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Young tableaux and crystal $\mathcal{B}(\infty)$ for finite simple Lie algebras ☆

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Abstract

We study the crystal base of the negative part of a quantum group. An explicit description of the crystal for quantum finite Lie algebras of types A_n, B_n, C_n, D_n, and G_2 is given in terms of Young tableaux.

Keywords: Crystal base; Crystal; Young tableau; Semi-standard tableau; Finite simple Lie algebra

1. Introduction

The quantum group $U_q(\mathfrak{g})$ is a q-deformation of the universal enveloping algebra over a Lie algebra $\mathfrak{g}$, and crystal bases reveal the structure of $U_q(\mathfrak{g})$-modules in a very simplified form. As these $U_q(\mathfrak{g})$-modules are known to be q-deformations of modules over the original Lie algebras, knowledge of these structures also affects the study of Lie algebras.

The crystal $\mathcal{B}(\infty)$, which is the crystal base of the negative part $U_q^-(\mathfrak{g})$ of a quantum group, has received attention since the very birth of crystal base theory [5]. This is not only because it is an essential part of the *grand loop* argument proving the existence of crystal bases, but because it gives insight into the structure of the quantum group itself.

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Much effort has been made to give a description of the crystals $\mathcal{B}(\infty)$ for various Kac–Moody algebras. In the current work, we restrict ourselves to finite simple Lie algebras of types A_n, B_n, C_n, D_n, and G_2. We shall realize $\mathcal{B}(\infty)$ as crystals consisting of marginally large semi-standard tableaux, which are semi-standard tableaux of special form.

We use the definitions of *semi-standard* tableaux as given by Kashiwara and Nakashima [7]. For the G_2 type, we shall take the Young tableau description of the highest weight crystal $\mathcal{B}(\lambda)$ given in [4] as the definition of semi-standard tableaux. The trickiest part of the notion semi-standard involves something called *configuration*, but the condition large ensures that no such configuration can occur, and so we obtain a vast simplification. In the process of obtaining our results, we describe new action of Kashiwara operators on the set of marginally large tableaux, in a manner which is uniform over all types.

Our result for A_n type will be equivalent to that of the previous work [8], which had relied on a description by Cliff [1], but it will be obtained through a completely different approach.

We remark that the current work has been used recently to create yet another description of $\mathcal{B}(\infty)$ based on Nakajima monomials [10,11]. While developing the standard module theory, Nakajima discovered that the set of monomials appearing in the t-analogues of q-characters $\chi_{q,t}(M(P))$ of a standard module $M(P)$ has a crystal structure. Motivated by this observation, Kashiwara and Nakajima independently defined a crystal structure on the set of Nakajima monomials and realized the crystals $\mathcal{B}(\lambda)$ [6,13]. The monomial set can be extended so that it contains the crystal $\mathcal{B}(\infty)$ in addition to $\mathcal{B}(\lambda)$ [9–11]. This extended Nakajima monomial description of $\mathcal{B}(\infty)$ is shown to be correct by relating it to our tableaux description. There is a natural correspondence between the monomial description and our Young tableau description.

The current result can also be extended in the affine direction [12] by considering Young walls. The Young wall combinatorial scheme consists of colored blocks of various shapes, and can be viewed as an extension of the Young tableaux.

There is a work [14] that gives the connection between geometric and combinatorial descriptions of $\mathcal{B}(\lambda)$. We expect to investigate such a connection between this work and geometric descriptions of $\mathcal{B}(\infty)$. For the A_n type, this was recently done in [15], based on the earlier work [8], which contained results equivalent to the current work for the A_n type.

The paper is organized as follows. We start by introducing the notion of large semi-standard tableaux. Then, an equivalence relation is given to a collection of large semi-standard tableaux, and a crystal structure is given to the resulting set of equivalence classes. In Section 4, this new crystal is shown to be isomorphic to $\mathcal{B}(\infty)$. Our main result is given in the last section, where a set of representatives for our new crystal, called marginally large semi-standard tableaux, is explicitly presented. This gives a new description of $\mathcal{B}(\infty)$.

2. Large semi-standard tableaux

Throughout this paper, we shall be dealing with finite Lie algebras $\mathfrak{g}$ of types A_n, B_n, C_n, D_{n+1}, and G_2. Unless explicitly stated otherwise, all our discussions will hold true for each of these types. Notice that the subscript for D-type is different from the others. This is to simplify our later writing, and does not imply any restriction on the range of D-types we are considering. For the G_2 case, $n = 2$ should be assumed.

We shall assume knowledge of the basic theory of crystal bases, and related standard notation, for example, as given in the books [2,3], will be used. The crystal base of $U_q^-(\mathfrak{g})$, first introduced in [5], will be denoted by $\mathcal{B}(\infty)$. For each of the finite classical types, we shall use the definitions of *semi-standard* tableaux as given by Kashiwara and Nakashima [7]. For the G_2 type, even though there is a slight risk of confusion, we shall take the Young tableau description of highest weight crystal $\mathcal{B}(\lambda)$ given in [4] as the definition of semi-standard tableaux. Since the first of these two works is a rather well-known result, and since the second is very similar in spirit to the first, we refer readers to the original papers and shall not repeat the complicated definitions here. The set of alphabets to be used inside the boxes constituting the Young tableaux for each type will be denoted commonly by J, and it will be equipped with an ordering $\prec$, as given in [4,7]. For example, in the C_n case, it would be

$$J = \{1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}.$$

Also, based on results of the same papers, we shall identify elements of the highest weight crystal $\mathcal{B}(\lambda)$ with semi-standard tableaux of shape λ, which we collectively denote as $\mathcal{T}(\lambda)$.

For later use, we recall the Kashiwara operator action on these tableaux. We first read the boxes in the tableau through the *far eastern reading* and write down the boxes in *tensor product form*. That is, we read through each column from top to bottom starting from the rightmost column, continuing to the left. The following diagram gives an example.

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & \bar{1} \\ \hline 2 & 3 & \bar{3} & \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & \bar{1} & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} = \boxed{\bar{1}} \otimes \boxed{3} \otimes \boxed{\bar{2}} \otimes \boxed{1} \otimes \boxed{\bar{3}} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{\bar{1}} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3}.$$

Then, we apply the *tensor product rule* to decide on which box to apply $\tilde{f}_i$ or $\tilde{e}_i$ to. After application of the Kashiwara operator to one of the boxes, they are gathered back into the original form.

In practice, the tensor product rule on multiple tensors can be applied through calculation of the i*-signature*. This is done as follows.

(1) First, under each tensor component x, write down $\varepsilon_i(x)$-many 1s followed by $\varphi_i(x)$-many 0s.
(2) Then, from the long sequence of mixed 0s and 1s, successively cancel out every occurrence of a $(0, 1)$ pair until we arrive at a sequence of 1s followed by 0s, reading from left to right. This is called the i-signature of the whole tensor product form.
(3) To apply $\tilde{f}_i$ to the whole product, apply it to the single tensor component corresponding to the leftmost 0 remaining in the i-signature. If no 0 remains, the result of $\tilde{f}_i$ action is set to zero.
(4) Similarly, for $\tilde{e}_i$, apply it to the component corresponding to the rightmost 1, or set it to zero when no 1 remains.

We wish to restrict the set of dominant integral weights P^+ slightly for some of the classical types.

- A_n case: $\hat{P}^+ := P^+$;
- B_n case: $\hat{P}^+ := \{\lambda \in P^+ \mid \lambda(h_n) \text{ is even}\}$;
- C_n case: $\hat{P}^+ := P^+$;
- D_{n+1} case: $\hat{P}^+ := \{\lambda \in P^+ \mid \lambda(h_n) = \lambda(h_{n+1})\}$;
- G_2 case: $\hat{P}^+ := P^+$.

$$A_2 \text{ case:} \quad \begin{array}{cccccc} 1&1&1&1&1&3 \\ 2&2&3&&& \end{array} \qquad \begin{array}{cccccc} 1&1&1&2&3&3 \\ 2&2&3&&& \end{array}$$

$$B_3 \text{ case:} \quad \begin{array}{cccccccc} 1&1&1&1&1&1&\bar{3}&\bar{1} \\ 2&2&2&\bar{2}&&&& \\ 3&0&&&&&& \end{array} \qquad \begin{array}{ccccc} 1&1&1&3&\bar{1} \\ 2&3&0&\bar{3}& \\ 3&0&&& \end{array}$$

$$C_4 \text{ case:} \quad \begin{array}{cccccccccc} 1&1&1&1&1&1&1&1&4&\bar{1} \\ 2&2&2&2&2&\bar{3}&\bar{2}&&& \\ 3&3&3&&&&&&& \\ 4&\bar{4}&&&&&&&& \end{array} \qquad \begin{array}{ccccccc} 1&1&1&1&2&4&\bar{3} \\ 2&2&3&&&& \\ 3&3&\bar{4}&&&& \\ \bar{4}&\bar{4}&&&&& \end{array}$$

$$D_4 \text{ case:} \quad \begin{array}{ccccccc} 1&1&1&1&1&1&4 \\ 2&2&2&\bar{4}&\bar{4}&& \\ 3&\bar{3}&&&&& \end{array} \qquad \begin{array}{cccccc} 1&1&2&2&3&\bar{4} \\ 3&4&4&\bar{3}&\bar{2}& \\ \bar{4}&&&&& \end{array}$$

$$G_2 \text{ case:} \quad \begin{array}{cccccccccc} 1&1&1&1&2&3&0&\bar{3}&\bar{2}&\bar{1} \\ 2&2&3&&&&&&& \end{array} \qquad \begin{array}{cccccc} 1&2&0&\bar{3}&\bar{3}&\bar{2} \\ 2&3&\bar{2}&&& \end{array}$$

Fig. 1. Large (left) and non-large (right) tableaux.

Notice that for $\lambda \in \hat{P}^+$, elements of $\mathcal{B}(\lambda)$ become the most usual tableaux, in the sense that they do not involve any half-size boxes or other complications. It is also clear that given any $\lambda \in P^+$, we may always find a larger $\mu \in \hat{P}^+$, that is, one such that $\mu - \lambda \in P^+$.

We borrow the notion of *large* semi-standard tableaux from [1]. For the remainder of this paper, the top row of a tableau shall always be counted as the *first* row.

Definition 2.1. A semi-standard tableau T of shape $\lambda \in \hat{P}^+$ is *large* if it consists of n non-empty rows, and if for each $1 \leqslant i \leqslant n$, the number of i-boxes in the ith row is strictly greater than the number of all boxes in the $(i+1)$th row. In particular the nth row of T contains at least one n-box. For each finite type, denote by $\mathcal{T}(\lambda)^L$, the set of all large semi-standard tableaux of shape λ.

Once again, we remind readers that we are giving this definition for each of the types A_n, B_n, C_n, D_{n+1}, and G_2. The n appearing in the definition is meant to be the same n used as subscripts for the algebra types, with $n = 2$ for the G_2 case.

In Fig. 1, for some of the finite types, we give examples of semi-standard tableaux. The ones on the left are large, and the ones on the right are not large.

3. The new crystal $\mathcal{T}(\infty)$

Let us collect all large tableaux into one set (separately for each finite type).

$$\mathcal{T}^L = \bigcup_{\lambda \in \hat{P}^+} \mathcal{T}(\lambda)^L. \tag{1}$$

We shall define an equivalence relation on this set.

Definition 3.1. Two tableaux $T_1, T_2 \in \mathcal{T}^L$ are *related*, written $T_1 \sim T_2$, if for each $1 \leqslant i \leqslant n$ and $j \in J$ such that $j \succ i$, the number of j-boxes appearing in the ith rows of T_1 and T_2 are equal.

It is trivial to verify that the above gives an equivalence relation. We fix a notation

$$\mathcal{T}(\infty) := \mathcal{T}^L/\sim \tag{2}$$

for the set of equivalence classes. As is customary, $\bar{T}$ will denote the equivalence class containing $T \in \mathcal{T}^L$. This section is devoted to providing $\mathcal{T}(\infty)$ with a crystal structure.

Let us start with the Kashiwara operators.

Lemma 3.2. *Fix $i \in I$.*

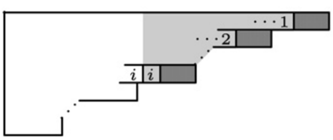
(1) *If tableau T is large, then $\tilde{f}_i T$ is never zero.*
(2) *Given any element of $\mathcal{T}(\infty)$, it is always possible to choose its representative $T \in \mathcal{T}^L$ in such a way that $\tilde{f}_i T$ is large.*
(3) *If $T_1, T_2 \in \mathcal{T}^L$ belong to the same equivalence class and $\tilde{f}_i T_1$ and $\tilde{f}_i T_2$ are both large, then $\tilde{f}_i T_1$ and $\tilde{f}_i T_2$ belong to the same equivalence class.*
(4) *If tableau T is large, then $\tilde{e}_i T$ is either zero or large.*
(5) *If $T_1, T_2 \in \mathcal{T}^L$ belong to the same equivalence class, then either $\tilde{e}_i T_1$ and $\tilde{e}_i T_2$ are both zero, or $\tilde{e}_i T_1$ and $\tilde{e}_i T_2$ belong to the same equivalence class.*

Proof. (1) It suffices to show that, after all canceling out, at least one 0 remains in the i-signature for T. Consider the rightmost i-block in the ith row of T. The signature to be written under it in the tensor form of T is 0. Notice that the condition *large* guarantees it to be the lowest block in its column. Such careful consideration of both the conditions *large* and *semi-standard* for each of the finite types will show that the signature 0 under that block will not be canceled out by signatures from blocks contained in any of the columns sitting to its left.

(2) Given any $T \in \mathcal{T}^L$ which is a representative for $b \in \mathcal{T}(\infty) = \mathcal{T}^L/\sim$, let us create a *larger* representative of b. First, construct a column consisting of i boxes, with k-box sitting in the kth row ($1 \leqslant k \leqslant i$). Consider the rightmost i-box sitting in the ith row of T and insert the constructed column to its left. It is clear that this new tableau T' is a (large) representative for b.

Now, during the proof of item (1) of this lemma, we saw that if we apply $\tilde{f}_i$ to T', it will act on either the rightmost i-block in the ith row of T', or on one of the boxes sitting in columns to its right. Due to the column we have inserted, neither case will affect the largeness of T', and hence the result is obtained.

(3) The tableaux T_1 and T_2 (or any other large tableaux) will take the following form, where we have shaded some of the blocks so that we may easily refer to them below.



We already know from the proof of item (1) of this lemma that $\tilde{f}_i$ will not act on any of the boxes contained in the unshaded part. It will act on either the light shaded i-box or on the dark shaded part. Notice that, for all finite types, j-boxes with $j < i$ do not contribute to i-signatures, hence except for the i-box, none of the light shaded part affects the i-signature for the two tableaux. Also, by definition of the equivalence relation, the dark shaded parts will be identical for the two.

Hence $\tilde{f}_i$ will act on two *corresponding* boxes contained in the two tableaux. This will result in the two tableaux being related even after $\tilde{f}_i$ action.

(4) For any large tableau T, whose i-signature does not contain any 1, such as the highest weight tableau u_λ, we have $\tilde{e}_i T = 0$. For the case when at least one 1 remains in the i-signature for T, careful consideration of both the conditions *large* and *semi-standard* will show that the rightmost 1 in the i-signature for T would have been written under a dark shaded block of the diagram for proof of item (3). Hence $\tilde{e}_i$ will act on one of the dark shaded parts of the tableau T. None of these cases would affect the largeness of T, and hence the result is obtained.

(5) This may be proved as in the proof for item (3) with $\tilde{f}_i$ changed to $\tilde{e}_i$, together with the proof of item (4). $\square$

It is now clear that, given $b \in \mathcal{T}(\infty)$ and $i \in I$, we may define

$$\tilde{f}_i b = \overline{\tilde{f}_i T} \in \mathcal{T}(\infty), \tag{3}$$

$$\tilde{e}_i b = \overline{\tilde{e}_i T} \in \mathcal{T}(\infty) \cup \{0\} \tag{4}$$

by choosing an appropriate representative T for b.

Lemma 3.3. *If $T_1 \in \mathcal{T}(\lambda_1)^L$ and $T_2 \in \mathcal{T}(\lambda_2)^L$ are related to each other with* $\mathrm{wt}(T_1) = \lambda_1 - \xi_1$ *and* $\mathrm{wt}(T_2) = \lambda_2 - \xi_2$*, then $\xi_1 = \xi_2$.*

Based on this trivial lemma, for $\bar{T} \in \mathcal{T}(\infty)$ with $T \in \mathcal{T}(\lambda)^L$, we can define

$$\mathrm{wt}(\bar{T}) = \mathrm{wt}(T) - \lambda. \tag{5}$$

To complete the description of the crystal structure, it only remains to define

$$\varepsilon_i(\bar{T}) = \varepsilon_i(T), \tag{6}$$

$$\varphi_i(\bar{T}) = \varepsilon_i(\bar{T}) + \mathrm{wt}(\bar{T})(h_i). \tag{7}$$

These may be shown to be well-defined with the help of Lemma 3.2(5).

Theorem 3.4. *The operators given by Eqs.* (3) *to* (7)*, define a crystal structure on $\mathcal{T}(\infty)$.*

Proof. This is proved through a step by step checking of the definition for an abstract crystal. Most parts of proving that $\mathcal{T}(\infty)$ is a crystal are straightforward. We will concentrate on showing

$$\tilde{f}_i(b) = b' \quad \text{if and only if} \quad b = \tilde{e}_i(b') \quad \text{for } b, b' \in \mathcal{T}(\infty),\ i \in I,$$

which is one of the conditions that should be satisfied by a crystal.

Let $b = \overline{T}$ and $b' = \overline{T'}$ with $T \in \mathcal{T}(\lambda)^L$, $T' \in \mathcal{T}(\lambda')^L$ and $\tilde{f}_i T$, $\tilde{e}_i T' \in \mathcal{T}^L$. Assuming $b = \tilde{e}_i(b')$, since $\mathcal{T}(\lambda')$ is a crystal, we can see that

$$\tilde{f}_i(b) = \tilde{f}_i\big(\tilde{e}_i(b')\big) = \tilde{f}_i\big(\tilde{e}_i(\overline{T'})\big) = \tilde{f}_i\big(\overline{\tilde{e}_i(T')}\big) = \overline{\tilde{f}_i\big(\tilde{e}_i(T')\big)} = \overline{T'} = b',$$

showing the if part. The only if part follows similarly from

$$\tilde{e}_i(b') = \tilde{e}_i(\tilde{f}_i(b)) = \tilde{e}_i(\tilde{f}_i(\overline{T})) = \tilde{e}_i(\overline{\tilde{f}_i(T)}) = \overline{\tilde{e}_i(\tilde{f}_i(T))} = \overline{T} = b,$$

which is true when $\tilde{f}_i(b) = b'$ is assumed. □

4. Crystal isomorphism

In this section, an isomorphism between crystal $\mathcal{B}(\infty)$ and the crystal $\mathcal{T}(\infty)$, constructed in the previous section, will be given. We start by recalling the following theorem from [5].

Theorem 4.1. *For weight $\lambda \in P^+$ and irreducible highest weight module $V(\lambda)$, let $\pi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ be the $U_q^-(\mathfrak{g})$-linear homomorphism sending 1 to the highest weight vector v_λ.*

(1) *We have $\pi_\lambda(L(\infty)) = L(\lambda)$, hence π_λ induces the surjective homomorphism*

$$\bar{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda).$$

(2) *The mapping $\bar{\pi}_\lambda$, which sends $\tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} u_\infty$ to $\tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} u_\lambda$, gives a bijection between $\{b \in \mathcal{B}(\infty);\ \bar{\pi}_\lambda(b) \neq 0\}$ and $\mathcal{B}(\lambda)$.*
(3) *$\tilde{f}_i \circ \bar{\pi}_\lambda = \bar{\pi}_\lambda \circ \tilde{f}_i$.*
(4) *If $b \in \mathcal{B}(\infty)$ satisfies $\bar{\pi}_\lambda(b) \neq 0$, then $\tilde{e}_i \bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_i b)$.*

We shall adopt the notation $\bar{\pi}_\lambda$ introduced in this theorem. Let us prepare for the definition of an explicit mapping from $\mathcal{B}(\infty)$ to $\mathcal{T}(\infty)$. Recall that we are identifying elements of the highest weight crystal $\mathcal{B}(\lambda)$ with the semi-standard tableaux of $\mathcal{T}(\lambda)$.

Lemma 4.2.

(1) *Given any $b \in \mathcal{B}(\infty)$, there exists $\lambda \in \hat{P}^+$, for which $\bar{\pi}_\lambda(b)$ is large.*
(2) *Given any $b \in \mathcal{B}(\infty)$, if both $\bar{\pi}_\lambda(b)$ and $\bar{\pi}_{\lambda'}(b)$ are large, then the two belong to the same equivalence class. In particular, any large highest weight elements u_λ and $u_{\lambda'}$ are related.*

Proof. (1) Simply put, depending on the distance of b (in terms of $\tilde{f}_i$) from the highest element u_∞ of $\mathcal{B}(\infty)$, we can always choose λ large enough so that $\bar{\pi}_\lambda(b)$ is still large.

Suppose that $b \in \mathcal{B}(\infty)_{-\xi} = \{b \in \mathcal{B}(\infty) \mid \mathrm{wt}(b) = -\xi\}$ with $\xi = \sum_{i \in I} n_i \alpha_i \in Q_+$. Fix $\lambda = \sum_{i \in I} \lambda_i \Lambda_i \in \hat{P}^+$ such that each $\lambda_i > n_i$. If we write $b = \tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} u_\infty$, we have n_i-many of the indices $i_1, \dots, i_k$ equal to i, for each $i \in I$. Since $\lambda_i > n_i$, after applying $\tilde{f}_{i_1}, \tilde{f}_{i_2}, \dots, \tilde{f}_{i_k}$ to u_λ, at least one column consisting of i-many boxes with a k-box sitting in the kth row for $1 \leqslant k \leqslant i$, still remains in $\bar{\pi}_\lambda(b)$. So the semi-standard tableau given by $\bar{\pi}_\lambda(b) \in \mathcal{B}(\lambda)$ is large. This completes the proof.

(2) That any two large highest weight elements u_λ and $u_{\lambda'}$ are related follows from the definition of the equivalence relation. Starting from this point, we may use induction with the help of Lemma 3.2(3) to obtain the result. □

We are now ready to define the mapping

$$\psi : \mathcal{B}(\infty) \to \mathcal{T}(\infty). \tag{8}$$

Given any $b \in \mathcal{B}(\infty)$, choose $\lambda \in \hat{P}^+$ for which $\bar{\pi}_\lambda(b)$ is large, and set

$$\psi(b) = \overline{\bar{\pi}_\lambda(b)}. \tag{9}$$

The above lemma shows that this is well defined. We thus arrive at one of our main results.

Theorem 4.3. *The mapping* (8) *is an isomorphism between* $\mathcal{B}(\infty)$ *and* $\mathcal{T}(\infty)$.

Proof. First, notice that in the definition of ψ, the case $\bar{\pi}_\lambda(b) = 0$ is never encountered, unless b itself is zero. Hence items (3) and (4) of Theorem 4.1 show that ψ commutes with the Kashiwara operators in the strict sense.

Since, in both of the crystals $\mathcal{B}(\infty)$ and $\mathcal{T}(\infty)$, the function ε_i counts the number of times $\tilde{e}_i$ can be applied to an element before reaching zero, strict commuting between ψ and $\tilde{e}_i$ implies that ψ preserves ε_i.

Next, for $b \in \mathcal{B}(\infty)$ with $\psi(b) \in \mathcal{T}(\infty)$, let $b = \tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} u_\infty$ and suppose $\bar{\pi}_\lambda(b)$ is large. We can check that

$$\begin{aligned} \mathrm{wt}\big(\psi(b)\big) &= \mathrm{wt}\big(\overline{\bar{\pi}_\lambda(b)}\big) = \mathrm{wt}\big(\bar{\pi}_\lambda(b)\big) - \lambda = \mathrm{wt}(\tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} u_\lambda) - \lambda \\ &= -(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}) = \mathrm{wt}(b), \end{aligned}$$

showing preservation of weight by ψ. This, together with preservation of ε_i by ψ, implies preservation of φ_i by ψ. Thus, so far, we have shown that ψ is a strict crystal morphism.

Now, item (2) of Theorem 4.1 shows that ψ is surjective. With any elements b and b' of $\mathcal{B}(\infty)$ of different weights, their images under the map ψ are different because $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$ and $\mathrm{wt}(\psi(b')) = \mathrm{wt}(b')$, as was shown above. The images under ψ of two elements having the same weight are also different since we know $|\mathcal{B}(\infty)_{-\xi}| = |\mathcal{T}(\lambda)_{\lambda-\xi}|$ for all sufficiently large λ (see Corollary 4.4.5 of [5]). We have thus shown the mapping ψ to be injective and hence bijective. $\square$

Remark 4.4. For all of the symmetrizable Kac–Moody algebras, Kashiwara [5] has shown the existence of an injective strict crystal morphism

$$\phi : \mathcal{B}(\infty) \to \mathcal{B}(\infty) \otimes \mathcal{B}_{i_k} \otimes \mathcal{B}_{i_{k-1}} \otimes \cdots \otimes \mathcal{B}_{i_1},$$

where $\mathcal{B}_i = \{b_i(k) \mid k \in \mathbf{Z}\}$ are certain abstract crystal and where $S = i_1, i_2, \ldots, i_k$ is any sequence of numbers in I. This is usually referred to as the Kashiwara embedding.

In the work [1], for each of the finite classical types, Cliff fixes an explicit choice of sequence S, and describes $\phi(b)$ for each $b \in \mathcal{B}(\infty)$. This description is given in terms of number of boxes appearing in a *large* $\bar{\pi}_\lambda(b)$, where the mapping $\bar{\pi}_\lambda$ is the one given in Theorem 4.1. By carefully collecting these image points, an explicit combinatorial description for $\mathcal{B}(\infty)$ was obtained.

Note that Cliff's description of $\mathcal{B}(\infty)$ *involves* large tableaux, whereas our result on $\mathcal{B}(\infty)$ of this section is given *in terms of* large tableaux. This difference was crucial in further developments [10–12] of this work.

5. An explicit description of $\mathcal{B}(\infty)$

To achieve our final goal of giving an explicit description of $\mathcal{B}(\infty)$ in terms of tableaux, it suffices to describe an explicit set of representatives for $\mathcal{T}(\infty) = \mathcal{T}^L/\sim$ and translate the various operators on $\mathcal{T}(\infty)$ to that on the representative set.

Definition 5.1. A tableau $T \in \mathcal{T}^L$ is *marginally* large, if for $1 \leqslant i \leqslant n$, the number of i-boxes in the ith row of T is greater than the number of all boxes in the $(i+1)$th row by exactly one. In particular, the nth row of T should contain one n-box.

It is clear that the set of marginally large tableaux forms a set of representatives for $\mathcal{T}(\infty) \cong \mathcal{B}(\infty)$. In passing, we remark that the difference of numbers considered above does not have to be *one* to obtain a representative set. It suffices to fix it to some positive number for each row.

We shall now describe this representative set more explicitly for each finite type. For each case, we shall present a set of alphabets to be used inside the boxes forming the tableaux, together with an ordering on the set. Next, a set of conditions that should be satisfied by the tableaux is presented. The set of all tableaux subject to the given conditions will be the set of representatives for $\mathcal{T}(\infty) \cong \mathcal{B}(\infty)$.

These descriptions were obtained by considering all conditions defining *semi-standard* tableaux together with the condition *marginally large*. The trickiest part of the notion semi-standard involves something called *configuration*, but the condition large ensures that no such configuration can occur, and we obtain a vast simplification. The final description we give below are thus much simpler than the definition of semi-standard tableaux.

After giving out the explicit representative sets, we shall describe the action of the Kashiwara operators on these sets, in a manner which is applicable commonly to all cases. The definition for operators wt, ε_i, and φ_i remain unchanged from those given by Eqs. (5)–(7).

Sections 5.1 to 5.6 can be seen as the main contribution of this paper.

5.1. A_n case

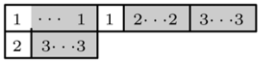
Alphabet:

$$J = \{1 \prec 2 \prec \cdots \prec n \prec n+1\}.$$

Conditions:

(1) Tableau consists of n rows.
(2) For $1 \leqslant i \leqslant n$, the ith row of the leftmost column is an i-box.
(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
(4) For $1 \leqslant i \leqslant n$, the number of i-boxes in the ith row is larger than the total number of boxes appearing in the $(i+1)$th row by exactly one.

Example 5.2. The set of representatives for $\mathcal{T}(\infty)$, in the A_2 case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$T =$  .

The element corresponding to the highest weight element u_∞ is

$$T_\infty = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

5.2. B_n case

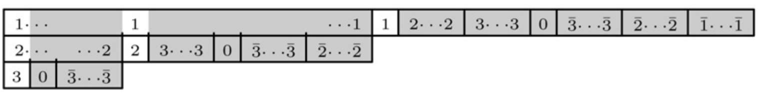
Alphabet:

$$J = \{1 \prec 2 \prec \cdots \prec n \prec 0 \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}.$$

Conditions:

(1) Tableau consists of n rows.
(2) For $1 \leqslant i \leqslant n$, the ith row of the leftmost column is an i-box.
(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
(4) For $1 \leqslant i \leqslant n$, the number of i-boxes in the ith row is larger than the total number of boxes appearing in the $(i+1)$th row by exactly one.
(5) All entries in the ith row are less than or equal to $\bar{i}$ (w.r.t. $\prec$).
(6) Index 0 appears as an entry at most once in each row.

Example 5.3. The set of representatives for $\mathcal{T}(\infty)$, in the B_3 case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$T =$  .

The element corresponding to highest weight element u_∞ is

$$T_\infty = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

5.3. C_n case

Alphabet:

$$J = \{1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}.$$

Conditions:

(1) Tableau consists of n rows.
(2) For $1 \leqslant i \leqslant n$, the ith row of the leftmost column is an i-box.
(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
(4) For $1 \leqslant i \leqslant n$, the number of i-boxes in the ith row is larger than the total number of boxes appearing in the $(i+1)$th row by exactly one.
(5) All entries in the ith row are less than or equal to $\bar{i}$ (w.r.t. $\prec$).

Example 5.4. The set of representatives for $\mathcal{T}(\infty)$, in the C_3 case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$$T = \begin{array}{|l|l|l|l|l|l|l|l|l|l|}
\hline
1 \cdots & & 1 & \cdots 1 & 1 & 2\cdots 2 & 3\cdots 3 & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} & \bar{1}\cdots\bar{1} \\
\hline
2 & \cdots\ 2 & 2 & 3\cdots 3 \quad \bar{3}\cdots\bar{3} \quad \bar{2}\cdots\bar{2} \\
\cline{1-4}
3 & \bar{3}\cdots\bar{3} \\
\cline{1-2}
\end{array}.$$

The element corresponding to highest weight element u_∞ is

$$T_\infty = \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 \\
\cline{1-2}
3 \\
\cline{1-1}
\end{array}.$$

5.4. D_{n+1} case

Alphabet:

$$J = \{1 \prec 2 \prec \cdots \prec n \prec \begin{matrix} \overline{n+1} \\ n+1 \end{matrix} \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}.$$

Conditions:

(1) Tableau consists of n rows.
(2) For $1 \leqslant i \leqslant n$, the ith row of the leftmost column is an i-box.
(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
(4) For $1 \leqslant i \leqslant n$, the number of i-boxes in the ith row is larger than the total number of boxes appearing in the $(i+1)$th row by exactly one.
(5) All entries in the ith row are less than or equal to $\bar{i}$ (w.r.t. $\prec$).
(6) $n+1$ and $\overline{n+1}$ do not appear in the same row.

Example 5.5. The set of representatives for $\mathcal{T}(\infty)$, in the D_4 case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size. Either one of 4 or $\bar{4}$ may take the place of each of the letters x, y, and z.

$$T = \begin{array}{|l|l|l|l|l|l|l|l|l|l|l|}
\hline
1 \cdots & & 1 & \cdots 1 & 1 & 2\cdots 2 & 3\cdots 3 & x\cdots x & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} & \bar{1}\cdots\bar{1} \\
\hline
2 \cdots & \cdots 2 & 2 & 3\cdots 3 \quad y\cdots y \quad \bar{3}\cdots\bar{3} \quad \bar{2}\cdots\bar{2} \\
\cline{1-4}
3 & z\cdots z \quad \bar{3}\cdots\bar{3} \\
\cline{1-2}
\end{array}.$$

The element corresponding to highest weight element u_∞ is

$$T_\infty = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,.$$

5.5. G_2 case

Alphabet:

$$J = \{1 \prec 2 \prec 3 \prec 0 \prec \bar{3} \prec \bar{2} \prec \bar{1}\}.$$

Conditions:

(1) Tableau consists of 2 rows.
(2) For $1 \leqslant i \leqslant 2$, the ith row of the leftmost column is an i-box.
(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
(4) For $1 \leqslant i \leqslant 2$, the number of i-boxes in the ith row is larger than the total number of boxes appearing in the $(i+1)$th row by exactly one.
(5) Only 2 and 3 appear as indices in the second row.
(6) Index 0 appears as an entry at most once in the first row.

Example 5.6. The set of representatives for $\mathcal{T}(\infty)$, in the G_2 case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots\ 1 & 1 & 2\cdots 2 & 3\cdots 3 & 0 & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} & \bar{1}\cdots\bar{1} \\ \hline 2 & 3\cdots 3 & \multicolumn{7}{c}{} \\ \cline{1-2}\end{array}\,.$$

The element corresponding to highest weight element u_∞ is

$$T_\infty = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

5.6. Kashiwara operators

To apply $\tilde{f}_i$ to one of the representatives, we go through the following procedure.

(1) Apply $\tilde{f}_i$ to the tableau as usual. That is, write it in tensor product form, apply tensor product rule, and assemble back into original tableau form.
(2) If the result is a large tableau, we are done. It is automatically marginally large.
(3) If the result is not large, then $\tilde{f}_i$ must have been applied to the rightmost i-box in the ith row. Insert one column consisting of i rows to the left of the box $\tilde{f}_i$ acted upon. The added column should have a k-box at the kth row for $1 \leqslant k \leqslant i$.

To apply $\tilde{e}_i$ to one of the representatives, we go through the following procedure.

(1) Apply $\tilde{e}_i$ to the tableau as usual.
(2) If the result is zero or a marginally large tableau, we are done.

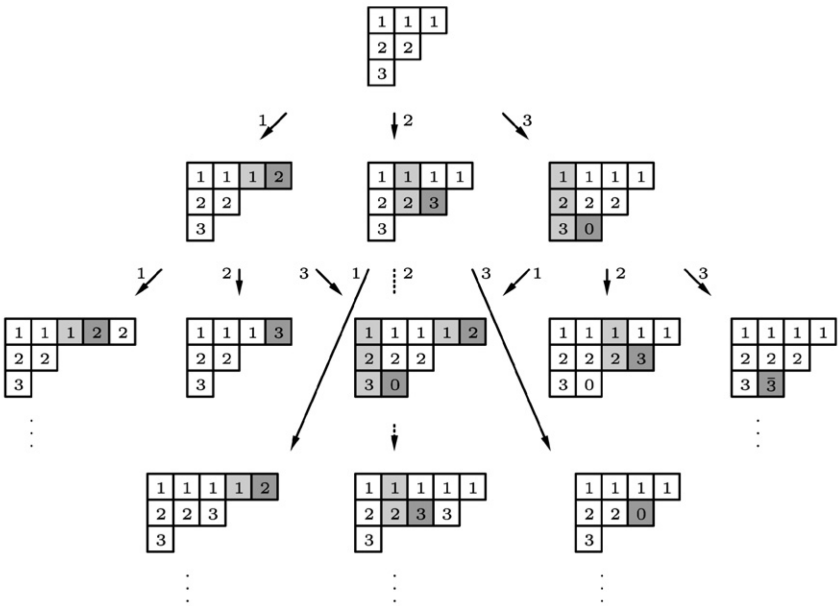


Fig. 2. Crystal $\mathcal{T}(\infty)$ for type B_3.

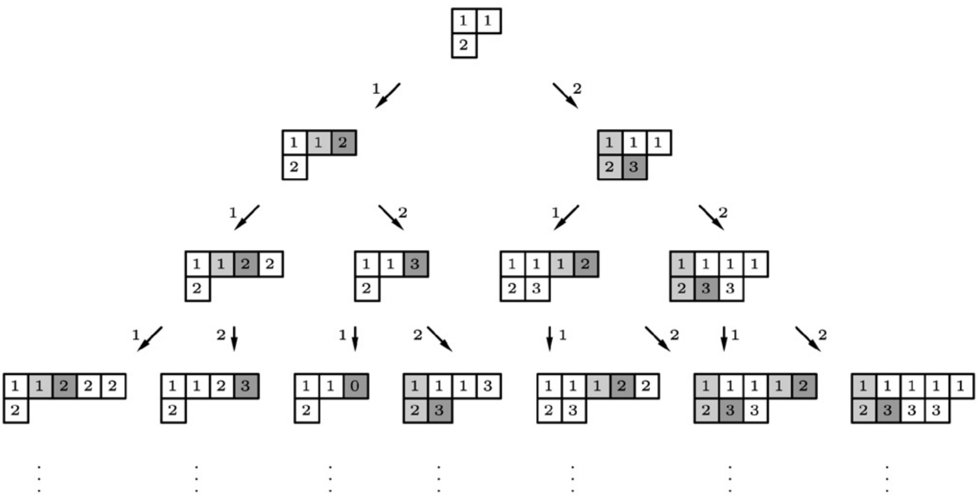


Fig. 3. Crystal $\mathcal{T}(\infty)$ for type G_2.

(3) Otherwise, the result is large but not marginally large. The $\tilde{e}_i$ operator has acted on the box sitting to the right of the rightmost i-box in the ith row. Remove the column containing the changed box. It will be of i rows and have a k-box at the kth row for $1 \leqslant k \leqslant i$.

Example 5.7. In Figs. 2 and 3, we illustrate the top part of crystal $\mathcal{T}(\infty)$ for finite types B_3 and G_2. The dark shaded blocks are the ones $\tilde{f}_i$ has acted upon, and the light shadings show columns inserted to preserve largeness.

5.7. *Comparison with a previous A_n result*

Our result on A_n-type can be found in an earlier work [8]. There, the approach was very different, relying on a work of Cliff [1], and the final result was written in a slightly different form.

The only difference with the current result is that, there, infinitely many copies of our left-most column were added to the left of each representative. This has the advantage of having the Kashiwara operators look slightly more natural. We do not *insert* or *remove* columns to remain marginally large, but *push* or *pull* infinite rows instead.

In the current work, we chose not to add these infinitely many columns, so as to keep our representatives within the frames of Young tableaux. The choice between these two presentations seems to be a matter of taste.

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