Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 210626, 13 pages
doi:10.1155/2008/210626

*Research Article*

# On the Stability of Generalized Additive Functional Inequalities in Banach Spaces

**Jung Rye Lee,[1] Choonkil Park,[2] and Dong Yun Shin[3]**

[1] *Department of Mathematics, Daejin University, Kyeonggi 487-711, South Korea*
[2] *Department of Mathematics, Hanyang University, Seoul 133-791, South Korea*
[3] *Department of Mathematics, University of Seoul, Seoul 130-743, South Korea*

Correspondence should be addressed to Choonkil Park, baak@hanyang.ac.kr

Received 18 February 2008; Accepted 2 May 2008

Recommended by Ram Verma

We study the following generalized additive functional inequality $\|af(x) + bf(y) + cf(z)\| \leq \|f(\alpha x + \beta y + \gamma z)\|$, associated with linear mappings in Banach spaces. Moreover, we prove the Hyers-Ulam-Rassias stability of the above generalized additive functional inequality, associated with linear mappings in Banach spaces.



## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Rassias [6] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [7] following the same approach as in Rassias [4] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [7] as well as by Rassias and Šemrl [8] that one cannot prove Rassias' theorem when $p = 1$. The counterexamples of Gajda [7] as well as of Rassias and Šemrl [8] have stimulated several mathematicians to create new definitions of *approximately additive* or *approximately linear* mappings (cf. Găvruţa [5], Jung [9] who among others studied the Hyers-Ulam stability of functional equations). The paper of Rassias [4] had great influence on the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of Czerwik [10], Hyers et al. [11]). During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam-Rassias stability to a number of functional equations and mappings (see [12–17]).

Gilányi [18] showed that if $f$ satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\|, \tag{1.1}$$

then $f$ satisfies the quadratic functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y), \tag{1.2}$$

see also [19]. Fechner [20] and Gilányi [21] proved the Hyers-Ulam-Rassias stability of the functional inequality (1.1). Park et al. [22] investigated the Jordan-von Neumann-type Cauchy-Jensen additive mappings and prove their stability, and Cho and Kim [23] proved the Hyers-Ulam-Rassias stability of the Jordan-von Neumann-type Cauchy-Jensen additive mappings.

The purpose of this paper is to investigate the generalized additive functional inequality in Banach spaces and the Hyers-Ulam-Rassias stability of generalized additive functional inequalities associated with linear mappings in Banach spaces.

Throughout this paper, we assume that $X$, $Y$ are Banach spaces and that $a$, $b$, $c$, $\alpha$, $\beta$, $\gamma$ are nonzero complex numbers.

## 2. Generalized additive functional inequalities

Consider a mapping $f : X \to Y$ satisfying the following functional inequality:

$$\|af(x)+bf(y)+cf(z)\| \le \|f(\alpha x+\beta y+\gamma z)\| \tag{2.1}$$

for all $x, y, z \in X$.

We investigate the generalized additive functional inequality in Banach spaces.

We will use that for an additive mapping $f$, we have $f((m/n)x) = (m/n)f(x)$ for any positive integers $n$, $m$ and all $x \in X$ and so $f(rx) = rf(x)$ for any rational number $r$ and all $x \in X$.

**Theorem 2.1.** *Let $f : X \to Y$ be a nonzero mapping satisfying $f(0) = 0$ and* (2.1). *Then the following hold:*

(a) *$f$ is additive;*

(b) *if $\alpha/\beta$, $\beta/\gamma$ are rational numbers, then $a/\alpha = b/\beta = c/\gamma$;*

(c) *if $\alpha$ is a rational number, then $|a| \le |\alpha|$.*

*Proof.* (a) Letting $y = -(\alpha/\beta)x$, $z = 0$ in (2.1), we get $af(x) + bf(-(\alpha/\beta)x) = 0$.

Letting $y = 0$, $z = -(\alpha/\gamma)x$ in (2.1), we get $af(x) + cf(-(\alpha/\gamma)x) = 0$.

Letting $x = 0$, $y = (\alpha/\beta)x$, $z = -(\alpha/\gamma)x$ in (2.1), we get $bf((\alpha/\beta)x) + cf(-(\alpha/\gamma)x) = 0$.

Thus, we get $f(-(\alpha/\beta)x) = -f((\alpha/\beta)x)$ and so $f(-x) = -f(x)$, $bf(x) = af((\beta/\alpha)x)$, and

$$\frac{b}{a}f\left(\frac{\alpha}{\beta}x\right) = \frac{c}{b}f\left(\frac{\beta}{\gamma}x\right) = \frac{a}{c}f\left(\frac{\gamma}{\alpha}x\right) = f(x) \tag{2.2}$$

for all $x \in X$.

On the other hand, letting $z = -(\alpha x + \beta y)/\gamma = -(\alpha/\gamma)(x + (\beta/\alpha)y)$ in (2.1), we get

$$af(x) + bf(y) + cf\left(-\frac{\alpha}{\gamma}\left(x + \frac{\beta}{\alpha}y\right)\right) = 0. \tag{2.3}$$

The facts that

$$cf\left(-\frac{\alpha}{\gamma}\left(x + \frac{\beta}{\alpha}y\right)\right) = c\left(-\frac{a}{c}\right)f\left(x + \frac{\beta}{\alpha}y\right) = -af\left(x + \frac{\beta}{\alpha}y\right) \tag{2.4}$$

and $bf(y) = af((\beta/\alpha)y)$ give that

$$f\left(x + \frac{\beta}{\alpha}y\right) = f(x) + f\left(\frac{\beta}{\alpha}y\right) \tag{2.5}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$, which implies that $f$ is additive.

(b) Since $f$ is additive by (a) and since $\alpha/\beta$ and $\beta/\gamma$ are rational numbers, the facts that $(b/a)f((\alpha/\beta)x) = f(x)$ and $(c/b)f((\beta/\gamma)x) = f(x)$ give that

$$\frac{b}{a}\cdot\frac{\alpha}{\beta}f(x) = \frac{c}{b}\cdot\frac{\beta}{\gamma}f(x) = f(x) \tag{2.6}$$

for all $x \in X$. Since $f$ is nonzero, we conclude that $a/\alpha = b/\beta = c/\gamma$.

(c) Letting $y = z = 0$ in (2.1), since $\alpha$ is a rational number, we get

$$\|af(x)\| \le \|f(\alpha x)\| = \|\alpha f(x)\| \tag{2.7}$$

for all $x \in X$. Since $f$ is nonzero, we conclude that $|a| \le |\alpha|$, as desired. □

As an application of Theorem 2.1, if we consider a mapping $f : X \to Y$ satisfying

$$\|f(x) + f(y) + f(z)\| \le \|f(x + 2y + 3z)\| \tag{2.8}$$

for all $x, y, z \in X$, then we conclude that $f \equiv 0$.

Actually, for a mapping $f : X \to Y$ satisfying $f(0) = 0$ and

$$\|af(x) + bf(y) + cf(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| \tag{2.9}$$

for all $x, y, z \in X$, when $\alpha/\beta$, $\beta/\gamma$ are rational numbers, the above theorem says that $f \equiv 0$ unless $a/\alpha = b/\beta = c/\gamma$.

Here, we consider functional inequalities similar to (2.1).

*Remark 2.2.* Let $f : X \to Y$ be a mapping with $f(0) = 0$. If $f$ satisfies

$$\|af(x) + bf(y) + cf(z)\| \le \|f(\alpha x + \beta y)\| \tag{2.10}$$

for all $x, y, z \in X$, then by letting $x = y = 0$, we get $cf(z) = 0$ for all $z \in X$ and so $f \equiv 0$. And if $f$ satisfies

$$\|af(x) + bf(y)\| \le \|f(\alpha x + \beta y + \gamma z)\| \tag{2.11}$$

for all $x, y, z \in X$, then by letting $y = 0$, $z = -\alpha x/\gamma$, we get $af(x) = 0$ for all $x \in X$ and so $f \equiv 0$.

In order to generalize the inequality (2.1), in the following corollaries, we assume that $a_k$'s and $\alpha_k$'s, $k = 1, 2, \ldots, n$ $(n \ge 3)$ are nonzero complex numbers.

**Corollary 2.3.** *Let $f : X \to Y$ be a nonzero mapping satisfying $f(0) = 0$ and*

$$\left\|\sum_{k=1}^{n} a_k f(x_k)\right\| \le \left\|f\left(\sum_{k=1}^{n} \alpha_k x_k\right)\right\| \tag{2.12}$$

*for all $x_k \in X$. Then the following hold:*

(a) *$f$ is additive;*

(b) *if $\alpha_j/\alpha_i$ is a rational number, then $a_i/\alpha_i = a_j/\alpha_j$;*

(c) *if $\alpha_i$ is a rational number, then $|a_i| \le |\alpha_i|$.*

*Proof.* (a) Let $x_k = 0$ in (2.12) except for three $x_k$'s. Then by the same reasoning as in the proof of Theorem 2.1, it is proved and so we omit the details.

(b) Letting $x_i = x$, $x_j = y$, by the same reasoning as in the corresponding part of the proof of Theorem 2.1, we can prove it.

(c) Letting $x_k = 0$ for all $k$ with $k \ne i$, (2.12) gives that

$$\|a_i f(x_i)\| \le \|f(\alpha_i x_i)\| = \|\alpha_i f(x_i)\|. \tag{2.13}$$

Since $f$ is nonzero, we conclude that $|a_i| \le |\alpha_i|$, as desired. □

In the above corollary, similar to Remark 2.2, we notice that if a mapping $f$ satisfies $f(0) = 0$ and

$$\left\|\sum_{k=1}^{p} a_k f(x_k)\right\| \le \left\|f\left(\sum_{k=1}^{q} \alpha_k x_k\right)\right\| \tag{2.14}$$

for some $p, q \in \{1, 2, \ldots, n\}$ with $p \ne q$ and all $x_k \in X$, then $f \equiv 0$.

**Corollary 2.4.** *For an invertible $3 \times 3$ matrix $(a_{ij})$ of complex numbers, let $f : X \to Y$ be a nonzero mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|af(a_{11}x + a_{12}y + a_{13}z) + bf(a_{21}x + a_{22}y + a_{23}z) + cf(a_{31}x + a_{32}y + a_{33}z)\| \\ &\quad \le \|f((\alpha a_{11} + \beta a_{21} + \gamma a_{31})x + (\alpha a_{12} + \beta a_{22} + \gamma a_{32})y + (\alpha a_{13} + \beta a_{23} + \gamma a_{33})z)\| \end{aligned} \tag{2.15}$$

*for all $x, y, z \in X$. Then the following hold:*

(a) *$f$ is additive;*

(b) *if $\alpha/\beta$, $\beta/\gamma$ are rational numbers, then $a/\alpha = b/\beta = c/\gamma$;*

(c) *if $\alpha$ is a rational number, then $|a| = |\alpha|$.*

*Proof.* If we let $s = a_{11}x + a_{12}y + a_{13}z$, $t = a_{21}x + a_{22}y + a_{23}z$, $u = a_{31}x + a_{32}y + a_{33}z$, then since a matrix $(a_{ij})$ is invertible and

$$(\alpha a_{11} + \beta a_{21} + \gamma a_{31})x + (\alpha a_{12} + \beta a_{22} + \gamma a_{32})y + (\alpha a_{13} + \beta a_{23} + \gamma a_{33})z = \alpha s + \beta t + \gamma u, \quad (2.16)$$

inequality (2.15) is equivalent to

$$\|af(s) + bf(t) + cf(u)\| \le \|f(\alpha s + \beta t + \gamma u)\| \quad (2.17)$$

for all $s, t, u \in X$. Thus by applying Theorem 2.1, our proofs are clear. □

By the same reasoning as in Remark 2.2, we obtain the following result.

*Remark 2.5.* For an invertible $3 \times 3$ matrix $(a_{ij})$ of complex numbers, let $f : X \to Y$ be a mapping with $f(0) = 0$. If $f$ satisfies

$$\begin{aligned} &\|af(a_{11}x + a_{12}y + a_{13}z) + bf(a_{21}x + a_{22}y + a_{23}z) + cf(a_{31}x + a_{32}y + a_{33}z)\| \\ &\quad \le \|f((\alpha a_{11} + \beta a_{21})x + (\alpha a_{12} + \beta a_{22})y + (\alpha a_{13} + \beta a_{23})z)\| \end{aligned} \quad (2.18)$$

or

$$\begin{aligned} &\|af(a_{11}x + a_{12}y + a_{13}z) + bf(a_{21}x + a_{22}y + a_{23}z)\| \\ &\quad \le \|f((\alpha a_{11} + \beta a_{21} + \gamma a_{31})x + (\alpha a_{12} + \beta a_{22} + \gamma a_{32})y + (\alpha a_{13} + \beta a_{23} + \gamma a_{33})z)\| \end{aligned} \quad (2.19)$$

for all $x, y, z \in X$, then $f \equiv 0$.

Now we investigate linearity of a mapping $f : X \to Y$. The following is a well-known and useful lemma.

**Lemma 2.6.** *Let $f : X \to Y$ be an additive mapping satisfying $\lim_{t \in \mathbb{R}, t \to 0} f(tx) = 0$ for all $x \in X$. Then $f$ is an $\mathbb{R}$-linear mapping.*

**Theorem 2.7.** *Let $f : X \to Y$ be a nonzero mapping satisfying (2.1) and $\lim_{t \in \mathbb{R}, t \to 0} f(tx) = 0$ for all $x \in X$. Then the following hold:*

(a) *$f$ is $\mathbb{R}$-linear;*

(b) *if $\alpha/\beta$, $\beta/\gamma$ are real numbers, then $a/\alpha = b/\beta = c/\gamma$.*

*Proof.* (a) For a mapping $f$ satisfying $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in X$, if we let $x = 0$, then we get $f(0) = 0$. Since $f$ satisfies (2.1), from (a) in Theorem 2.1 and Lemma 2.6 we conclude that $f$ is $\mathbb{R}$-linear.

(b) Since $f$ is $\mathbb{R}$-linear by (a) and $\alpha/\beta$, $\beta/\gamma$ are real numbers, by the same reasoning as in the proof of Theorem 2.1(b), we can prove it. □

## 3. Stability of generalized additive functional inequalities

In this section, we study the Hyers-Ulam-Rassias stability of generalized additive functional inequalities in Banach spaces.

First of all, we introduce $\alpha$-additivity of a mapping and investigate its properties.

*Definition 3.1.* For a mapping $f : X \to Y$, we say that $f$ is $\alpha$-additive if

$$f(x + \alpha y) = f(x) + \alpha f(y) \tag{3.1}$$

for all $x, y \in X$.

**Proposition 3.2.** *If a mapping $f : X \to Y$ is $\alpha$-additive, then $f$ is additive and $1/\alpha$-additive.*

*Proof.* Let $f : X \to Y$ be an $\alpha$-additive mapping. Letting $x = y = 0$ in (3.1), we get $f(0) = 0$. Letting $x = 0$ in (3.1), we get $f(\alpha y) = \alpha f(y)$ for all $y \in X$. Moreover, letting $x = 0$ and replacing $y$ by $y/\alpha$ in (3.1), we get $f(y/\alpha) = (1/\alpha) f(y)$ for all $y \in X$. Hence we obtain

$$f(x + y) = f\left(x + \alpha \cdot \frac{y}{\alpha}\right) = f(x) + \alpha f\left(\frac{y}{\alpha}\right) = f(x) + f(y) \tag{3.2}$$

for all $x, y \in X$ and so $f$ is additive.

On the other hand, we have

$$f\left(x + \frac{1}{\alpha} y\right) = f\left(\frac{1}{\alpha}(y + \alpha x)\right) = \frac{1}{\alpha} f(y + \alpha x) = f(x) + \frac{1}{\alpha} f(y) \tag{3.3}$$

for all $x, y \in X$ and so $f$ is $1/\alpha$-additive. □

*Remark 3.3.* If a mapping $f : X \to Y$ is $\alpha$-additive and $\beta$-additive, then we have

$$f(x + \alpha\beta y) = f(x) + \alpha f(\beta y) = f(x) + \alpha\beta f(y) \tag{3.4}$$

for all $x, y \in X$, which implies that $f$ is $\alpha\beta$-additive.

In the following lemma, we give conditions for a mapping $f : X \to Y$ to be $\mathbb{C}$-linear.

**Lemma 3.4.** *Let $f : X \to Y$ be an $\alpha$-additive mapping satisfying $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in X$. If $\alpha$ is not a real number, then $f$ is a $\mathbb{C}$-linear mapping.*

*Proof.* Let $f$ be an $\alpha$-additive mapping satisfying $\lim_{t\in\mathbb{R},t\to 0} f(tx) = 0$ for all $x \in X$. Since $f$ is additive, by Lemma 2.6, $f$ is $\mathbb{R}$-linear. When $\alpha$ is not real, if we let $\alpha = a + bi$ for some real numbers $a$, $b$ $(b \neq 0)$, then since $f$ is additive and $\mathbb{R}$-linear, we have

$$(a + bi) f(x) = f\big((a + bi)x\big) = f(ax) + f(bix) = af(x) + bf(ix) \tag{3.5}$$

and so $f(ix) = if(x)$ for all $x \in X$, which implies that $f$ is $\mathbb{C}$-linear. □

Now we are ready to investigate the Hyers-Ulam-Rassias stability of generalized additive functional inequality associated with a linear mapping. Here, we give a lemma for our main result.

**Lemma 3.5.** *Let $f : X \to Y$ be a mapping. If there exists a function $\varphi : X \to [0, \infty)$ satisfying*

$$\|f(\alpha x) - \alpha f(x)\| \leq \varphi(x), \tag{3.6}$$

$$\sum_{j=0}^{\infty} \frac{\varphi(\alpha^j x)}{|\alpha|^j} < \infty \tag{3.7}$$

*for all $x \in X$, then there exists a unique mapping $L : X \to Y$ satisfying $L(\alpha x) = \alpha L(x)$ and*

$$\|f(x) - L(x)\| \leq \frac{1}{|\alpha|} \sum_{j=0}^{\infty} \frac{\varphi(\alpha^j x)}{|\alpha|^j} \tag{3.8}$$

*for all $x \in X$. If, in addition, $f$ is additive, then $L$ is $\alpha$-additive.*

Note that this lemma is a special case of the results of [24].

*Proof.* Replacing $x$ by $\alpha^j x$ in (3.6), we get $\|f(\alpha^{j+1}x) - \alpha f(\alpha^j x)\| \leq \varphi(\alpha^j x)$. Dividing by $|\alpha|^{j+1}$ in the above inequality, we get

$$\left\| \frac{f(\alpha^{j+1}x)}{\alpha^{j+1}} - \frac{f(\alpha^j x)}{\alpha^j} \right\| \leq \frac{\varphi(\alpha^j x)}{|\alpha|^{j+1}} \tag{3.9}$$

for all $x \in X$. From the above inequality, we have

$$\left\| \frac{f(\alpha^{n+1}x)}{\alpha^{n+1}} - \frac{f(\alpha^q x)}{\alpha^q} \right\| \leq \sum_{j=q}^{n} \left\| \frac{f(\alpha^{j+1}x)}{\alpha^{j+1}} - \frac{f(\alpha^j x)}{\alpha^j} \right\| \leq \sum_{j=q}^{n} \frac{1}{|\alpha|} \frac{\varphi(\alpha^j x)}{|\alpha|^j} \tag{3.10}$$

for all $x \in X$ and all nonnegative integers $q, n$ with $q < n$. Thus by (3.7), the sequence $\{f(\alpha^n x)/\alpha^n\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{f(\alpha^n x)/\alpha^n\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n \to \infty} \frac{f(\alpha^n x)}{\alpha^n} \tag{3.11}$$

for all $x \in X$.

In order to prove that $L$ satisfies (3.8), if we put $q = 0$ and let $n \to \infty$ in the above inequality, then we obtain

$$\|f(x) - L(x)\| \leq \sum_{j=0}^{\infty} \frac{1}{|\alpha|} \frac{\varphi(\alpha^j x)}{|\alpha|^j} \tag{3.12}$$

for all $x \in X$.

On the other hand,

$$L(\alpha x) = \lim_{n\to\infty} \frac{f(\alpha^n \alpha x)}{\alpha^n} = \alpha \lim_{n\to\infty} \frac{f(\alpha^{n+1} x)}{\alpha^{n+1}} = \alpha L(x) \tag{3.13}$$

for all $x \in X$, as desired.

Now to prove the uniqueness of $L$, let $L' : X \to Y$ be another mapping satisfying $L'(\alpha x) = \alpha L'(x)$ and (3.8). Then we have

$$\begin{aligned}\|L(x) - L'(x)\| &= \frac{1}{|\alpha|^n} \|L(\alpha^n x) - L'(\alpha^n x)\| \\ &\le \frac{1}{|\alpha|^n} (\|L(\alpha^n x) - f(\alpha^n x)\| + \|L'(\alpha^n x) - f(\alpha^n x)\|) \\ &\le \frac{2}{|\alpha|^n} \cdot \frac{1}{|\alpha|} \sum_{j=0}^{\infty} \frac{\varphi(\alpha^j \alpha^n x)}{|\alpha|^j} \\ &= \frac{2}{|\alpha|} \sum_{j=n}^{\infty} \frac{\varphi(\alpha^j x)}{|\alpha|^j}\end{aligned} \tag{3.14}$$

which goes to zero as $n \to \infty$ for all $x \in X$ by (3.7). Consequently, $L$ is a unique desired mapping.

In addition, when $f$ is additive, $L$ is also additive and so the fact of $L(\alpha x) = \alpha L(x)$ for all $x \in X$ gives that $L$ is $\alpha$-additive. □

According to Theorem 2.1, the inequality (2.1) can be reduced as the following additive functional inequality

$$\|\alpha f(x) + \beta f(y) + \gamma f(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| \tag{3.15}$$

for all $x, y, z \in X$.

In the following theorem, we prove the Hyers-Ulam-Rassias stability of the above additive functional inequality.

**Theorem 3.6.** *Let $\xi = -\alpha/\beta$ and let $f : X \to Y$ be a mapping satisfying $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ for all $x \in X$. If there exists a function $\varphi : X^3 \to [0,\infty)$ satisfying*

$$\|\alpha f(x) + \beta f(y) + \gamma f(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| + \varphi(x, y, z), \tag{3.16}$$

$$\sum_{j=0}^{\infty} \frac{\varphi(\xi^j x, \xi^j y, \xi^j z)}{|\xi|^j} < \infty, \tag{3.17}$$

$$\lim_{t\in\mathbb{R}, t\to 0} \sum_{j=0}^{\infty} \frac{\varphi(\xi^j tx, \xi^{j+1} tx, 0)}{|\xi|^j} = 0 \tag{3.18}$$

*for all $x, y, z \in X$, then there exists a unique $\mathbb{R}$-linear and $\xi$-additive mapping $L : X \to Y$ satisfying*

$$\|f(x) - L(x)\| \le \frac{1}{|\alpha|} \sum_{j=0}^{\infty} \frac{\varphi(\xi^j x, \xi^{j+1} x, 0)}{|\xi|^j} \tag{3.19}$$

*for all $x \in X$. If, in addition, $\xi$ is not a real number, then $L$ is a $\mathbb{C}$-linear mapping.*

*Proof.* Replacing $y = -(\alpha/\beta)x$, $z = 0$ in (3.16), since

$$\left\|\alpha f(x) + \beta f\left(-\frac{\alpha}{\beta}x\right)\right\| \le \varphi\left(x, -\frac{\alpha}{\beta}x, 0\right), \tag{3.20}$$

we get

$$\|f(\xi x) - \xi f(x)\| \le \frac{1}{|\beta|}\varphi(x, \xi x, 0) \tag{3.21}$$

for all $x \in X$. If we replace $\varphi(x)$ in Lemma 3.5 by $(1/|\beta|)\varphi(x, \xi x, 0)$, then by (3.17) and Lemma 3.5, there exists a unique mapping $L : X \to Y$ satisfying $L(\xi x) = \xi L(x)$ for all $x \in X$ and (3.19). In fact, $L(x) := \lim_{n\to\infty}(f(\xi^n x)/\xi^n)$ for all $x \in X$. Moreover, by $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ for all $x \in X$ and (3.18), we get

$$\lim_{t\in\mathbb{R}, t\to 0}\|L(tx) - f(tx)\| \le \lim_{t\in\mathbb{R}, t\to 0}\frac{1}{|\alpha|}\sum_{j=0}^{\infty}\frac{\varphi(\xi^j tx, \xi^{j+1}tx, 0)}{|\xi|^j} = 0 \tag{3.22}$$

and so $\lim_{t\in\mathbb{R}, t\to 0} L(tx) = 0$ for all $x \in X$. Since (3.16) and (3.17) give

$$\begin{aligned}\|\alpha L(x) + \beta L(y) + \gamma L(z)\| &= \lim_{n\to\infty}\left\|\frac{\alpha f(\xi^n x) + \beta f(\xi^n y) + \gamma f(\xi^n z)}{\xi^n}\right\| \\ &\le \lim_{n\to\infty}\left\|\frac{f(\xi^n(\alpha x + \beta y + \gamma z))}{\xi^n}\right\| + \lim_{n\to\infty}\frac{\varphi(\xi^n x, \xi^n y, \xi^n z)}{|\xi|^n} \\ &= \|L(\alpha x + \beta y + \gamma z)\| + 0 \\ &= \|L(\alpha x + \beta y + \gamma z)\|,\end{aligned} \tag{3.23}$$

we conclude that by Theorem 2.1 and Lemma 2.6, a mapping $L$ is $\mathbb{R}$-linear and $\xi$-additive. When $\xi$ is not a real number, by Lemma 3.4, a mapping $L$ is $\mathbb{C}$-linear. □

In the above theorem, we remark that when $\xi$ is $-\gamma/\beta$ or $-\alpha/\gamma$, we obtain the same result as in Theorem 3.6.

As an application of Theorem 3.6, we obtain the following stability.

**Corollary 3.7.** *Let $f : X \to Y$ be a mapping satisfying $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ for all $x \in X$ and $\xi = -\alpha/\beta$. When $|\alpha| > |\beta|$ and $0 < p < 1$, or $|\alpha| < |\beta|$ and $p > 1$, if there exists a $\theta \ge 0$ satisfying*

$$\|\alpha f(x) + \beta f(y) + \gamma f(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.24}$$

*for all $x, y, z \in X$, then there exists a unique $\mathbb{R}$-linear and $\xi$-additive mapping $L : X \to Y$ satisfying*

$$\|f(x) - L(x)\| \le \frac{\theta(|\alpha|^p + |\beta|^p)}{|\alpha||\beta|(|\beta|^{p-1} - |\alpha|^{p-1})}\|x\|^p \tag{3.25}$$

*for all $x \in X$.*

*Proof.* If we define $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, then $\varphi$ satisfies the conditions of (3.17) and (3.18). Thanks to Theorem 3.6, it is proved. □

Before closing this section, we establish another stability of generalized additive functional inequalities.

**Lemma 3.8.** *Let $f : X \to Y$ be a mapping. If there exists a function $\psi : X \to [0,\infty)$ satisfying* (3.6) *and*

$$\sum_{j=1}^{\infty} |\alpha|^j \psi\left(\frac{x}{\alpha^j}\right) < \infty \tag{3.26}$$

*for all $x \in X$, then there exists a unique mapping $L : X \to Y$ satisfying $L(\alpha x) = \alpha L(x)$ and*

$$\|f(x) - L(x)\| \le \frac{1}{|\alpha|}\sum_{j=1}^{\infty} |\alpha|^j \psi\left(\frac{x}{\alpha^j}\right) \tag{3.27}$$

*for all $x \in X$. If, in addition, $f$ is additive, then $L$ is $\alpha$-additive.*

Note that this lemma is a special case of the results of [24].

*Proof.* Replacing $x$ by $x/\alpha^j$ in (3.6), we get $\|f(x/\alpha^{j-1}) - \alpha f(x/\alpha^j)\| \le \psi(x/\alpha^j)$. Multiplying by $|\alpha|^{j-1}$ in the above inequality, we get

$$\left\|\alpha^{j-1} f\left(\frac{x}{\alpha^{j-1}}\right) - \alpha^j f\left(\frac{x}{\alpha^j}\right)\right\| \le |\alpha|^{j-1}\psi\left(\frac{x}{\alpha^j}\right) \tag{3.28}$$

for all $x \in X$. From the above inequality, we have

$$\left\|\alpha^n f\left(\frac{x}{\alpha^n}\right) - \alpha^{q-1} f\left(\frac{x}{\alpha^{q-1}}\right)\right\| \le \sum_{j=q}^{n}\left\|\alpha^j f\left(\frac{x}{\alpha^j}\right) - \alpha^{j-1} f\left(\frac{x}{\alpha^{j-1}}\right)\right\| \le \sum_{j=q}^{n}\frac{1}{|\alpha|}|\alpha|^j\psi\left(\frac{x}{\alpha^j}\right) \tag{3.29}$$

for all $x \in X$ and all nonnegative integers $q$, $n$ with $q < n$. Thus by (3.26) the sequence $\{\alpha^n f(x/\alpha^n)\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{\alpha^n f(x/\alpha^n)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \alpha^n f\left(\frac{x}{\alpha^n}\right) \tag{3.30}$$

for all $x \in X$. In order to prove that $L$ satisfies (3.27), if we put $q = 1$ and let $n \to \infty$ in the above inequality, then we obtain

$$\|f(x) - L(x)\| \le \frac{1}{|\alpha|}\sum_{j=1}^{\infty}|\alpha|^j\varphi\left(\frac{x}{\alpha^j}\right) = \frac{1}{|\alpha|}\sum_{j=1}^{\infty}|\alpha|^j\psi\left(\frac{x}{\alpha^j}\right) \tag{3.31}$$

for all $x \in X$.

On the other hand,

$$L(\alpha x) = \lim_{n\to\infty} \alpha^n f\left(\frac{\alpha x}{\alpha^n}\right) = \alpha \lim_{n\to\infty} \alpha^{n-1} f\left(\frac{x}{\alpha^{n-1}}\right) = \alpha L(x) \tag{3.32}$$

for all $x \in X$, as desired.

Now to prove the uniqueness of $L$, let $L' : X\to Y$ be another mapping satisfying $L'(\alpha x) = \alpha L'(x)$ and (3.27). Then we have

$$\begin{aligned}
\|L(x) - L'(x)\| &= |\alpha|^n \left\| L\left(\frac{x}{\alpha^n}\right) - L'\left(\frac{x}{\alpha^n}\right) \right\| \\
&\le |\alpha|^n \left( \left\| L\left(\frac{x}{\alpha^n}\right) - f\left(\frac{x}{\alpha^n}\right) \right\| + \left\| L'\left(\frac{x}{\alpha^n}\right) - f\left(\frac{x}{\alpha^n}\right) \right\| \right) \\
&\le 2|\alpha|^n \cdot \frac{1}{|\alpha|} \sum_{j=1}^{\infty} |\alpha|^j \varphi\left(\frac{x}{\alpha^j \alpha^n}\right) \\
&= \frac{2}{|\alpha|} \sum_{j=1}^{\infty} |\alpha|^{n+j} \varphi\left(\frac{x}{\alpha^{n+j}}\right) \\
&= \frac{2}{|\alpha|} \sum_{j=n+1}^{\infty} |\alpha|^j \varphi\left(\frac{x}{\alpha^j}\right)
\end{aligned} \tag{3.33}$$

which goes to zero as $n\to\infty$ for all $x \in X$ by (3.26). Consequently, $L$ is a unique desired mapping. □

**Theorem 3.9.** *Let $\xi = -\alpha/\beta$ and let $f : X\to Y$ be a mapping satisfying $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ for all $x \in X$. If there exists a function $\varphi : X^3\to[0,\infty)$ satisfying (3.16) and*

$$\sum_{j=1}^{\infty} |\xi|^j \varphi\left(\frac{x}{\xi^j}, \frac{y}{\xi^j}, \frac{z}{\xi^j}\right) < \infty, \tag{3.34}$$

$$\lim_{t\in\mathbb{R}, t\to 0} \sum_{j=1}^{\infty} |\xi|^j \varphi\left(\frac{tx}{\xi^j}, \frac{tx}{\xi^{j-1}}, 0\right) = 0 \tag{3.35}$$

*for all $x, y, z \in X$, then there exists a unique $\mathbb{R}$-linear and $\xi$-additive mapping $L : X\to Y$ satisfying*

$$\|f(x) - L(x)\| \le \frac{1}{|\alpha|} \sum_{j=1}^{\infty} |\xi|^j \varphi\left(\frac{x}{\xi^j}, \frac{x}{\xi^{j-1}}, 0\right) \tag{3.36}$$

*for all $x \in X$. If, in addition, $\xi$ is not a real number, then $L$ is a $\mathbb{C}$-linear mapping.*

*Proof.* Replacing $y = -(\alpha/\beta)x$, $z = 0$ in (3.16), we get

$$\|f(\xi x) - \xi f(x)\| \le \frac{1}{|\beta|} \varphi(x, \xi x, 0) \tag{3.37}$$

for all $x \in X$. Thus by (3.34) and Lemma 3.8, there exists a unique mapping $L : X\to Y$ satisfying (3.36) and $L(\xi x) = \xi L(x)$ for all $x \in X$. Since $L(x) := \lim_{n\to\infty} \xi^n f(x/\xi^n)$ for all $x \in X$, by $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ and (3.35), we get

$$\lim_{t\in\mathbb{R}, t\to 0} \|L(tx) - f(tx)\| \le \lim_{t\in\mathbb{R}, t\to 0} \frac{1}{|\alpha|} \sum_{j=1}^{\infty} |\xi|^j \varphi\left(\frac{tx}{\xi^j}, \frac{tx}{\xi^{j-1}}, 0\right) = 0 \tag{3.38}$$

and so $\lim_{t\in\mathbb{R}, t\to 0} L(tx) = 0$ for all $x \in X$. It follows from (3.16) and (3.34) that

$$\begin{aligned} \left\|\alpha L(x) + \beta L(y) + \gamma L(z)\right\| &= \lim_{n\to\infty}\left\|\xi^n\left(\alpha f\left(\frac{x}{\xi^n}\right) + \beta f\left(\frac{y}{\xi^n}\right) + \gamma f\left(\frac{z}{\xi^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|\xi^n f\left(\frac{\alpha x}{\xi^n} + \frac{\beta y}{\xi^n} + \frac{\gamma z}{\xi^n}\right)\right\| + \lim_{n\to\infty}|\xi|^n\varphi\left(\frac{x}{\xi^n}, \frac{y}{\xi^n}, \frac{z}{\xi^n}\right) \\ &= \left\|L(\alpha x + \beta y + \gamma z)\right\| + 0 \\ &= \left\|L(\alpha x + \beta y + \gamma z)\right\| \end{aligned} \tag{3.39}$$

for all $x, y, z \in X$. The rest of the proof is the same as in the corresponding part of the proof of Theorem 3.6. □

**Corollary 3.10.** *Let $f : X \to Y$ be a mapping satisfying $\lim_{t\in\mathbb{R}, t\to 0} f(tx) = 0$ for all $x \in X$. When $|\alpha| > |\beta|$ and $p > 1$, or $|\alpha| < |\beta|$ and $0 < p < 1$, if there exists a $\theta \ge 0$ satisfying*

$$\left\|\alpha f(x) + \beta f(y) + \gamma f(z)\right\| \le \left\|f(\alpha x + \beta y + \gamma z)\right\| + \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{3.40}$$

*for all $x, y, z \in X$, then there exists a unique $\mathbb{R}$-linear and $\xi$-additive mapping $L : X \to Y$ satisfying*

$$\left\|f(x) - L(x)\right\| \le \frac{\theta\left(|\alpha|^p + |\beta|^p\right)}{|\alpha||\beta|\left(|\alpha|^{p-1} - |\beta|^{p-1}\right)}\|x\|^p \tag{3.41}$$

*for all $x \in X$.*

*Proof.* If we define $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, then $\varphi$ satisfies the conditions of (3.34) and (3.35). Thanks to Theorem 3.9, it is proved. □


## Acknowledgments

The first author was supported by Daejin University grants in 2008. The authors would like to thank the referees for a number of valuable suggestions regarding a previous version of this paper.


## References


[1] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[6] Th. M. Rassias, "Problem 16; 2, Report of the 27th International Symposium on Functional Equations," *Aequationes Mathematicae*, vol. 39, no. 2-3, pp. 292–293, 1990.

[7] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[8] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[9] S.-M. Jung, "On the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 204, no. 1, pp. 221–226, 1996.

[10] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[11] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[12] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[13] Th. M. Rassias, "The problem of S. M. Ulam for approximately multiplicative mappings," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 352–378, 2000.

[14] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[15] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[16] Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[17] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[18] A. Gilányi, "Eine zur Parallelogrammgleichung äquivalente Ungleichung," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 303–309, 2001.

[19] J. Rätz, "On inequalities associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 66, no. 1-2, pp. 191–200, 2003.

[20] W. Fechner, "Stability of a functional inequality associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 71, no. 1-2, pp. 149–161, 2006.

[21] A. Gilányi, "On a problem by K. Nikodem," *Mathematical Inequalities & Applications*, vol. 5, no. 4, pp. 707–710, 2002.

[22] C. Park, Y. S. Cho, and M.-H. Han, "Functional inequalities associated with Jordan-von Neumann-type additive functional equations," *Journal of Inequalities and Applications*, vol. 2007, Article ID 41820, 13 pages, 2007.

[23] Y.-S. Cho and H.-M. Kim, "Stability of functional inequalities with Cauchy-Jensen additive mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 89180, 13 pages, 2007.

[24] G. L. Forti, "Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 127–133, 2004.