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A New Phase at a Finite Quark Density from AdS/CFT

Shin Nakamura and K. P. Yogendran
Department of physics, BK21 Program Division, Hanyang University, Seoul 133-791 and Center for Quantum Spacetime, Sogang University, Seoul 121-742

Yunseok Seo* and Sang-Jin Sin
Department of physics, BK21 Program Division, Hanyang University, Seoul 133-791

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We explore the phases of the $\mathcal{N}=2$ super Yang-Mills theory at a finite quark density by introducing a quark chemical potential in a D3-D7 setup. We formulate the thermodynamics of brane embeddings and we find that the density versus chemical potential equation of state has a rich structure. This yields *two* distinct first-order phase transitions in a small window of the quark density. In other words, there is a new first-order phase transition in the region of deconfined quarks. In this new phase, the chemical potential is a decreasing function of the density. We suggest that this might be relevant to the difference in the sQGP-wQGP phases of QCD.

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I. INTRODUCTION

There has been much hope that one might be able to use AdS/CFT [1] to describe the real systems after a certain number of deformations. For example, it has been suggested that the fireball in the Relativistic Heavy Ion Collider(RHIC) be viewed as a strongly interacting system [2,3] and be studied using dual gravity models There have been many attempts to construct models phenomenologically closer to QCD [9].

More recently, there has been renewed interest in $\mathcal{N}=2$ super Yang-Mills (SYM) systems with quenched fundamental quark flavors studied by using a holographic description with probe D7-branes in the AdS_5 black-hole background [10–15]. The key observation is that we have confinement of quarks even in the absence of gluon confinement or the area law [10]. The phases of this theory are characterized by brane embeddings: whether the D7-brane touches the black-hole horizon (black-hole embedding) or not (Minkowski embedding). Different types of embedding lead to different meson spectra.

In this letter, we explore the phases of this theory at *finite quark density* by introducing a quark chemical potential along the lines of that in [16,17]. We will first establish a clear formulation of the thermodynamics of brane embeddings. We find that we need to renormalize the finite chemical potential due to the divergence of the thermodynamic potentials. We will also find that apart from the type of first-order phase transition described in [10,11] at zero chemical potential, there is another class of first order phase transition within the black-hole embedding category: It is indicated in Figure 1 as a hopping between two black-hole embeddings.

Since black-hole embeddings correspond to a deconfined phase, we cautiously suggest that this new type of first-order phase transitions might be relevant to the difference between sQGP-wQGP in RHIC experiments. In particular, we find that the chemical potential in this new phase is a decreasing function of the density.

We emphasize that, depending on whether we control the system by using the chemical potential (grand canonical ensemble) or by using the density (canonical ensemble), the phase diagram is different. In this letter,

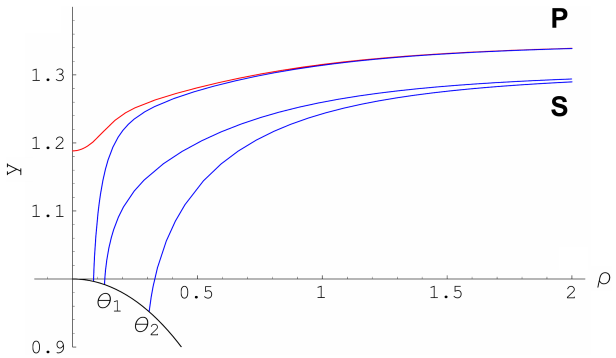

Fig. 1. Brane embedding and phase transitions. Solid lines: Minkowski embedding (red) to black-hole embedding (blue). Dashed lines: Hopping from a black-hole embedding to another one.

*E-mail: yseo@hanyang.ac.kr

we present an analysis of and the results for a system based on the canonical ensemble. The details, including a description of the other process based on the grand canonical ensemble, will be reported in a detailed publication [18].

The Euclidean AdS black hole metric is given by

$$ds^2 = \frac{U^2}{R^2}\left(f(U)dt^2 + d\vec{x}^2\right) + R^2\left(\frac{dU^2}{f(U)U^2} + d\Omega_5^2\right), \quad (1)$$

where $f(U) = 1-(U_0/U)^4$. The Hawking temperature of this geometry is given by $T = U_0/\pi R^2 = U_0/(\sqrt{2\lambda}\pi\alpha')$, where $\lambda = g_{YM}^2 N_c$. We introduce a dimensionless coordinate ξ defined by $d\xi^2/\xi^2 = dU^2/(U^2 f)$ so that the bulk geometry is

$$\begin{aligned} ds^2 &= \frac{U^2(\xi)}{R^2}\left(f(\xi)dt^2 + d\vec{x}^2\right) + \frac{R^2}{\xi^2}ds_6^2, \quad \text{with} \quad (2)\\ ds_6^2 &= d\xi^2 + \xi^2 d\Omega_5^2 = d\rho^2 + \rho^2 d\Omega_3^2 + dy^2 + y^2 d\varphi^2, \end{aligned}$$

where we have defined $\xi^2 \equiv y^2 + \rho^2$ and ρ is the radius of the 3-sphere. The black hole's horizon is located at $\xi = 1$. The *induced metric* on the D7-brane is

$$ds_{D7}^2 = \frac{U^2}{R^2}\left(f dt^2 + d\vec{x}^2\right) + \frac{R^2}{\xi^2}\left((1+y'^2)d\rho^2 + \rho^2 d\Omega_3^2\right), \quad (3)$$

where $y' = \partial_\rho y(\rho)$. It is interesting to notice that the bulk metric in Eq. (1) and the induced metric in Eq. (3) have the same Hawking temperature. This means that the bulk and the brane are in equilibrium. The Euclidean DBI action of the D7-brane in the presence of the gauge field strength $F_{\rho t}$ is

$$\begin{aligned} S &= N_f \mu_7 \int dt d^3x d\rho d\Omega_3 \sqrt{\det(G + 2\pi\alpha' F)} \\ &= \beta V_3 \int d\rho \mathcal{L}, \end{aligned} \quad (4)$$

with

$$\mathcal{L} = \tau_7 \rho^3 \omega_+^{3/2}\sqrt{\frac{\omega_-^2}{\omega_+}(1+y'^2) - (F/m_T)^2}, \quad (5)$$

where $\tau_7 = N_f N_c^2 T^4 g_{YM}^2/32$, $\omega_\pm(\xi) = 1 \pm \xi^{-4}$ and $m_T = \frac{1}{2}\sqrt{\lambda}T$ (the Chern-Simons term vanishes in the present case).

II. CONSERVED CHARGE AND EQUATIONS OF MOTION

Since $\mathcal{L}$ does not depend on A_0 explicitly, its conjugate momentum is a *conserved quantity*:

$$\Pi_{A_0} = \frac{\partial \mathcal{L}}{\partial F_{\rho t}} \equiv -Q, \quad (6)$$

in terms of which we can write

$$F_{\rho t} = m_T \tilde{Q}\omega_- \sqrt{(1+y'^2)}/\sqrt{\omega_+(\tilde{Q}^2 + \omega_+^3\rho^6)}, \quad (7)$$

where $\tilde{Q} = \frac{m_T}{\tau_7}Q$. Since we have a constraint, Eq. (6), to obtain a Lagrangian for y, we should not substitute $F_{\rho t}$ into the original Lagrangian, Eq. (5). The correct procedure is to perform a Legendre transformation

$$\mathcal{H} = \mathcal{L} - \Pi_{A_0} A_0' \quad (8)$$

and then impose the conservation equation, Eq. (6), to eliminate the electric field completely. The resulting "Hamiltonian" is given by

$$\begin{aligned} \mathcal{H} &= \mathcal{T}(y,\rho)\sqrt{1+y'^2}, \\ \mathcal{T}(y,\rho) &= \tau_7\sqrt{\frac{\omega_-^2}{\omega_+}(\tilde{Q}^2 + \omega_+^3\rho^6)}. \end{aligned} \quad (9)$$

We can take this Hamiltonian as our effective Lagrangian for y and $\mathcal{T}$ may be regarded as the effective tension of the D7-brane. The resulting equation of motion is

$$\frac{y''}{1+y'^2} + \frac{\partial \log \mathcal{T}}{\partial \rho}y' - \frac{\partial \log \mathcal{T}}{\partial y} = 0. \quad (10)$$

One can check that both the original and the effective Lagrangian give the same equation of motion for y.

III. CHEMICAL POTENTIAL IN GRAVITY DUAL

In [16,17], the quark chemical potential was introduced as the value of A_0 on the D7-brane world volume. Here, we define the chemical potential in a gauge-invariant fashion:

$$\mu \equiv \int_{\rho_{min}}^{\infty} d\rho\, F_{\rho t} = \lim_{\rho\to\infty} A_0. \quad (11)$$

For the last equality, we need to gauge fix $A_\rho = 0$ and set $A_0|_{\rho_{min}} = 0$, which agrees with [16,17].

Notice that $\mu = \int F_{\rho t}$ is the work to bring a unit charge from the UV region ($\rho = \infty$) to the IR region against an electric field $F_{\rho t}$. This definition in Eq. (11) agrees with our intuition of the chemical potential as work done to add a quark to the system.

For the Minkowski embedding, $\rho_{min} = 0$. For the black hole embedding, $\rho_{min} = \cos\theta$ where θ is the angular coordinate on the y-ρ plane. [1]

[1] After this paper was uploaded, the authors of [19] pointed out that in the deconfinig phase, there might be no Minkowski embedding.

IV. THERMODYNAMIC POTENTIALS

A generic grand potential (density) is defined by $e^{-\beta V_3 \Omega(\mu)} = \mathrm{Tr} e^{-\beta V_3 (H-\mu N)}$. Here, we identify the DBI action, which is a functional of A_0, as the grand potential $S_{DBI} = \beta V_3 \Omega$. Then, integrating the Legendre transformation, Eq. (8),

$$\int d\rho \mathcal{H} = \int d\rho \mathcal{L} - \int d\rho \Pi_{A_0} A_0'. \tag{12}$$

Using the fact that $\Pi_{A_0} = -Q$ is a constant (in ρ evolution), one can rewrite the above as

$$F(Q) = \Omega(\mu) + \mu Q. \tag{13}$$

It is remarkable that the Legendre transformation in the bulk classical field theory is reinterpreted as the Legendre transformation between the canonical and the grand canonical ensembles in the boundary thermodynamics.

The chemical potential enters the Hamiltonian density of the gauge theory at the boundary as a coupling to the baryon number density:

$$\Delta \mathcal{H}_{YM} = -\mu \langle \psi^\dagger \psi \rangle. \tag{14}$$

Therefore, Q, which has been originally defined as a first integral of the DBI action, should be identified as the number density of quarks/baryons. Notice that the effective tension of the brane increases as we increase the quark density, which should have been expected. More precisely, after considering various scale factors, we have

$$Q = \langle \psi^\dagger \psi \rangle. \tag{15}$$

V. RENORMALIZATION OF CHEMICAL POTENTIAL

The Helmholtz free energy F and the grand potential are in fact divergent quantities because they contain a divergent ρ integral. Therefore, we need to regularize them. We choose to subtract $F_0(Q)$, the value of F, for the D7-brane configuration that touches the black hole on the equatorial plane ($y = 0$). This is like a Pauli-Villars regularization in the brane setup. We call this brane the reference brane, so the renormalized free energy, which we will calculate, is defined by

$$F_R(Q) = F(Q) - F_0(Q). \tag{16}$$

This has a far reaching effect on the chemical potential. To see this, notice

$$\begin{aligned} F_R(Q) &= \Omega(\mu) - \Omega(\mu_0) + (\mu - \mu_0) Q : \\ &= \Omega_R(\mu_R) + \mu_R Q, \end{aligned} \tag{17}$$

where we have used the fact that $F_0(Q) = \Omega(\mu_0) + \mu_0 Q$. Notice that μ_0 is the chemical potential at the reference-brane configuration. The point is that when we deal with renormalized quantities such as F_R and Ω_R, we also have to use the renormalized chemical potential, although μ itself is a finite quantity from the beginning. In all numerical analysis, we need to use μ_R for the chemical potential. From now on, we delete the subindex R unless it is confusing.

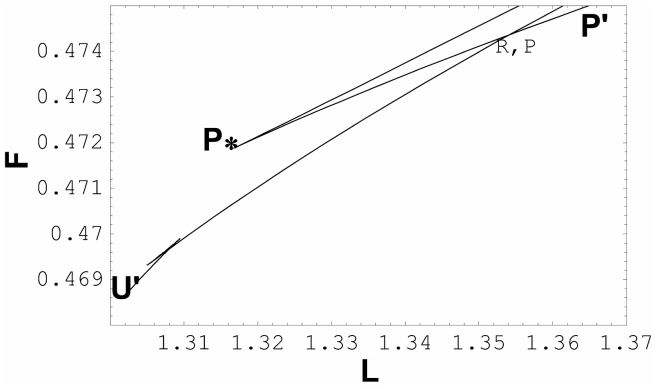

Fig. 2. Free energy vs. L for fixed Q. The second phase transition is indicated by a small kink near U'.

VI. THE PHASE STRUCTURE OF D3-D7

In this article, we study a system based on the canonical ensemble, where the number density Q is continuous while the chemical potential is allowed to jump. Let us expand y and A_0 in the form of $y(\rho) = L + \frac{\tilde{c}}{\rho^2} + O(\rho^{-4})$, $A_0(\rho) = \mu + \frac{c'}{\rho^2} + O(\rho^{-4})$. We have $L = \frac{m_q}{\sqrt{2\lambda}\pi T}$ and $\tilde{c} = c\frac{\sqrt{2}\pi m_T}{\tau_7}$, where m_q is a quark mass and $c = \langle \bar{\psi}\psi \rangle$ [14,15]. One can interpret L as the quark mass at fixed temperature or as an inverse temperature at fixed quark mass. We also find that $c' = -\frac{1}{2}\frac{m_T^2}{\tau_7} Q$. The standard AdS/CFT dictionary establishes that we have two pairs of conjugate variables, namely $(\tilde{c}, L)$ and (Q, μ).

To analyse the phase transition, we plot the free energy F/T^4 as a function of L for a given Q (Figure 2) and track the least free energy configuration. In Figure 2, we start from P' on the Minkowski branch. As L decreases, the free energy decreases until we intersect the black-hole branch at P. For smaller values of L, the black-hole configurations have lower free energy. At P, therefore, the D7-brane jumps from a Minkowski embedding to touch the black-hole horizon. This is the same type of first-order phase transition first found in [10] for zero chemical potential. Decreasing L further decreases the latitude (θ) of the intersection of the brane with the horizon smoothly. For large Q, the story ends here.

For very small Q, however, as we decrease L further, initially the latitude of the intersection of the brane-black hole goes down smoothly. However, at a critical latitude θ_1, the embedding suddenly jumps to a smaller latitude θ_2, as indicated schematically in Figure 1 (the

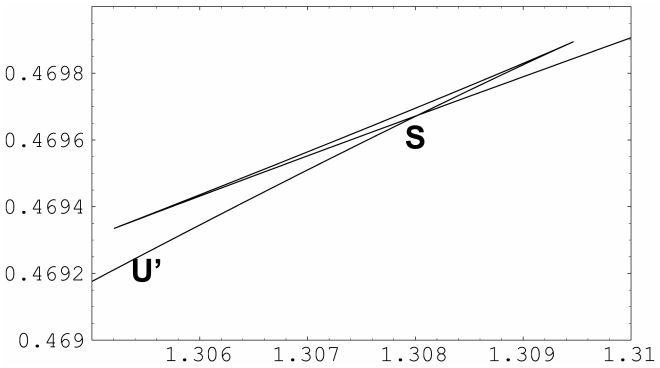

Fig. 3. Free energy vs. L for fixed Q around the second phase transition point.

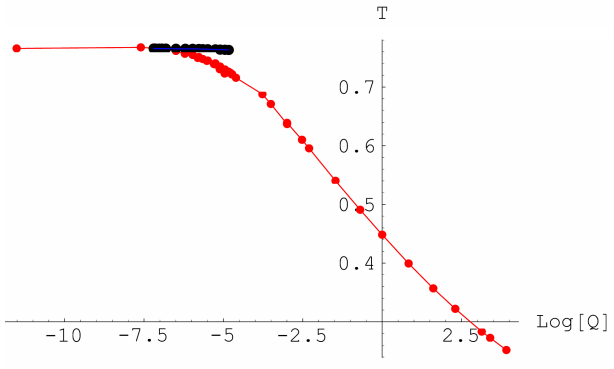

Fig. 4. Phase diagram in the T-$\tilde{Q}$ plane.

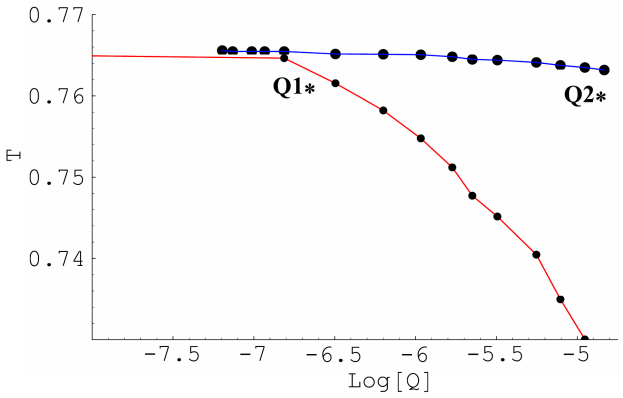

Fig. 5. T-$\tilde{Q}$ phase diagram around the new phase boundary.

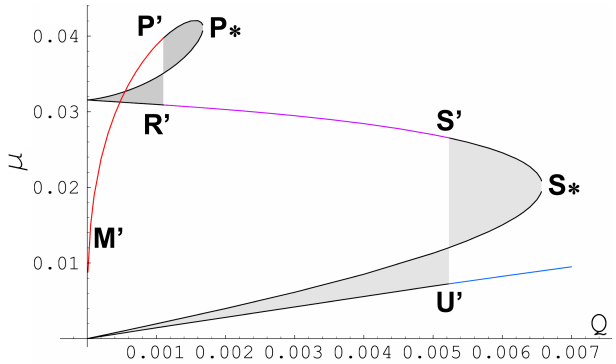

Fig. 6. μ-Q relation: We plot μ/T vs. Q/T^3.

kink near U' in Figure 2). An embedding with touching latitudes between these two values has higher free energy, as seen in Figure 3 and, hence, is never realized. This phenomenon happens only at a finite density within a small density window,

$$\log \tilde{Q}_{1*} = -6.812 < \log \tilde{Q} < -4.726 = \log \tilde{Q}_{2*}. \quad (18)$$

The full phase diagram, therefore, looks as in Figure 4. Notice that the horizontal axis is $\log \tilde{Q}$.

The chemical potential jumps across the phase boundary. For $\tilde{Q} > \tilde{Q}_{2*}$, there is no kink, hence, the second phase transition disappears as we can see in Figure 5. Furthermore, we have a second-order phase transition at $\tilde{Q} = \tilde{Q}_{2*}$. The slopes of the phase boundary lines are discontinuous at $\tilde{Q} = \tilde{Q}_{1*}$, where three lines meet.

VII. EQUATIONS OF STATE

The relations between the variables $(\tilde{c}, L)$ (equivalently (c, m_q)) and (Q, μ) can be considered as the diagram of equation of state, which is much like the P-V diagram of Van der Waals in a liquid-gas phase transition. We may determine these by using the thermodynamic relations

$$c = -\frac{\partial F}{\partial m_q}, \qquad \mu = \frac{\partial F}{\partial Q}.$$

Numerical analysis shows that the relationship between μ and Q given by $Q = -\frac{\partial \Omega(\mu)}{\partial \mu}$ is not monotonic; further, μ is not a single-valued function of Q.

In Figures 6 and 7, we show the relation between $\tilde{c}$ and L for a representative value of Q and the relation between Q and μ for a representative value of L. The equations of state represented in these diagrams show that we have a much richer structure than that in a Van der Waals P-V diagram.

We track the phase diagram from M' in Figure 6 at a fixed value of L, which may be thought of an inverse temperature (or quark mass). The first phase transition takes place at a certain critical value of Q at point P'. Here, the brane embedding jumps from a Minkowski embedding to a black-hole embedding, which results in a jump of the chemical potential from P' to R' on the diagram. If we increase Q further, we have a second phase transition, which is realized as a jump from S' to U'. We have also indicated Maxwell's construction, which allows us to determine the location of the phase transition (i.e., equality of the areas of the shaded regions on either side of a particular jump). For larger values of L, however, we have only a single phase transition.

The point to be noted is that in a small window between R' and S', the chemical potential is a *decreasing* function of the density (in the deconfined phase). This

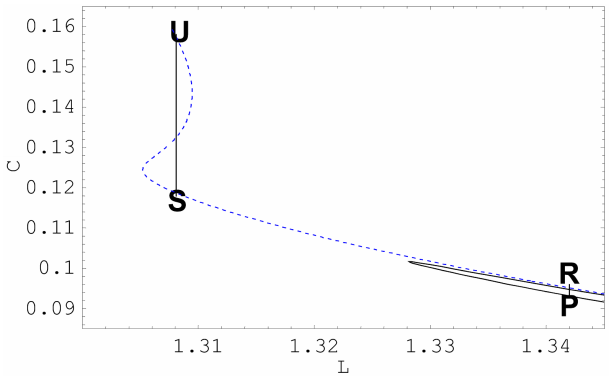

Fig. 7. $\tilde{c}$-L relation: Here, we interpret L as the quark mass.

window seems to be thermodynamically unstable. The instability arises because we did not include the bound states explicitly. In terms of quarks and gluons only, such bound states should appear here as an instability because the degree of freedom is changed by the formation of bound states. However, this does not mean that the new phase does not exist. Within the deconfined phase (black-hole embeddings), there are regions that allows a bound state, while the majority of the high-temperature regions do not allow such bound states. Therefore, our proposal is that it is natural to identify the region with bound states as the sQGP.

Referring to the $\tilde{c}$-L equation of state in Figure 7, the two phase transitions are as labelled. In this case, however, the chiral condensate decreases uniformly as a function of L (excepting of course, for the jumps). For large quark mass L, it seems to vanish, as expected.

VIII. DISCUSSION

Although characterized by a nonstandard behavior of the chemical potential, the nature of the second phase is not very clear from the gauge theory point of view. However, since both phases belong to the black-hole embedding corresponding to deconfined quarks, it might be relevant to the famous difference between sQGP and wQGP discussed within RHIC physics [2]. In fact, if we take the temperature and the $\tilde{Q}$ of the relevant region to be 200 MeV and 5×10^{-3}, respectively and choose $N_f N_c \sqrt{\lambda}/16 \sim O(1)$, then the density $Q \sim 1$ fm^{-3}, which is in the RHIC ball park.

An immediate question is the universality of this second phase. Since most of the properties of the branes for small Q are determined in the neighborhood of the horizon, we might expect this phenomenon persist in (perhaps more realistic) models so long as we have a black hole.

One can also look at a process that corresponds to a horizontal Maxwell's construction in the μ-Q diagram. This is a process where the chemical potential is used as a control parameter. Such a process is more typical in the literature [20]. Some other related questions are about the effect of the density and the temperature on the meson spectrum and on the heavy quark potentials. A limitation of this model is the absence of a gluon confinement. We can also ask whether what we found in this research is a universal feature of black-hole embeddings. These issues are currently under investigation [18].

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