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Dense holographic QCD in the Wigner-Seitz approximation

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Abstract: We investigate cold dense matter in the context of Sakai and Sugimoto's holographic model of QCD in the Wigner-Seitz approximation. In bulk, baryons are treated as instantons on $S^3 \times R^1$ in each Wigner-Seitz cell. In holographic QCD, Skyrmions are instanton holonomies along the conformal direction. The high density phase is identified with a crystal of holographic Skyrmions with restored chiral symmetry at about $4M_{\rm KK}^3/\pi^5$. As the average density goes up, it approaches to uniform distribution while the chiral condensate approaches to p-wave over a cell. The chiral symmetry is effectively restored in long wavelength limit since the chiral order parameter is averaged to be zero over a cell. The energy density in dense medium varies as $n_B^{5/3}$, which is the expected power for non-relativistic fermion. This shows that the Pauli exclusion effect in boundary is encoded in the Coulomb repulsion in the bulk.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence.

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Contents

1. Introduction

The AdS/CFT approach [1] provides a powerful framework for discussing large N_c gauge theories at strong coupling $\lambda = g^2 N_c$. The model suggested by Sakai and Sugimoto (SS) [2] offers a specific holographic realization that includes N_f flavors and is chiral. For $N_f \ll N_c$, chiral QCD is obtained as a gravity dual to N_f D8-$\overline{\text{D8}}$ branes embedded into a D4 background in 10 dimensions where supersymmetry is broken by the Kaluza-Klein (KK) mechanism. The SS model yields a holographic description of pions, vectors, axials and baryons that is in good agreement with experiment [2−4]. The SS model at finite temperature has been discussed in [5] and at finite baryon density in [6−8]. The finite baryon chemical potential problem in D3/D7 model was discussed in [9] and Isospin chemical potential and glueball decay also has been discussed in [10].

Cold and dense hadronic matter in QCD is difficult to track from first principles in current lattice simulations owing to the sign problem. In large N_c QCD baryons are solitons and a dense matter description using Skyrme's chiral model [11] was originally suggested by Skyrme and others [12]. At large N_c and high density matter consisting of solitons crystallizes, as the ratio of potential to kinetic energy $\Gamma = V/K \approx N_c^2$ is much larger than 1. QCD matter at large N_c was recently revisited in [14].

The many-soliton problem can be simplified in the crystal limit by first considering all solitons to be the same and second by reducing the crystal to a single cell with boundary conditions much like the Wigner-Seitz approximation in the theory of solids. A natural way to describe the crystal topology is through T^3 with periodic boundary conditions. In so far, this problem can only be addressed numerically. A much simpler and analytically tractable approximation consists of treating each Wigner-Seitz cell as S^3 with no boundary condition involved. The result is dense Skyrmion matter on S^3 [13]. Interestingly enough, the energetics of this phase is only few percent above the energetics of a more involved numerical analysis based on T^3. Skyrmions on S^3 restore chiral symmetry on the average above a critical density. While Skyrmions on S^3 are unstable against T^3, they still capture the essentials of dense matter and chiral restoration in an analytically tractable framework.

Cold dense matter in holographic QCD is a crystal of instantons with $\Gamma = \sqrt{\lambda}/v_F \gg 1$ where $v_F \approx 1/N_c$ is the Fermi velocity. (In contrast hot holographic QCD has $\Gamma = \sqrt{\lambda} \gg 1$). When the wigner-Seitz cell is *approximated* by S^3, the pertinent instanton is defined on $S^3 \times R$. In this paper, we investigate cold QCD matter using instantons on $S^3 \times R$ in bulk. As a result the initial D4 background is *deformed* to accomodate for the S^3 which is just the back reaction of the flavour crystal structure on the pure gauge theory. Holographic dense matter can be organized in $1/\lambda$ at large N_c. The deformation is only valid for large S^3 or low densities for otherwise the metric is no longer a solution to the supergravity equations.

Below we show that the energy per unit cell ε of a D4 instanton on $S^3 \times R$ with fixed radius $\tilde{\mathcal{R}}$ is

$$\varepsilon = M_0 \left[n_B + \frac{a}{\lambda} n_B^{1/3} + \frac{b}{\lambda} Z_c n_B^2 \right] \tag{1.1}$$

with $Z_c \leq \tilde{\mathcal{R}} \sim \lambda^0$ a cutoff in the holographic direction. So effectively D4 instantons living in $S^3 \times [0, Z_c]$ carry larger energy density than D4 instantons living in $R^3 \times R$ (low density). When $Z_c = \tilde{\mathcal{R}} \sim 1/n_B^{1/3}$, the n_B^2 softens to $n_B^{5/3}$ reverting the situation (high density). Indeed, we find that for $Z_c = \tilde{\mathcal{R}}$ the transition takes place for $n_B^c = \frac{4}{\pi^5} M_{\rm KK}^3 \approx 1.26 n_0$ with n_0 the nuclear matter density($M_{\rm KK} = 500$MeV). The resulting crystal is effectively four dimensional. The validity of the metric deformation just noted suggests that the transition occurs at sufficiently low baryonic densities just like in nuclear matter.

In section 2, we define this deformation and discuss the D8 brane embedding structure. The instantons on $S^3 \times R$ in the flavour D8 brane is discussed in section 3. In section 4,5,6 we derive the equation of state of cold holographic matter using the small size instanton expansion and in general. In section 7 we show how the holographic small instantons in bulk transmute large size Skyrmions on the boundary. The comparison to other models of nuclear matter is carried in section 8. Our conclusions are in section 9.

2. D8 brane action

We consider crystallized skyrmions at finite density in the Wigner Seitz approximation. Spatial R^3 is naturally converted to T^3 with periodic boundary conditions. As a result the

D4 background geometry is deformed. The baryons are then instantons on $T^3 \times R$. Most solutions are only known numerically on the lattice. A simpler and analytically tractable analysis that captures the essentials of dense matter is to substitute T^3 by S^3 in bulk with no boundary conditions altogether. As a result, the D4 background dual to the crystal is modified with the boundary special space as S^3. Specifically, the 10 dimensional space is that of $(R^1 \times S^3) \times R^1 \times S^4$. The ensuing metric on D4 is therefore

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(-dt^2 + \mathcal{R}^2 d\Omega_3^2 + f(U) d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right) , \quad (2.1)$$

$$d\Omega_3 \equiv d\psi^2 + \sin^2\psi \; d\theta^2 + \sin^2\psi \sin^2\theta \; d\phi^2 , \qquad f(U) \equiv 1 - \frac{U_{\rm KK}^3}{U^3} , \quad (2.2)$$

$$e^{\phi} = g_s \left(\frac{U}{R}\right)^{3/4} , \qquad F_4 \equiv dC_3 = \frac{2\pi N_c}{V_4} \epsilon_4 , \quad (2.3)$$

While this compactified metric is not an exact solution to the general relativity (GR) equations for small size S^3, it can be regarded as an approximate solution for large size S^3. Indeed, in this case, the GR equations are seen to be sourced by terms wich are down by the size of S^3. Here, (2.3) can be regarded as an approximation to the stable metric with T^3 for a dense matter analysis. Clearly, the former is unstable against decay to the latter, which will be reflected by the fact that the energy of dense matter on S^3 is higher than that on T^3. As indicated in the introduction, the Skyrmion analysis shows that the energy on S^3 is only few percent that of T^3. So we expect the current approximation to capture the essentials of dense matter in holographic QCD. Specifically, the nature and strength of the attraction and repulsion in dense matter. Indeed, this will be the case as we will detail below.

Now, consider N_f probe D8-branes in the N_c D4-branes background. With $U(N_f)$ gauge field A_M on the D8-branes, the effective action consists of the DBI action and the Chern-Simons action

$$S_{\rm D8} = S_{\rm DBI} + S_{\rm CS} ,$$

$$S_{\rm DBI} = -T_8 \int d^9x \; e^{-\phi} \; {\rm tr} \sqrt{-\det(g_{MN} + 2\pi\alpha' F_{MN})} , \quad (2.4)$$

$$S_{\rm CS} = \frac{1}{48\pi^3} \int_{D8} C_3 {\rm tr}\, F^3 \; . \quad (2.5)$$

where $T_8 = 1/((2\pi)^8 l_s^9)$, the tension of the D8-brane, $F_{MN} = \partial_M A_N - \partial_N A_M - i\,[A_M, A_N]$ $(M, N = 0, 1, \cdots, 8)$, and g_{MN} is the induced metric on the D8-branes

$$ds_{D8}^2 = \left(\frac{U}{R}\right)^{3/2} (-dt^2 + \mathcal{R}^2 d\Omega_3^2) + g_{\sigma\sigma} d\sigma^2 + \left(\frac{R}{U}\right)^{3/2} U^2 d\Omega_4^2 , \quad (2.6)$$

$$g_{\sigma\sigma} \equiv G_{\tau\tau} \partial_\sigma\tau \partial_\sigma\tau + G_{UU} \partial_\sigma U \partial_\sigma U , \quad (2.7)$$

where G_{MN} refer to the background metric (2.1) and the profile of the D8 brane is parameterized by $U(\sigma)$ and $\tau(\sigma)$.

The gauge field A_M has nine components, $\mathcal{A}_0$, $A_i = A_{1,2,3}$, $A_\sigma(= A_4)$, and $A_\alpha(\alpha = 5,6,7,8$, the coordinates on the S^4). We assume

$$\mathcal{A}_0 = \mathcal{A}_0(\sigma) \in U(1)\,, \tag{2.8}$$

$$(A_i = A_i(x^i,\sigma), \quad A_\sigma = A_\sigma(x^i,\sigma)) \in SU(N_f)\,, \tag{2.9}$$

$$A_\alpha = 0\;. \tag{2.10}$$

Then the action becomes 5-dimensional:

$$S_{\rm DBI} = -\frac{8\pi^2 T_8 R^3}{3g_s}\,{\rm tr}\int dt\epsilon_3 d\sigma\; U \left[\left\{\left(\frac{U}{R}\right)^{3/2} g_{\sigma\sigma} - (2\pi\alpha')^2(\partial_\sigma\mathcal{A}_0)^2\right\}\left\{\left(\frac{U}{R}\right)^3 + \frac{1}{2}(2\pi\alpha')^2 F_{ij}F^{ij}\right\} + \left(\frac{U}{R}\right)^3 (2\pi\alpha')^2 F_{\sigma i}F_\sigma{}^i + \frac{1}{4}(2\pi\alpha')^4(\epsilon_{ijk}F_{i\sigma}F_{jk})(\epsilon_{ijk}F^i{}_\sigma F^{jk})\right]^{1/2}, \tag{2.11}$$

$$S_{\rm CS} = \frac{N_c}{24\pi^2}{\rm tr}\int \mathcal{A}\wedge F\wedge F\,, \tag{2.12}$$

where ϵ_3 is the volume form of S^3 space and the indices $i,j,k(\in\{\psi,\theta,\phi\})$ are raised by the metric $\tilde{g}^{ij}$ defined by

$$\tilde{g}^{ij} = \left(\frac{1}{\mathcal{R}^2}, \frac{1}{\mathcal{R}^2\sin^2\psi}, \frac{1}{\mathcal{R}^2\sin^2\psi\sin^2\theta}\right)\;. \tag{2.13}$$

3. Instanton in $S^3 \times R^1$

Only $\mathcal{A}_0$ will be determined dynamically in the given instanton background A_i, A_σ. The exact background instanton solution is unknown. Thus we start with an approximate solution which is the SU(2) Yang-Mills instanton solution in the space with metric,

$$ds^2 = d\sigma^2 + \mathcal{R}^2 d\Omega_3\;. \tag{3.1}$$

This metric is different from our metric in (2.6) and (2.11), where there are warping factors. Furthermore our action is the nonlinear DBI action and not a Yang-Mills action. However it can be shown that the Yang-Mills instanton in the space (3.1) is the leading order solution of $1/\lambda$ expansion of the full metric and the DBI action as shown in [3]. So the solution can be used in the leading order calculation.

We summarize here the (anti) self dual instanton solution obatained in [15]. Using the ansatz,

$$A = f(\sigma)U^{-1}dU\,, \qquad U \equiv \cos\psi + i\tau_a\hat{r}^a(\theta,\phi)\sin\psi\,, \tag{3.2}$$

we get the field strength, in terms of vielbein whose relation to the coordinate ψ,θ,ϕ, is specified in [15],

$$F = \frac{(\partial_\sigma f)\tau_a}{\mathcal{R}}\; e^0\wedge e^a + \frac{1}{2}\left[\frac{2(f^2-f)\tau_d\epsilon^d{}_{bc}}{\mathcal{R}^2}\right]\; e^b\wedge e^c\,, \tag{3.3}$$

where we used $L_a = U^{-1}\partial_a U = \tau_a/\mathcal{R}$. If we require (anti) self-duality,

$$\partial_\sigma f = \pm \frac{2(f^2 - f)}{\mathcal{R}}, \tag{3.4}$$

then f is determined as

$$f_\pm \equiv \frac{1}{1 + e^{\mp 2(\sigma - \sigma_0)/\mathcal{R}}}, \tag{3.5}$$

so the field strength of one (anti) instanton solution is

$$F^\pm = (\partial_\sigma f_\pm)\frac{\tau_a}{\mathcal{R}}\left(e^0 \wedge e^a \pm \frac{1}{2}\epsilon^a{}_{bc} e^b \wedge e^c\right). \tag{3.6}$$

4. D8 brane plus Instanton

Now that we have the background instanton configurations, the remaining dynamical variables are τ and $\mathcal{A}_0$. However it can be shown that $\partial_\sigma \tau(\sigma) = 0$ is always a solution of the Euler-Lagrange equation regardless of the gauge field. For simplicity we will work with this specific configuration so that the only dynamical variable is $\mathcal{A}_0$. Let us parameterize $\mathcal{A}_0$ by Z defined as

$$\begin{aligned} U &\equiv (U_{\rm KK}^3 + U_{\rm KK}\sigma^2)^{1/3}, \\ Z &\equiv \frac{\sigma}{U_{\rm KK}}, \quad K \equiv 1 + Z^2. \end{aligned} \tag{4.1}$$

Then the field strength is expressed in terms of Z and the dimensionless radius $\widehat{\mathcal{R}} \equiv \mathcal{R}/U_{\rm KK}$,

$$\begin{aligned} F_{Za} &= \frac{1}{U_{\rm KK}} f' \, \frac{\tau_a}{\widehat{\mathcal{R}}}, \\ F_{ab} &= \frac{1}{U_{\rm KK}^2} f' \, \frac{\epsilon_{ab}{}^c \tau_c}{\widehat{\mathcal{R}}}, \end{aligned} \tag{4.2}$$

where $f' \equiv \partial_Z f$. The instanton configuration is

$$f_\pm = \frac{1}{1 + e^{\mp 2(Z - Z_0)/\widehat{\mathcal{R}}}}. \tag{4.3}$$

The DBI action reads

$$\begin{aligned} S_{\rm DBI} = &-\frac{8\pi^2 T_8 R^3}{3g_s} \,{\rm tr} \int dt \epsilon_3 dZ K^{1/3} \\ &\left[\left\{\left(\frac{4}{9}\right) U_{\rm KK}^2 K^{-1/3} - (2\pi\alpha')^2(\mathcal{A}_0')^2\right\}\left\{K\left(\frac{U_{\rm KK}}{R}\right)^3 + \frac{1}{2}(2\pi\alpha')^2 F_{ab}^2\right\}\right. \\ &\left. + K\left(\frac{U_{\rm KK}}{R}\right)^3 (2\pi\alpha')^2 F_{Za}^2 + \frac{1}{4}(2\pi\alpha')^4(\epsilon_{abc}F_{aZ}F_{bc})^2 \right]^{1/2}, \end{aligned} \tag{4.4}$$

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where $\mathcal{A}_0' \equiv \partial_Z \mathcal{A}_0$. We are using the same vielbein coordinates as (3.3). Since the instanton size ($\mathcal{R}$) is of order $\mathcal{O}(\lambda^{-1/2})$ we define a new dimensionless parameter $\widetilde{\mathcal{R}}$, which is order of (λ^0), as

$$\widetilde{\mathcal{R}} \equiv \sqrt{\lambda}\widehat{\mathcal{R}} = \sqrt{\lambda}\frac{\mathcal{R}}{U_{\rm KK}} \ . \tag{4.5}$$

Furthermore we rescale the coordinate and the instanton field strength for a systematic $1/\lambda$ expansion

$$\begin{aligned} x^a &\to \lambda^{-1/2} x^a \,, \quad Z \to \lambda^{-1/2} Z \,, \quad t \to t \,, \\ F_{ab} &\to \lambda F_{ab} \,, \quad F_{aZ} \to \lambda F_{aZ} \,, \quad \mathcal{A}_0 \to \mathcal{A}_0 \,, \\ K &= (1+Z^2) \to \left(1 + \frac{1}{\lambda}Z^2\right) \equiv K_\lambda, \end{aligned} \tag{4.6}$$

so all coordinates and gauge fields become of order of $\mathcal{O}(\lambda^0)$ and we

By using the instanton solution (4.3) we get

$$\begin{aligned} S_{\rm DBI} = &-\frac{N_c\lambda}{3^9\pi^5 U_{\rm KK} M_{\rm KK}^{-3}} \ {\rm tr} \int dt\epsilon_3 dZ K_\lambda^{1/3} \\ &\left[\left\{1 + \frac{3^7\pi^2}{4M_{\rm KK}^2 U_{\rm KK}^2} K_\lambda^{1/3}\frac{f'^2}{\widetilde{\mathcal{R}}^2} - \frac{1}{\lambda}\frac{3^6\pi^2}{4M_{\rm KK}^2} K_\lambda^{1/3} (\mathcal{A}_0')^2\right\}\right. \\ &\quad \left.\left\{M_{\rm KK}^2 U_{\rm KK}^2 K_\lambda^{4/3} + \frac{3^7\pi^2}{4M_{\rm KK}^2 U_{\rm KK}^2} K_\lambda^{1/3}\frac{f'^2}{\widetilde{\mathcal{R}}^2}\right\} \quad \right]^{1/2} \end{aligned} \tag{4.7}$$

If we let $U_{\rm KK} = M_{\rm KK}^{-1}$ for simplicity, then the DBI action yields

$$\begin{aligned} S_{\rm DBI} &= -dN_c\lambda \int dt\epsilon_3 dZ \sqrt{\left\{1 + K_\lambda^{1/3}\widetilde{F}^2 - \frac{1}{\lambda}K_\lambda^{1/3}(\widetilde{\mathcal{A}}_0')^2\right\}\left\{K_\lambda^{3/4} + K_\lambda^{1/3}\widetilde{F}^2\right\}} \\ &= -dN_c\lambda \int dt\epsilon_3 dZ \left[1 + \frac{3Z^2}{8\lambda} + \widetilde{F}^2 + \frac{Z^2}{3\lambda}\widetilde{F}^2 - \frac{1}{2\lambda}(\widetilde{\mathcal{A}}_0')^2 + \mathcal{O}((1/\lambda)^2)\right] \,, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} d &\equiv \frac{2M_{\rm KK}^4}{3^9\pi^5} \,, \quad \widetilde{\mathcal{A}}_0 \equiv \frac{3^3\pi}{2M_{\rm KK}}\mathcal{A}_0 \,, \quad \widetilde{F}^2 \equiv \frac{3^7\pi^2}{4}J \,, \\ J &\equiv \frac{f'^2}{\widetilde{\mathcal{R}}^2} = \frac{{\rm sech}^4(Z/\widetilde{\mathcal{R}})}{4\widetilde{\mathcal{R}}^4} \sim \frac{1}{3\widetilde{\mathcal{R}}^3}\delta(Z) \,, \\ \partial_Z\mathcal{K} &\equiv \partial_Z \frac{1}{6\widetilde{\mathcal{R}}^3}\left[\tanh(Z/\widetilde{\mathcal{R}})\left(1 + \frac{1}{2}{\rm sech}^2(Z/\widetilde{\mathcal{R}})\right)\right] \sim \frac{1}{6\widetilde{\mathcal{R}}^3}{\rm sgn}(Z) \end{aligned} \tag{4.9}$$

The Chern-Simons action does not change by the recaling (4.6), and it is order of λ^0. With the instanton solution (4.3) the Chern-Simons action reduces to

$$\begin{aligned} S_{\rm CS} &= \frac{N_c}{24\pi^2}{\rm tr} \int \mathcal{A}\wedge F \wedge F = \frac{N_c}{8\pi^2}{\rm tr}\int dt\epsilon_3 dZ \mathcal{A}_0 \frac{1}{2}\left(24M_{\rm KK}^3\frac{f'^2}{\widetilde{\mathcal{R}}^2}\right) \\ &= cN_c \int dt\epsilon_3 dZ \widetilde{\mathcal{A}}_0 \widetilde{F}^2 \,, \end{aligned} \tag{4.10}$$

where

$$c \equiv \frac{4M_{\rm KK}^4}{3^9\pi^5}\ . \tag{4.11}$$

It also can be written as

$$S_{\rm CS} = 3N_c\widetilde{\mathcal{R}}^3 \int dZ \mathcal{A}_0 \partial_Z \mathcal{K} \to N_c\,, \qquad \text{for } \mathcal{A}_0 = 1\,, \tag{4.12}$$

which confirms that the field configuration (4.2), and (4.3) describe the single (anti) self dual instanton since $S_{\rm CS}$ corresponds to $N_c \times$ the Pontryagin index when $\mathcal{A}_0 = 1$.

5. Equation of state in $1/\lambda$

The equation of state of cold holographic matter is the energy following from the action functional. The total action up to order of λ^0 is

$$\begin{aligned} S &\equiv \int dt\epsilon_3 dZ(\mathcal{L}_{\rm DBI} + \mathcal{L}_{\rm CS}) \\ &= -dN_c \int dt\epsilon_3 dZ\ \left[\lambda\widetilde{F}^2 + \frac{Z^2}{3}\widetilde{F}^2 - \frac{1}{2}(\widetilde{\mathcal{A}}_0')^2\right] + cN_c\int d^4x dZ\widetilde{\mathcal{A}}_0\widetilde{F}^2\ . \end{aligned} \tag{5.1}$$

where $\mathcal{A}_0$ is an auxillary field with no time-dependence that can be eliminated by the equation of motion or Gauss law,

$$\Pi' = cN_c\widetilde{F}^2\,, \tag{5.2}$$

with

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\widetilde{\mathcal{A}}_0'} = dN_c\widetilde{\mathcal{A}}_0'\,, \tag{5.3}$$

The integral of the equation of motion with $\widetilde{F}^2$ in (4.9) is

$$\begin{aligned} \Pi(Z) &= \Pi(\infty)\left[\tanh(Z/\widetilde{\mathcal{R}})\left(1+\frac{1}{2}\mathrm{sech}^2(Z/\widetilde{\mathcal{R}})\right)\right]\,, \\ \Pi(\infty) &= \frac{M_{\rm KK}^4}{54\pi^3}\frac{N_c}{\widetilde{\mathcal{R}}^3}\,, \end{aligned} \tag{5.4}$$

where we have set $\Pi(0) = 0$.

The energy of one cell is

$$\begin{aligned} \mathcal{E}_{\rm cell} &= -\int \epsilon_3 dZ(\mathcal{L}_{\rm DBI} + \mathcal{L}_{\rm CS}) \\ &= dN_c\int\epsilon_3\ dZ\ \left[\lambda\widetilde{F}^2 + \frac{Z^2}{3}\widetilde{F}^2 - \frac{1}{2}\frac{\Pi^2}{(dN_c)^2}\right] - \int\epsilon_3\ dZ\widetilde{\mathcal{A}}_0\Pi'\,, \\ &= dN_c\ \int\epsilon_3\ dZ\ \left[\lambda\widetilde{F}^2 + \frac{Z^2}{3}\widetilde{F}^2 + \frac{1}{2}\frac{\Pi^2}{(dN_c)^2}\right] - \int\epsilon_3\ \widetilde{\mathcal{A}}_0(Z)\Pi(Z)\Big|_{-\infty}^{\infty}\ . \end{aligned} \tag{5.5}$$

Thus the energy density (ε) of the crystalline structure is

$$
\begin{aligned}
\varepsilon &\equiv \frac{N\mathcal{E}_{\rm cell}}{V} \approx \frac{\mathcal{E}_{\rm cell}}{\int \epsilon_3} \\
&= dN_c \int dZ \left[\lambda \widetilde{F}^2 + \frac{Z^2}{3}\widetilde{F}^2 + \frac{1}{2}\frac{\Pi^2}{(dN_c)^2}\right] - \widetilde{\mathcal{A}}_0(Z)\Pi(Z)\Big|_{-\infty}^{\infty}\,, \qquad (5.6)
\end{aligned}
$$

where N is the total number of baryons(cells) and V is the total volume which is approximated by $N\int \epsilon_3$. Interestingly the second term in (5.6) is equal to $\mu_B n_B$ since the density and the baryon chemical potential is given by

$$
n_B = \frac{1}{\int \epsilon_3} = \frac{1}{2\pi^2(U_{\rm KK}\widetilde{\mathcal{R}})^3} = \frac{1}{2\pi^2(\sqrt{\lambda}\mathcal{R})^3}\,, \qquad \mu_B \equiv N_c \mathcal{A}_0(\infty)\,. \qquad (5.7)
$$

respectively and because

$$
\widetilde{\mathcal{A}}_0(Z)\Pi(Z)\Big|_{-\infty}^{\infty} = 2\widetilde{\mathcal{A}}_0(\infty)\Pi(\infty) = N_c\mathcal{A}_0(\infty)\frac{1}{2\pi^2(U_{\rm KK}\widetilde{\mathcal{R}})^3} = \mu_B n_B\,, \qquad (5.8)
$$

Notice that $\widetilde{\mathcal{R}}$ is of order of $(\lambda)^0$ from (4.5) so the baryon density is of order $(N_c\lambda)^0$. Since the action is finite and concentrated in a finite size, we can restrict the integral to the region $Z \le Z_c$ and expand the action in $1/\lambda$.

$$
\begin{aligned}
\varepsilon &= dN_c \int_0^{Z_c} dZ \left[\lambda \widetilde{F}^2 + \frac{Z^2}{3}\widetilde{F}^2 + \frac{1}{2}\frac{\Pi^2}{(dN_c)^2}\right] \qquad (5.9)\\
&= M_0\left[n_B + \frac{a}{\lambda}n_B^{1/3} + \frac{b}{\lambda}Z_c n_B^2\right]\,, \qquad (5.10)
\end{aligned}
$$

where

$$
\begin{aligned}
M_0 &\equiv 8\pi^2\kappa M_{\rm KK}\,, \quad \kappa \equiv \frac{\lambda N_c}{216\pi^3}\,, \\
a &\equiv \frac{(\pi^2-6)M_{\rm KK}^2}{36(2\pi^2)^{2/3}}\,, \quad b \equiv \frac{3^6\pi^4}{2M_{\rm KK}^3}\,, \qquad (5.11)
\end{aligned}
$$

A few remarks are in order.

1. Z_c is introduced as an arbitrary cut-off which is bigger than the instanton size. However in section 8, we will argue that Z_c should be identified as baryon size by explicitly contructing the Skyrmion out of the instanton Therefore it is not an arbitrary number.

2. Even in the case the instanton size is small, the baryon size on the boundary is not. It is of order $(N_c\lambda)^0$ and large in units of M_{KK}. This point is important. While the instanton size in bulk is of the order of the string length and thus small as $1/\sqrt{\lambda}$ in units of M_{KK}, its image on the boundary is a large Skyrmion.

3. The position of the instanton Z_0 in the conformal direction is set to zero by parity.

The various density contributions in (5.10) can be understood from the zero density and finite instanton calculation discussed by Sakai and Sugimoyo to order $N_c\lambda^0$. For that, we recall that the energy balance for a holographic instanton with flat $\mathbb{R}^3$ directions reads schematically as [3]

$$N_c \left(\mathbf{A}\,\lambda\,\rho^2 + \mathbf{B}\,\frac{1}{\lambda\,\rho^2} \right) \tag{5.12}$$

leading to an instanton size in bulk of order $\rho \sim (\mathbf{B}/\mathbf{A})^{1/4}/\sqrt{\lambda}$. The Coulomb repulsion $\mathbf{B}$ is 10^4 times the gravitational attraction $\mathbf{A}$ resulting into a size that is of order $\rho \sim 10/\sqrt{\lambda}$. This parametrically huge repulsion results in a stiffer equation of state in holographic QCD.

The linear term in n_B in (5.10) is just the topological winding of the $U(N_f)$ flavored instanton in D8 on S^4 due to the self duality of the instanton configuration. It is leading and of order $N_c\lambda$. Geometry is unaffected by matter. A point-like instanton in bulk corresponds to a very large Skyrmion on the boundary. The term of order $n_B^{1/3}$ is of order $N_c\lambda^0$. It corresponds to the *attraction* due to gravity in bulk at finite size. Indeed, the energy of this term is of order $\lambda\rho^2 = \widetilde{\mathcal{R}}^2$, as in (5.12) favoring smaller and smaller instanton. The energy per volume for this term is of order $1/\widetilde{\mathcal{R}}$. Since in matter the cell size is of the order of the interparticle distance $1/n_B^{1/3}$, the $n_B^{1/3}$ follows. The term of order n_B^2 is also of order $N_c\lambda^0$. It stems from the Coulomb repulsion in bulk which is of order $1/\lambda\rho^2 = 1/\widetilde{\mathcal{R}}^2$ since the instanton is static in 4-space (space-plus-conformal). This contribution is *repulsive* and favors larger size instanton. The corresponding energy per cell is of order $(Z_c/\widetilde{\mathcal{R}})(1/\widetilde{\mathcal{R}}^5)$, since the warping in the conformal direction is subleading in $1/\lambda$. The n_B^2 contribution follows.

For a Skyrmion with a size $Z_c \ll \widetilde{\mathcal{R}}$, (5.10) describes the low density regime. In this regime the use of the $S^3 \times \mathbb{R}$ instanton is likely to give higher energy than a localized but flat instanton at the pole of S^3 say. Dilute holographic matter is made out of flat $\mathbb{R}^3$ instantons with (5.10) providing an upper bound on the energy per unit volume. This phase breaks spontaneously chiral symmetry. In the point particle limit, the equation of state at low density was discussed in [8]

$$\epsilon_{\rm p} \sim N_c \frac{27\pi^4}{4M_{\rm KK}^2} n_B^2 \tag{5.13}$$

for low densities after re-scaling $\sqrt{\lambda}n_B/\lambda^{3/2} \rightarrow n_B$ [8]. The point-like and flat space instanton contribution (5.13) at low density is lower in energy than (5.10) and therefore favored. This will be made more explicit below. The $n_B^{1/3}$ is absent in the point like limit (finite size effect).

As the density is increased (or equivalently as $\widetilde{\mathcal{R}}$ approaches down to Z_c), there is a change in the equation of state (5.10). For $Z_c = \widetilde{\mathcal{R}}$,

$$\varepsilon = M_0 \left[n_B + \frac{a}{\lambda} n_B^{1/3} + \frac{b'}{\lambda} n_B^{5/3} \right] , \tag{5.14}$$

with b changing to b'

$$b' \equiv \frac{3^6 (2\pi^2)^{5/3}}{2^3 M_{\rm KK}^2} \; . \tag{5.15}$$

The softening of the equation of state at higher density from n_B^2 to $n_B^{5/3}$ follows from a transition from a dilute gas/liquid phase to a dense solid/crystal phase. This transition effectively restores chiral symmetry as we will show later. An estimate of the chiral transition density follows by comparing the n_B^2 term from (5.13) to the leading $n_B^{5/3}$ in (5.14)

$$\epsilon_{\rm s} \sim N_c \frac{27\pi^{7/3}}{2^{4/3} M_{\rm KK}} n_B^{5/3} \tag{5.16}$$

By setting $\epsilon_p = \epsilon_s$, the critical transition density follows

$$n_B^c = \frac{4M_{\rm KK}^3}{\pi^5} \ . \tag{5.17}$$

6. Numbers

To give some estimates of the numbers emerging from the current discussion, we first recall that in holographic QCD the mass of one baryon at next to leading order is not unique. We refer to [3] for a more thorough discussion. In particular, the baryon mass to order $N_c\lambda^0$ is

$$M_B = M_0 \left(1 + \frac{c}{\lambda}\right) , \tag{6.1}$$

where $c = 27\pi\sqrt{2/15}$. Thus the interaction energy per unit volume for the dilute case is

$$E_{\rm int}^{\rm Dilute} \equiv \varepsilon - n_B M_B = \frac{M_0}{\lambda} \left(a n_B^{1/3} - c n_B + b\, Z_c\, n_B^2\right) , \tag{6.2}$$

while for the denser case it is,

$$E_{\rm int}^{\rm Dense} \equiv \varepsilon - n_B M_B = \frac{M_0}{\lambda} \left(a n_B^{1/3} - c n_B + b'\, n_B^{5/3}\right) \ . \tag{6.3}$$

For numerical estimates, we use $M_{\rm KK} = 500\,{\rm MeV}$ and $M_0 = 940\,{\rm MeV}$ for $N_c = 3$ [3]. Our parameters are

$$\lambda \sim 53.2\,, \quad a \sim 0.095\,{\rm fm}^{-2}\,, \quad b \sim 2172\,{\rm fm}^3\,, \quad b' \sim 2039\,{\rm fm}^2\,, \quad c \sim 31\ .$$

The interaction energies are then

$$S^3\ : E_{\rm int}^{\rm Dilute}({\rm GeV\,fm}^{-3}) = 0.00168 n_B^{1/3} - 0.548 n_B + 192 n_B^2\,, \tag{6.4}$$
$$S^3\ : E_{\rm int}^{\rm Dense}({\rm GeV\,fm}^{-3}) = 0.00168 n_B^{1/3} - 0.548 n_B + 36.0\, n_B^{5/3}\,, \tag{6.5}$$
$$R^3\ : E_{\rm int}^{\rm Dilute,P}({\rm GeV\,fm}^{-3}) = \qquad -0.548 n_B + 60.3\, n_B^2 \tag{6.6}$$

where we used $Z_c = 5$. Notice that due to the smallness of the coefficients of the first two terms, the last term is dominant even in the relatively small baryon density if it is not much smaller than $10^{-2}{\rm fm}^{-3}$.

The results on S^3: $E_{\rm int}^{\rm Dilute}$ and $E_{\rm int}^{\rm Dense}$ are compared in figure 1 for $Z_c = 5$ ($Z_c = 5/M_{\rm KK} = 1$ fm with restored dimensions). The results on S^3 and R^3 are compared in figure 2. In R^3 the instantons are point-like or $Z_c \sim \infty$ [8]. The crossing from R^3 to S^3 occurs at relatively small densities $n_B^c \approx 1.26\, n_0$ with $n_0 = 0.17$ ${\rm fm}^{-1/3}$ the nuclear matter density.

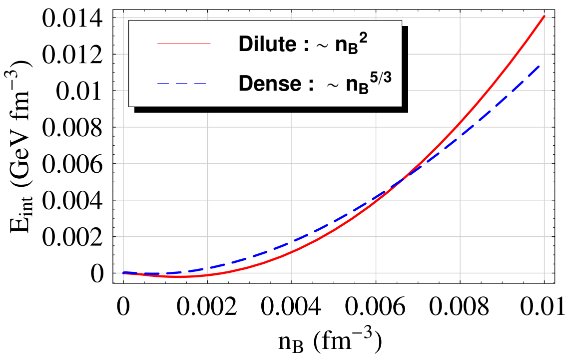

Figure 1: The energy per unit volume: $Z_c = 5$ (red) and $Z_c = \widetilde{\mathcal{R}}$ (blue). See text.

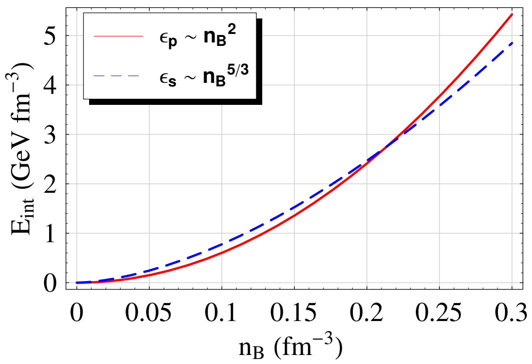

Figure 2: The energy per unit volume: $Z_c = \infty$ (red) and $Z_c = \widetilde{\mathcal{R}}$ (blue). See text.

7. Equation of state in general

The approximation of T^3 by S^3 suggested at the beginning of the paper was justified in way in the dilute limit or for small densities. Phenomenologically, we have found that the chiral phase transition from R^4 to $S^3 \times R$ occurs at few times nuclear matter density in holographic QCD, which is reasonable. The small size instantons dominate dense matter. This means that higher order corrections to both the DBI action and the starting D4 metric are important. While we do not know how to assess them, we now suggest that they may conspire to be small. Indeed, if we were not to expand the DBI plus CS actions, that is if we were to include only these class of higher order corrections our numerical results change only mildly.

Consider, the total (DBI + CS) action is written as

$$S = -dN_c\lambda \int d^4xdZ\sqrt{A - \frac{1}{\lambda}B(\mathcal{A}_0')^2} + \tilde{c}N_c \int d^4xdZJ\mathcal{A}_0\,, \tag{7.1}$$

where

$$A \equiv K_\lambda^{4/3} + \frac{3b}{M_{\rm KK}^2 U_{\rm KK}^4} K_\lambda^{1/3} J + \frac{3b}{U_{\rm KK}^2} K_\lambda^{5/3} J + \frac{9b^2}{M_{\rm KK}^2 U_{\rm KK}^6} K_\lambda^{2/3} J^2 \,, \tag{7.2}$$

$$B \equiv bK_\lambda^{5/3} + \frac{3b^2}{M_{\rm KK}^2 U_{\rm KK}^4} K_\lambda^{2/3} J \,, \qquad J = \frac{\mathrm{sech}^4(Z/\widetilde{\mathcal{R}})}{4\widetilde{\mathcal{R}}^4} \,,$$

$$b \equiv \frac{3^6\pi^2}{4M_{\rm KK}^2} \,, \qquad \widetilde{c} \equiv \frac{3}{2\pi^2 U_{\rm KK}^3} \,, \qquad d = \frac{2M_{\rm KK}^4}{3^9\pi^5} \,. \tag{7.3}$$

The equation of motion is

$$\widetilde{\Pi}' = \widetilde{c} N_c J \,, \tag{7.4}$$

with

$$\widetilde{\Pi} \equiv \frac{\partial \mathcal{L}}{\partial \mathcal{A}_0'} = \frac{dN_c B \mathcal{A}_0'}{\sqrt{A - \frac{1}{\lambda} B (\mathcal{A}_0')^2}} \,. \tag{7.5}$$

The integral of of motion is

$$\widetilde{\Pi}(Z) = \widetilde{\Pi}(\infty) \left[\tanh(Z/\widetilde{\mathcal{R}}) \left(1 + \frac{1}{2} \mathrm{sech}^2(Z/\widetilde{\mathcal{R}}) \right) \right] \,,$$

$$\widetilde{\Pi}(\infty) \equiv \frac{\widetilde{c} N_c}{6\widetilde{\mathcal{R}}^3} = \frac{N_c}{4\pi^2 (U_{\rm KK} \widetilde{\mathcal{R}})^3} = \sqrt{b}\,\Pi(\infty) \,. \tag{7.6}$$

The energy per cell is

$$\begin{aligned} \mathcal{E}_{\rm cell} &= -\int \epsilon_3 dZ (\mathcal{L}_{\rm DBI} + \mathcal{L}_{\rm CS}) \\ &= dN_c \lambda \int \epsilon_3 \; dZ \sqrt{\frac{A}{B + \frac{\widetilde{\Pi}^2}{\lambda d^2 N_c^2}}} - \int \epsilon_3 \; dZ \mathcal{A}_0 \widetilde{\Pi}' \\ &= dN_c \lambda \int \epsilon_3 \; dZ \sqrt{A + \frac{A \widetilde{\Pi}^2}{\lambda N_c^2 d^2 B}} - \int \epsilon_3 \; \mathcal{A}_0(Z) \widetilde{\Pi}(Z) \Big|_{-\infty}^{\infty} \,. \end{aligned} \tag{7.7}$$

The energy density (ε) of the crystalline structure is then

$$\begin{aligned} \varepsilon &\equiv \frac{N \mathcal{E}_{\rm cell}}{V} \approx \frac{\mathcal{E}_{\rm cell}}{\int \epsilon_3} \\ &= dN_c \lambda \int dZ \sqrt{A + \frac{A \widetilde{\Pi}^2}{\lambda N_c^2 d^2 B}} - \mathcal{A}_0(Z) \widetilde{\Pi}(Z) \Big|_{-\infty}^{\infty} \,, \end{aligned} \tag{7.8}$$

where N is the total number of baryons (cells) and V is the total volume which is approximated by $N \int \epsilon_3$. The second term in (5.6) is

$$\mathcal{A}_0(Z) \widetilde{\Pi}(Z) \Big|_{-\infty}^{\infty} = 2\mathcal{A}_0(\infty) \widetilde{\Pi}(\infty) = N_c \mathcal{A}_0(\infty) \frac{1}{2\pi^2 (U_{\rm KK} \widetilde{\mathcal{R}})^3} = \mu_B n_B \,, \tag{7.9}$$

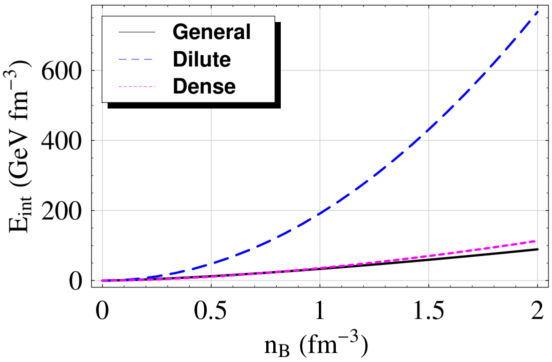

Figure 3: The energy per unit volume.

where

$$n_B = \frac{1}{\int \epsilon_3} = \frac{1}{2\pi^2(U_{\rm KK}\widetilde{\mathcal{R}})^3} = \frac{1}{2\pi^2(\sqrt{\lambda}\mathcal{R})^3}\,, \qquad \mu_B \equiv N_c \mathcal{A}_0(\infty)\ . \tag{7.10}$$

Since n_B is fixed we may set $\mu_B = 0$. Then the energy density is

$$\varepsilon = dN_c\lambda \int dZ \left(\sqrt{A + \frac{A\widetilde{\Pi}^2}{\lambda N_c^2 d^2 B}} - K_\lambda^{2/3} \right)\ , \tag{7.11}$$

where we subtracted the vacuum value. Thus the interaction energy per unit volume is

$$E_{\rm int} \equiv \varepsilon - n_B M_B\ . \tag{7.12}$$

In figure 3 we show the equation of state for the expanded and unexpanded actions. As expected the corrections are of order $1/\lambda$ for a finite and small size instanton. The unexpanded energy is finite for any size of the instanton due to the gravitational warping factors which are subleading in $1/\lambda$ after rescaling. The unexpanded results are similar to the expanded ones in the range of densities explored as it should. For extremely small n_B, $E_{\rm int} \sim 0.00251 n_B^{1/3}$, however for reasonably large density $n_B \sim 1{\rm fm}^{-3}$, $E_{\rm int} \sim 33.9\, n_B^{5/3}$. This power is consistent and expected from the expansion in eq. (5.14).

All general expressions in this section are consistent with the results quoted above to order $\mathcal{O}(\lambda^0)$. Indeed, if for simplicity we set $U_{\rm KK} = M_{\rm KK}^{-1}$ with $M_{\rm KK} = 1$, then A and B reduce to

$$\begin{aligned} A &= (1+3bJ)^2 + \frac{1}{\lambda}\frac{2Z^2}{3}(1+3bJ)(2+3bJ) + \mathcal{O}(\lambda^{-2})\,, \\ B &= b(1+3bJ) + \mathcal{O}(\lambda^{-1})\,, \end{aligned} \tag{7.13}$$

For example, by considering $3bJ = \widetilde{F}^2$ and $\widetilde{\Pi} = \sqrt{b}\Pi$, we can readily show that (7.11) reduces to (5.9)

$$\varepsilon = dN_c \int dZ \left[\lambda \widetilde{F}^2 + \frac{Z^2}{3}\widetilde{F}^2 + \frac{1}{2}\frac{\Pi^2}{(dN_c)^2} \right]\ .$$

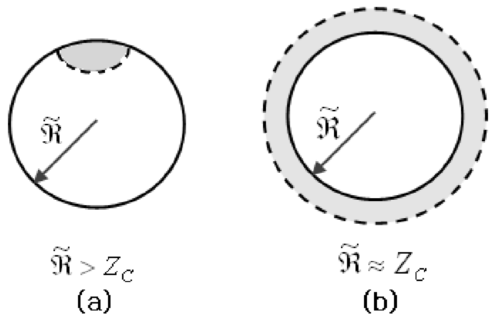

Figure 4: Holographic Skyrmion on S^3 on the boundary

8. Holographic Skyrmions from instantons

The $S^3 \times R$ instanton used in bulk has a very simple Skyrmion picture on the boundary. From (3.2) it follows that the gauge field at the boundary is $A(\infty, \vec{x}) = U^{-1}dU$. Following [3] we note that $U(\vec{x})$ is just the pion field at the boundary. When we have a cut-off in Z, we replace $A(\infty, \vec{x})$ by $A(Z_c, \vec{x})$. U is the boundary Skyrmion field originating from the bulk instanton. Thus U is just the *holonomy of the bulk instanton along the conformal direction*:

$$U(x; Z_c) = P \exp[i \int_0^{Z_c} dZ A_Z^{(instanton)}(Z)] \tag{8.1}$$

When the density is large and $Z_c \sim \mathcal{R}$, the instanton has a support covering the whole three sphere, therefore the resulting Skyrmion should be

$$U(\vec{x}) \simeq \sigma(\vec{x}) + i\tau_a \Pi^a(\vec{x}) = e^{i\tau_a \hat{r}^a(\theta,\phi)\,\psi}\,, \tag{8.2}$$

which is the identity map as (ψ, θ, ϕ) are the canonical angles for the unit S^3. The local Jacobian matrix for this map from S^3 to S^3 is $J^{ai} = \partial\Pi^a/\partial x_i = \mathbf{1}^{ai}/R$, proportional to the identity. The baryon density for this map is $\mathrm{detJ/volS^3} = 1/(2\pi^2 R^3)$ in agreement with bulk holography. The scalar field $\sigma(\vec{x}) = \cos\psi$ measures the *chiral condensate* and averages to zero on S^3

$$\frac{<\bar{q}q>_{S^3}}{<\bar{q}q>_{R^3}} =< \sigma(x) >_{S^3}= \frac{2}{\pi}\int_0^\pi d\psi \sin^2\psi \cos\psi = 0\ . \tag{8.3}$$

The $S^3 \times R$ instanton in (3.2) corresponds to a boundary Skyrmion on S^3 with restored chiral symmetry on the average. We should notice that the chiral condensation is p-wave over a cell while the density in this case is approximately constant over a cell. But it is certainly not a constant. In fact this is a result consistent with ref. [7] where it was argued that there can not be an uniform distribution. In figure 4, we show schematically how a Skyrmion of size Z_c looks on S^3 as a function of $\mathcal{R}$. (a) corresponds to the dilute phase with broken chiral symmetry, while (b) describes the dense phase with restored chiral symmetry.

In previous section, Z_c was introduced as a cut-off of the action bigger than the instanton size. Here we give interpretation of Z_c as the size of the Skyrmion on the boundary.

Note that the R^4 BPS instanton used in bulk in [3] for the description of a single baryon, yields a boundary Skyrmion as

$$U(\vec{x}) = Z_c/\xi_c - i\vec{\tau}\cdot\vec{x}/\xi_c \tag{8.4}$$

with $\xi_c^2 = Z_c^2 + \vec{x}^2 + \rho^2$ and this is the analogue of the unit map (8.2) with $\tan\psi = x/\xi_c$. Notice that *while the size of the instanton is ρ, the size of the Skyrmion is $\sqrt{Z_c^2+\rho^2}$.* If $\rho \ll Z_c$, Z_c itself is the size of the Skyrmion, hence our interpretation above comes. Holography transmutes a small size instanton ρ in bulk to a large size Skyrmion on the boundary.

At small densities with $\widetilde{\mathcal{R}} \gg Z_c$, one can replace the spherical cell by a flat space and the map (8.4) is relevant, while at high density $\widetilde{\mathcal{R}} \leq Z_c$ the map (8.2) is relevant. On S^3 this is pictorially depicted in figure 4. Notice also that (a) has broken chiral symmetry while (b) has restored chiral symmetry effectively (see eq. (8.3)). Again, in this case, our $S^3 \times R$ instanton in bulk describes the high density phase in holographic QCD with restored chiral symmetry. At low densities the energy density is about n_B^2 as discussed by many [6] in qualitative agreement with our figure here. The n_B^2 term is sourced by Coulomb's repulsion in both cases. The description on S^3 carries larger energy density than on R^3 and is therefore unfavorable energitically. It is favorable at higher densities. The transition occurs at about $\mathcal{R} = Z_c$, or $n_B^c = 1/(2\pi^2\, Z_c)$, resulting into an energy density of $n_B^{5/3}$. The value of n_B^c was estimated above.

The determination of Z_c or equivalently the critical size of $\widetilde{\mathcal{R}}$ depends on the energetics of the SS model. It is worth pointing that the single baryon mass analysis on S^3 as discussed in [13] allows a considerable simplification of this issue when the Skyrme model is used. We now note that this is justified in holographic QCD as small size instantons in bulk with $\rho = \widetilde{\mathcal{R}}/\sqrt{\lambda}$ map onto a large size Skyrmion on the boundary with $Z_c \gg \rho$. So the small size instanton expansion in bulk maps onto the gradient expansion in $1/Z_c$ on the boundary. Limiting the SS model on the boundary to the Skyrme model with f_π and e_S fixed by holography yields the specifics of the Skyrmion on the boundary to order λ.

In figure 5 we show how the holgraphic Skyrmion mass on S^3 to order λ changes with $\widetilde{\mathcal{R}}$ the radius of S^3 following [13]. The units of mass and length are respectively [3]

$$\begin{aligned} \frac{f_\pi}{2\sqrt{2}e_S} &= (\lambda N_c)\, M_{\rm KK}\, \frac{\sqrt{\mathbf{b}/2\pi}}{54\pi^5} \\ \frac{\sqrt{2}}{e_S f_\pi} &= (1/M_{\rm KK})\, \sqrt{8\mathbf{b}}/\pi^3\ . \end{aligned} \tag{8.5}$$

with $\mathbf{b} = 15.25$ and $L = \mathcal{R}$. We note that the mass $M_0 = 8\pi^2\kappa M_{\rm KK}$ corresponds to the point 0.95 at $\widetilde{\mathcal{R}} = \infty$ which matches the unit map result as expected. In figure 6 we show the same curve to order $1/\lambda$. Here the energetics is determined in bulk as the chiral Lagrangian in the SS model is not known beyond the order λ. Specifically,

$$\frac{M_0}{\lambda}\left(\mathbf{A}\widetilde{\mathcal{R}}^2 + \frac{\mathbf{B}}{\widetilde{\mathcal{R}}^2}\right) \tag{8.6}$$

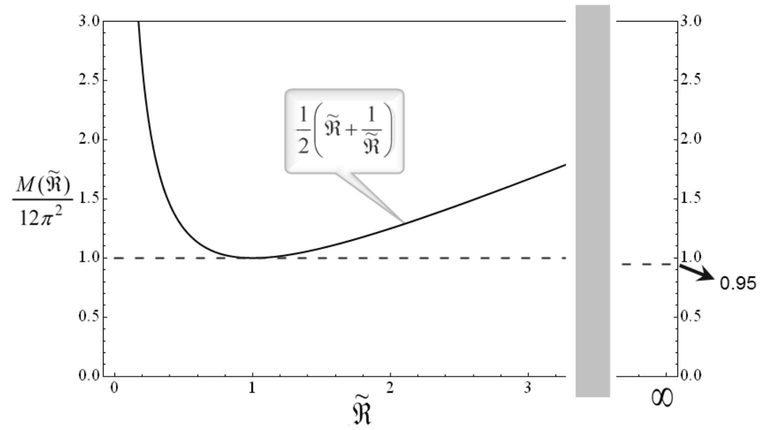

Figure 5: Holographic Skyrmion mass on S^3: order λ

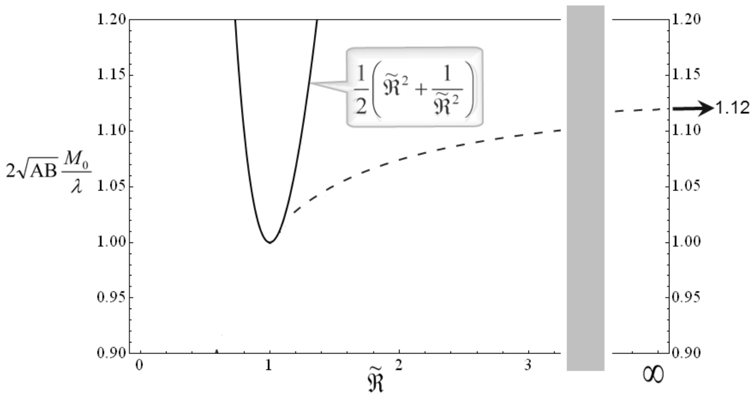

Figure 6: Holographic Skyrmion mass on S^3: order λ^0

with $\mathbf{A} = (\pi^2 - 6)/36 \sim 0.11$ and $\mathbf{B} = (3^6\pi^2)/4 \sim 1799$. The units of mass and lengths are

$$(\mathbf{B}/\mathbf{A})^{1/4} \sim 11.4$$
$$2\sqrt{\mathbf{AB}} \sim 27.8 \tag{8.7}$$

The point 1.12 is the $1/\lambda$ corrected mass (6.1) in these units. Finally, it is interesting to note that the holographic Skyrme model on S^3 yields *naively* the following equation of state

$$\varepsilon = M_0(n_B + a_S\, n_B^{2/3} + b_S\, n_B^{4/3})\,, \tag{8.8}$$

as first noted in the context of the canonical Skyrme model [16]. The $n_B^{2/3}$ for the Skyrmion stems from the universal current algebra $(\nabla\Pi)^2$ term which is attractive and scales as $1/\rho^2$ as opposed to $1/\rho$ from the finite size instanton in bulk. The $n_B^{4/3}$ for the Skyrmion stems from the repulsive Coulomb contribution per unit 3-volume $(1/\rho)/\rho^3$ from the Skyrme term as opposed to the repulsive Coulomb contribution per unit 3-volume $(1/\rho^2)/\rho^3$ in the instanton in bulk. We recall that Coulomb's law in 1+D dimensions is $1/\rho^{D-2}$.

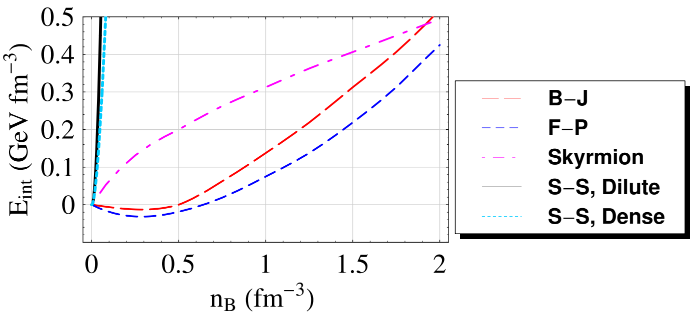

Figure 7: The energy per unit volume as a function of baryon density, for pure skyrmions, for the calculations of Bethe and Johnson [16, 17] and of Friedman and Pandharipande [16, 18], and for our instanton model based on Sakai-Sugimoto model for dilute and dense case.

At high density the *naive* scalings in (8.8) obtained at the boundary differs from (5.14) obtained in bulk in two essential ways: i) a_S and b_S are of order $N_c^0\lambda^0$ on the boundary while their bulk contributions are of order N_c^0/λ; ii) the scaling with n_B appears to differ by an extra (spatial) dimension, $D = 3$ on the boundary and $D = 4$ in bulk. These differences can be understood by noting that the size of the holographic Skyrmion is Z_c. This means that the chiral gradients $L_i = U^{-1}\partial_i U$ are nearly zero on the boundary with $U \sim \mathbf{1}$, except on an the shell $|\vec{x}| \approx Z_c$ of thickness $1/\sqrt{\lambda}$ to ensure that the topological baryon charge is finite.[1] This renders a_S and b_S in (8.8) effectively of order $1/\lambda$ as noted in bulk.

9. Comparison with nuclear models

In figure 7 we compare the interaction energy (6.2) and (6.3) with other hadronic models including Skyrme's chiral model. Holographic matter is subtantially stiffer as explained through the energy budget in (5.12). The reason can be traced back to the fact that for a *single* baryon the repulsion already dwarfs the attraction in holographic QCD

At high densities ε in (6.3) is approximated as

$$\varepsilon \sim \frac{N_c 3^3 (2\pi^2)^{5/3}}{2^3 \pi M_{\rm KK}} n_B^{5/3} \sim 36 n_B^{5/3}\ (\text{GeV fm}^{-3}) \quad \text{for } N_c = 3\,, \tag{9.1}$$

and whatever N_f since the flavoured instanton in bulk is always 2×2. This behaviour is different from that of free massless quarks in $D = 3$ (ε_3) but similar to *massive* quarks in $D = 3$ (ε_3'). Specifically,

$$\varepsilon_3 = \frac{N_c}{N_f^{1/3}} \frac{3^{4/3}\pi^{2/3}}{4} n_B^{4/3} \sim 5.52\, n_B^{4/3} (\text{GeV fm}^{-3}) \qquad \text{for } N_c = 3, N_f = 2\,, \tag{9.2}$$

$$\varepsilon_3' = \frac{N_c}{N_f^{2/3}} \frac{3^{3/5}\pi^{4/3}}{10} \frac{1}{m} n_B^{5/3} \sim 1.68 \frac{1}{m} n_B^{5/3} (\text{GeV fm}^{-3}) \quad \text{for } N_c = 3, N_f = 2\ . \tag{9.3}$$

[1] We note that for $U \sim \mathbf{1}$ the Skyrmion obeys the Faddeev-Bogomolnyi bound since the classical equations of motion are fulfilled.

So at strong coupling

$$\frac{\varepsilon}{\varepsilon_3'} = N_f^{2/3} \left(\frac{9^{6/5}5}{2^{1/3}}\right) \frac{m}{M_{\rm KK}} \quad \sim \quad \frac{88m}{M_{\rm KK}} \quad \text{for } N_f = 2\,, \tag{9.4}$$

independently of λ and N_c. As chiral symmetry is restored in the high density phase, the comparison to the the free massive quark phase in $D = 3$ suggests that the mass $m \sim M_{\rm KK}/88$ is a chirally symmetric screening mass. While the chiral transition restores chiral symmetry it still confines baryons.

10. Conclusions

We have provided a holographic description of dense and cold hadronic matter using the brane model put forward by Sakai and Sugimoto [3]. At large N_c the matter crystallizes and can be treated in the Wigner-Seitz approximation on T^3. For simplicity, the Wigner-Seitz cell was further approximated by S^3 in space leading to a simple instanton configuration on $S^3 \times R$ with R the conformal space. The resulting equation of state at next to leading order in λ shows a free quark behavior at high density, although the overall coefficient is cutoff sensitive and large resulting into a stiff euation of state.

At high densities the gauge gradients are of order $\sqrt{\lambda}$ so the DBI action may not be enough to fix the brane dynamics at order $N_c\lambda^0$ [3]. Also our simplification of T^3 by S^3 while justified at low density, involves curvature corrections at high densities. However, we believe that the essentials of dense matter in holographic QCD are already exposed on S^3 with a small attraction leading $n_B^{1/3}$ and a large Coulomb repulsion leading $n_B^{5/3}$, where 5/3 is the power of non-relativistic fermion. It is interesting to notice that the coulomb interaction in the bulk counts the fermi statistics in the boundary. The repulsion is 10^4 times the attraction resulting into a very stiff equation of state. Changing S^3 to T^3 will not affect the outcome quantitativaly we believe. Indeed, this is the case for dense Skyrmions [13].

The present work expands on the original ideas developed in [6]. Our calculations with finite size instantons are closer to those presented in reference [7] where finite size and homogeneous instantons were used through a variational estimate in $R^3 \times R$. Their arguments yield n_B instead of the $n_B^{1/3}$ we have reported in the equation of state at next to leading order with our $S^3 \times R$ instanton.

The inhomegeneous $S^3 \times R$ description of the crystal suggests that at high density, chiral symmetry is restored on the average. Indeed, since the dual of the instanton cell is the Skyrmion cell with a pion field restricted to S^3 in space. High density matter corresponds to small size S^3 where the pion field becomes just the unit map [13]. The corresponding chiral condensate on S^3 is seen to vanish as half of S^3 carries positive chiral condensate, while the other half carries negative chiral condensate so that on the average the chiral condensate is zero. This restoration of chiral symmetry is due to the formation of the crystal in the spatial direction in holographic QCD even though the D8-$\overline{\rm D8}$ configuration is still attached. In other words, the left and right D8 branes cease to talk to each other through the spatial directions not the conformal direction when they *crystallize at large* N_c.

The present crystal analysis is classical in bulk. A quantum analysis including vibrational and rotational motion is needed. These corrections are subleading in $1/N_c$ and should be estimated for a more thorough phenomenological discussion. Also, the inhomogeneous phase can be probed approximatly by a dilute gas of instantons on T^3 allowing for a lower energetics than on S^3. These issues and others will be discussed elsewhere.

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References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200]; for review, see O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[2] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843 [hep-th/0412141]; *More on a holographic dual of QCD*, *Prog. Theor. Phys.* **114** (2005) 1083 [hep-th/0507073].

[3] H. Hata, T. Sakai, S. Sugimoto and S. Yamato, *Baryons from instantons in holographic QCD*, hep-th/0701280.

[4] K. Nawa, H. Suganuma and T. Kojo, *Baryons in holographic QCD*, *Phys. Rev.* **D 75** (2007) 086003 [hep-th/0612187];
D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Chiral dynamics of baryons from string theory*, *Phys. Rev.* **D 76** (2007) 061901 [hep-th/0701276]; *Dynamics of baryons from string theory and vector dominance*, *JHEP* **09** (2007) 063 [arXiv:0705.2632]; *Nucleon form factors and hidden symmetry in holographic QCD*, *Phys. Rev.* **D 77** (2008) 014030 [arXiv:0710.4615].

[5] O. Aharony, J. Sonnenschein and S. Yankielowicz, *A holographic model of deconfinement and chiral symmetry restoration*, *Ann. Phys. (NY)* **322** (2007) 1420 [hep-th/0604161];
A. Parnachev and D.A. Sahakyan, *Chiral phase transition from string theory*, *Phys. Rev. Lett.* **97** (2006) 111601 [hep-th/0604173];
K. Peeters, J. Sonnenschein and M. Zamaklar, *Holographic melting and related properties of mesons in a quark gluon plasma*, *Phys. Rev.* **D 74** (2006) 106008 [hep-th/0606195].

[6] K.-Y. Kim, S.-J. Sin and I. Zahed, *Dense hadronic matter in holographic QCD*, hep-th/0608046;
N. Horigome and Y. Tanii, *Holographic chiral phase transition with chemical potential*, *JHEP* **01** (2007) 072 [hep-th/0608198];
A. Parnachev and D.A. Sahakyan, *Photoemission with chemical potential from QCD gravity dual*, *Nucl. Phys.* **B 768** (2007) 177 [hep-th/0610247];
O. Bergman, G. Lifschytz and M. Lippert, *Holographic nuclear physics*, *JHEP* **11** (2007) 056 [arXiv:0708.0326].

[7] M. Rozali, H.-H. Shieh, M. Van Raamsdonk and J. Wu, *Cold nuclear matter in holographic QCD*, *JHEP* **01** (2008) 053 [arXiv:0708.1322].

[8] K.-Y. Kim, S.-J. Sin and I. Zahed, *The chiral model of Sakai-Sugimoto at finite baryon density*, *JHEP* **01** (2008) 002 [arXiv:0708.1469].

[9] S. Nakamura, Y. Seo, S.-J. Sin and K.P. Yogendran, *A new phase at finite quark density from AdS/CFT*, hep-th/0611021;
S. Kobayashi, D. Mateos, S. Matsuura, R.C. Myers and R.M. Thomson, *Holographic phase transitions at finite baryon density*, *JHEP* **02** (2007) 016 [hep-th/0611099];
S. Nakamura, Y. Seo, S.-J. Sin and K.P. Yogendran, *Baryon-charge Chemical Potential in AdS/CFT*, *Prog. Theor. Phys.* **120** (2008) 51 [arXiv:0708.2818].

[10] A. Parnachev, *Holographic QCD with isospin chemical potential*, *JHEP* **02** (2008) 062 [arXiv:0708.3170];
O. Aharony, K. Peeters, J. Sonnenschein and M. Zamaklar, *Rho meson condensation at finite isospin chemical potential in a holographic model for QCD*, *JHEP* **02** (2008) 071 [arXiv:0709.3948];
K. Hashimoto, C.-I. Tan and S. Terashima, *Glueball decay in holographic QCD*, *Phys. Rev.* **D 77** (2008) 086001 [arXiv:0709.2208].

[11] T.H.R. Skyrme, *A Nonlinear field theory*, *Proc. Roy. Soc. Lond.* **A 260** (1961) 127; *A unified field theory of mesons and baryons*, *Nucl. Phys.* **31** (1962) 556;
I. Zahed and G.E. Brown, *The Skyrme model*, *Phys. Rept.* **142** (1986) 1.

[12] I.R. Klebanov, *Nuclear matter in the Skyrme model*, *Nucl. Phys.* **B 262** (1985) 133.

[13] N.S. Manton and P.J. Ruback, *Skyrmions in flat space and curved space*, *Phys. Lett.* **B 181** (1986) 137;
H. Forkel et al., *Chiral symmetry restoration and the Skyrme model*, *Nucl. Phys.* **A 504** (1989) 818;
B.-Y. Park, D.-P. Min, M. Rho and V. Vento, *Atiyah-Manton approach to Skyrmion matter*, *Nucl. Phys.* **A 707** (2002) 381 [nucl-th/0201014].

[14] L. McLerran and R.D. Pisarski, *Phases of cold, dense quarks at large-N_c*, *Nucl. Phys.* **A 796** (2007) 83 [arXiv:0706.2191].

[15] H. Forkel, *Yang-Mills instantons in closed Robertson-Walker space-time*, hep-th/0407166.

[16] M. Kutschera, C.J. Pethick and D.G. Ravenhall, *Dense matter in the chiral soliton model*, *Phys. Rev. Lett.* **53** (1984) 1041.

[17] H.A. Bethe and M.B. Johnson, *Dense baryon matter calculations with realistic potentials*, *Nucl. Phys.* **A 230** (1974) 1.

[18] B. Friedman and V.R. Pandharipande, *Hot and cold, nuclear and neutron matter*, *Nucl. Phys.* **A 361** (1981) 502.

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