac{Cb}{2-a} u \left(2 + 3u - 4au^2 \right). \tag{A·4}$$

A general solution is

$$F_1(u) = C_1 + D_1 \left(\frac{1+2a-2a^2}{2\sqrt{1+4a}\,(2-a)} \log \left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2au}{\sqrt{1+4a}}} \right) \right.$$
$$\left. -\frac{1}{u} + \frac{1}{2(2-a)} \log \left(1+u-au^2\right) \right)$$
$$+\frac{1}{2-a} \left(iCb - D_1 \right) \log \left(1-u \right). \tag{A·5}$$

Again, removing the singularity at the horizon, the constant D_1 should be

$$D_1 = iCb.$$

We also impose a boundary condition $F_1(u = 1) = 0$, so as to fix the constant C_1,

$$C_1 = -iCb\left\{ \frac{1 + 2a - 2a^2}{2\sqrt{1+4a}\,(2-a)} \log\left(\frac{1 - \frac{1-2a}{\sqrt{1+4a}}}{1 + \frac{1-2a}{\sqrt{1+4a}}} \right) - 1 + \frac{1}{2(2-a)} \log\left(2 - a \right) \right\}.$$

Therefore the final form is

$$F_1(u) = iCb\left\{ \frac{1 + 2a - 2a^2}{2\sqrt{1+4a}\,(2-a)} \left(\log\left(\frac{1 - \frac{1-2au}{\sqrt{1+4a}}}{1 - \frac{1-2a}{\sqrt{1+4a}}} \right) - \log\left(\frac{1 + \frac{1-2au}{\sqrt{1+4a}}}{1 + \frac{1-2a}{\sqrt{1+4a}}} \right) \right) \right.$$
$$\left. + 1 - \frac{1}{u} + \frac{1}{2(2-a)} \log\left(\frac{1 + u - au^2}{2-a} \right) \right\}. \qquad \text{(A·6)}$$

A differential equation for $G_1(u)$ is

$$0 = \left(u^2(1-u)(1+u-au^2)G_1' \right)' - Cb^2 u\left(1 - \frac{3a}{2(1+a)}u \right). \qquad \text{(A·7)}$$

A general solution is

$$G_1(u) = \widetilde{C}_1 - \frac{\widetilde{D}_1}{u}$$
$$+ \frac{\left(1 + 2a - 2a^2 \right)\left(Cb^2 + 2\widetilde{D}_1(1+a) \right)}{4\sqrt{1+4a}\,(1+a)(2-a)} \log\left(\frac{1 - \frac{1-2au}{\sqrt{1+4a}}}{1 + \frac{1-2au}{\sqrt{1+4a}}} \right)$$
$$- \frac{Cb^2 + 2\widetilde{D}_1(1+a)}{2(1+a)(2-a)} \left(\log\left(1 - u \right) - \frac{1}{2} \log\left(1 + u - au^2 \right) \right), \qquad \text{(A·8)}$$

and the constant $\widetilde{D}_1$ might be fixed as

$$\widetilde{D}_1 = -\frac{Cb^2}{2(1+a)}.$$

From the condition $G_1(u = 1) = 0$, we can fix the constant $\widetilde{C}_1$ as

$$\widetilde{C}_1 = -\frac{Cb^2}{2(1+a)}.$$

So we obtain the final form,

$$G_1(u) = \frac{Cb^2}{2(1+a)}\left(-1 + \frac{1}{u} \right). \qquad \text{(A·9)}$$

Appendix B
—— *Perturbative Solutions for Φ_+* ——

From the equation (4·10), we have a differential equation for $\widetilde{F}_0(u)$,

$$0 = \left(\left(1 - u\right)\left(1 + u - au^2\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2 \widetilde{F}_0' \right)'. \tag{B·1}$$

A general solution is given by

$$\widetilde{F}_0(u) = C_0 - \frac{D_0}{2(2-a)^3}\Bigg\{ \frac{18a(2-a)}{(1+4a)(2+2a-3au)} $$
$$ - \frac{1 - 10a - 2a^2}{(1+4a)^{3/2}} \log\left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2au}{\sqrt{1+4a}}} \right) $$
$$ + 2\log\left(1 - u\right) - \log\left(1 + u - au^2\right) \Bigg\}. \tag{B·2}$$

Since the function $\widetilde{F}_0$ should be regular at the horizon, we choose $D_0 = 0$ and get

$$\widetilde{F}_0(u) = C_0 = \widetilde{C}. \quad \text{(const)} \tag{B·3}$$

Substituting the solution to the equation (4·10), we get an equation for $\widetilde{F}_1(u)$,

$$0 = \Bigg(\left(1 - u\right)\left(1 + u - au^2\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2 \widetilde{F}_1' $$
$$ + i\frac{\widetilde{C}b}{2-a}\left(1 + u - au^2\right)\left(1 - \frac{3a}{2(1+a)}u\right)^2 \Bigg)'. \tag{B·4}$$

A general solution is given as

$$\widetilde{F}_1(u) = C_1 + \frac{2(1+a)^2}{(2-a)^3}D_1\Bigg\{ -\frac{18a(2-a)}{(1+4a)(2+2a-3au)} $$
$$ + \frac{(1 - 10a - 2a^2)}{(1+4a)^{3/2}} \log\left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2au}{\sqrt{1+4a}}} \right) $$
$$ + \log\left(1 + u - au^2\right) \Bigg\} $$
$$ + \frac{i\widetilde{C}(2-a)^2 b - 4D_1(1+a)^2}{(2-a)^3} \log\left(1 - u\right). \tag{B·5}$$

The constant of integration D_1 should be

$$D_1 = i\widetilde{C}\frac{(2-a)^2 b}{4(1+a)^2},$$

so that the singularity at the horizon would be removed. In addition, we require the condition $\widetilde{F}_1(u=1) = 0$ to fix the constant C_1,

$$C_1 = i\frac{\widetilde{C}b}{2-a}\left\{\frac{9a}{1+4a} - \frac{1-10a-2a^2}{2(1+4a)^{3/2}}\log\left(\frac{1-\frac{1-2a}{\sqrt{1+4a}}}{1+\frac{1-2a}{\sqrt{1+4a}}}\right) - \frac{1}{2}\log\left(2-a\right)\right\}.$$

Then we get the final form of the solution

$$\begin{aligned}
\widetilde{F}_1(u) = i\frac{\widetilde{C}b}{2-a}&\left\{\frac{27a^2}{1+4a}\left(\frac{1-u}{2+2a-3au}\right)\right. \\
&+\frac{1-10a-2a^2}{2(1+4a)^{3/2}}\left(\log\left(\frac{1-\frac{1-2au}{\sqrt{1+4a}}}{1-\frac{1-2a}{\sqrt{1+4a}}}\right) - \log\left(\frac{1+\frac{1-2au}{\sqrt{1+4a}}}{1+\frac{1-2a}{\sqrt{1+4a}}}\right)\right) \\
&\left.+\frac{1}{2}\log\left(\frac{1+u-au^2}{2-a}\right)\right\}.
\end{aligned}$$
(B·6)

Similarly we have a differential equation for $\widetilde{G}_1(u)$,

$$\begin{aligned}
0 =& \left(\left(1-u\right)\left(1+u-au^2\right)\left(1-\frac{3a}{2(1+a)}u\right)^2\widetilde{G}_1'\right)' \\
&-\frac{\widetilde{C}b^2}{u}\left(1+\frac{3a}{2(1+a)}u\right)\left(1-\frac{3a}{2(1+a)}u\right)^2.
\end{aligned}$$
(B·7)

A general solution of this equation can be obtained,

$$\begin{aligned}
\widetilde{G}_1(u) = \widetilde{C}_1 & \\
&-\frac{9a\left(\widetilde{D}_1 + 4\widetilde{C}(1+a)^2b^2\log(3a)\right)}{(2-a)^2(1+4a)}\left(\frac{1}{2+2a-3au}\right) \\
&+\frac{\widetilde{C}(2-a)(1+4a)(14+15a+42a^2+14a^3)b^2 + 6\widetilde{D}_1(1+a)(1-10a-2a^2)}{12(2-a)^3(1+a)(1+4a)^{3/2}} \\
&\quad\times\log\left(\frac{1-\frac{1-2au}{\sqrt{1+4a}}}{1+\frac{1-2au}{\sqrt{1+4a}}}\right) \\
&-\frac{\widetilde{C}b^2\left((2-a)(14+31a+8a^2) + 24(1+a)^3\log(3a)\right) + 6\widetilde{D}_1(1+a)}{6(2-a)^3(1+a)}
\end{aligned}$$

$$\times \log \left(1 - u\right)$$

$$- \frac{4\widetilde{C}(1+a)^2 b^2}{(2-a)^3} \log u \log \left(1 - u\right)$$

$$+ \frac{\widetilde{C}(2-a)(14 - 21a - 84a^2 - 76a^3)b^2 + 6\widetilde{D}_1(1+a)(1+4a)}{12(2-a)^3(1+a)(1+4a)}$$

$$\times \log \left(1 + u - au^2\right)$$

$$- \frac{54\widetilde{C}a^2(1+a)b^2}{(2-a)^2(1+4a)} \left(\frac{u \log u}{2 + 2a - 3au} \right)$$

$$- \frac{4\widetilde{C}(1+a)^2 b^2}{(2-a)^3} \mathrm{Li}_2(u)$$

$$+ \frac{2\widetilde{C}(1+a)^2 b^2}{(2-a)^3(1+4a)^{3/2}}$$

$$\times \Bigg\{ \left(1 - 10a - 2a^2 + (1+4a)^{3/2}\right)$$

$$\times \left(\log(3au) \log \left(1 - \frac{2au}{1 - \sqrt{1+4a}}\right) + \mathrm{Li}_2 \left(\frac{2au}{1 - \sqrt{1+4a}}\right) \right)$$

$$- \left(1 - 10a - 2a^2 - (1+4a)^{3/2}\right)$$

$$\times \left(\log(3au) \log \left(1 - \frac{2au}{1 + \sqrt{1+4a}}\right) + \mathrm{Li}_2 \left(\frac{2au}{1 + \sqrt{1+4a}}\right) \right) \Bigg\}.$$

$$(\mathrm{B\cdot 8})$$

The constant of integration $\widetilde{D}_1$ might be fixed to remove the singularity $u = 1$,

$$\widetilde{D}_1 = -\frac{\widetilde{C}b^2}{6(1+a)} \Big((2-a)(14 + 31a + 8a^2) + 24(1+a)^3 \log(3a) \Big).$$

Another constant of integration $\widetilde{C}_1$ is used to satisfy the condition $\widetilde{G}_1(u = 1) = 0$. The final expression of the solution is

$$\widetilde{G}_1(u) = \widetilde{C}b^2 \Bigg\{ -\frac{9a^2(14 + 31a + 8a^2)}{2(1+a)(1+4a)(2-a)^2} \left(\frac{1-u}{2 + 2a - 3au} \right)$$

$$+ \frac{(1+a)\Big(3a(2-a)(5 + 2a) - 2(1+a)(1 - 10a - 2a^2) \log(3a)\Big)}{(2-a)^3(1+4a)^{3/2}}$$

$$\times \left(\log \left(\frac{1 - \dfrac{1 - 2au}{\sqrt{1+4a}}}{1 - \dfrac{1 - 2a}{\sqrt{1+4a}}} \right) - \log \left(\frac{1 + \dfrac{1 - 2au}{\sqrt{1+4a}}}{1 + \dfrac{1 - 2a}{\sqrt{1+4a}}} \right) \right)$$

$$-\frac{4(1+a)^2}{(2-a)^3}\log u\log\left(1-u\right)$$

$$-\frac{(1+a)\Big(9a(2-a)+2(1+a)(1+4a)\log(3a)\Big)}{(2-a)^3(1+4a)}\log\left(\frac{1+u-au^2}{2-a}\right)$$

$$-\frac{54a^2(1+a)}{(2-a)^2(1+4a)}\left(\frac{u\log u}{2+2a-3au}\right)$$

$$-\frac{4(1+a)^2}{(2-a)^3}\left(\mathrm{Li}_2(u)-\frac{\pi^2}{6}\right)$$

$$+\frac{2(1+a)^2}{(1+4a)^{3/2}(2-a)^3}$$

$$\times\Bigg(\left(1-10a-2a^2+(1+4a)^{3/2}\right)$$

$$\times\bigg(\log u\log\left(1-\frac{2au}{1-\sqrt{1+4a}}\right)$$

$$+\log(3a)\log\left(\frac{1-\dfrac{2au}{1-\sqrt{1+4a}}}{1-\dfrac{2a}{1-\sqrt{1+4a}}}\right)$$

$$+\mathrm{Li}_2\left(\frac{2au}{1-\sqrt{1+4a}}\right)-\mathrm{Li}_2\left(\frac{2a}{1-\sqrt{1+4a}}\right)\bigg)$$

$$-\left(1-10a-2a^2-(1+4a)^{3/2}\right)$$

$$\times\bigg(\log u\log\left(1-\frac{2au}{1+\sqrt{1+4a}}\right)$$

$$+\log(3a)\log\left(\frac{1-\dfrac{2au}{1+\sqrt{1+4a}}}{1-\dfrac{2a}{1+\sqrt{1+4a}}}\right)$$

$$+\mathrm{Li}_2\left(\frac{2au}{1+\sqrt{1+4a}}\right)-\mathrm{Li}_2\left(\frac{2a}{1+\sqrt{1+4a}}\right)\bigg)\Bigg)\Bigg\}. \tag{B·9}$$

Appendix C
—— *Perturbative Solutions for h_{xy}* ——

From the equation of motion (5·2), one can get an equation for $F_0(u)$,

$$\left(\frac{1}{u}\big(1-u\big)\big(1+u-au^2\big)F_0'\right)'=0. \tag{C·1}$$

A general solution is given by

$$F_0(u) = C_0 + \frac{D_0}{2-a}\left\{ -\frac{3}{2\sqrt{1+4a}}\log\left(\frac{1-\dfrac{1-2au}{\sqrt{1+4a}}}{1+\dfrac{1-2au}{\sqrt{1+4a}}}\right)\right.$$

$$\left. -\log\left(1-u\right) + \frac{1}{2}\log\left(1+u-au^2\right)\right\}. \qquad (\text{C·2})$$

Constants of integration C_0 and D_0 should be determined to be a regular function at the horizon. So we here choose $D_0 = 0$ and get

$$F_0(u) = C_0 = C. \text{ (const)} \qquad (\text{C·3})$$

By using this solution, one can get an equation for $F_1(u)$,

$$\left(\frac{1}{u}\left(1-u\right)\left(1+u-au^2\right)F_1'\right)' = i\frac{Cb}{2-a}\frac{1}{u^2}\left(1+au^2\right). \qquad (\text{C·4})$$

A general solution is

$$F_1(u) = C_1 - \frac{1}{2\left(2-a\right)^2}\left(iCb + \left(2-a\right)D_1\right)$$

$$\times\left\{\frac{3}{\sqrt{1+4a}}\log\left(\frac{1-\dfrac{1-2au}{\sqrt{1+4a}}}{1+\dfrac{1-2au}{\sqrt{1+4a}}}\right) - \log\left(1+u-au^2\right)\right\}$$

$$+\frac{1}{\left(2-a\right)^2}\left(iCb\left(1-a\right) - \left(2-a\right)D_1\right)\log\left(1-u\right). \qquad (\text{C·5})$$

Removing the singularity at the horizon, the constant D_1 should be

$$D_1 = iC\frac{1-a}{2-a}b.$$

We also impose a boundary condition $F_1(u=1) = 0$, so as to fix the constant C_1,

$$C_1 = i\frac{Cb}{2\left(2-a\right)}\left\{\frac{3}{\sqrt{1+4a}}\log\left(\frac{1-\dfrac{1-2a}{\sqrt{1+4a}}}{1+\dfrac{1-2a}{\sqrt{1+4a}}}\right) - \log\left(2-a\right)\right\}.$$

Therefore the final form is

$$F_1(u) = i\frac{Cb}{2\left(2-a\right)}\left\{-\frac{3}{\sqrt{1+4a}}\left(\log\left(\frac{1-\dfrac{1-2au}{\sqrt{1+4a}}}{1-\dfrac{1-2a}{\sqrt{1+4a}}}\right) - \log\left(\frac{1+\dfrac{1-2au}{\sqrt{1+4a}}}{1+\dfrac{1-2a}{\sqrt{1+4a}}}\right)\right)\right.$$

$$\left. +\log\left(\frac{1+u-au^2}{2-a}\right)\right\}. \qquad (\text{C·6})$$

A differential equation for $G_1(u)$ is

$$\left(\frac{1}{u}\big(1-u\big)\big(1+u-au^2\big)G_1'\right)' = \frac{Cb^2}{u^2}. \tag{C.7}$$

The equation gives us the solution

$$
G_1(u) = \widetilde{C}_1 - \frac{C\big(1-2a\big)b^2 + 3\widetilde{D}_1}{2\sqrt{1+4a}\big(2-a\big)} \log\left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2au}{\sqrt{1+4a}}}\right)
$$
$$
+ \frac{Cb^2 - \widetilde{D}_1}{2\big(2-a\big)}\Big(2\log\big(1-u\big) - \log\big(1+u-au^2\big)\Big), \tag{C.8}
$$

and the constant $\widetilde{D}_1$ might be fixed as

$$\widetilde{D}_1 = Cb^2.$$

We can also fix the constant $\widetilde{C}_1$ as

$$
\widetilde{C}_1 = \frac{Cb^2}{\sqrt{1+4a}} \log\left(\frac{1 - \dfrac{1-2a}{\sqrt{1+4a}}}{1 + \dfrac{1-2a}{\sqrt{1+4a}}}\right).
$$

Then we obtain the result,

$$
G_1(u) = -\frac{Cb^2}{\sqrt{1+4a}}\left\{ \log\left(\frac{1 - \dfrac{1-2au}{\sqrt{1+4a}}}{1 - \dfrac{1-2a}{\sqrt{1+4a}}}\right) - \log\left(\frac{1 + \dfrac{1-2au}{\sqrt{1+4a}}}{1 + \dfrac{1-2a}{\sqrt{1+4a}}}\right) \right\}. \tag{C.9}
$$

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