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Crystals and Nakajima monomials for quantum generalized Kac–Moody algebras

Kyeonghoon Jeong [a,1], Seok-Jin Kang [a,1], Jeong-Ah Kim [b,2], Dong-Uy Shin [c,*,3]

[a] Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea
[b] Department of Mathematics, University of Seoul, Seoul 130-743, Republic of Korea
[c] Department of Mathematics Education, Hanyang University, Seoul 133-791, Republic of Korea

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Abstract

We introduce the notion of Nakajima monomials for quantum generalized Kac–Moody algebras and construct the crystals $B(\infty)$ and $B(\lambda)$ in terms of Nakajima monomials. We also give an explicit description of the Nakajima monomials in the crystals $B(\infty)$ and $B(\lambda)$ for the rank 2 quantum generalized Kac–Moody algebras and for the quantum Monster algebra.

Keywords: Crystals; Nakajima monomials; Generalized Kac–Moody algebras

Introduction

The *crystal basis theory* was introduced by Kashiwara for the quantum groups associated with Kac–Moody algebras [7]. Among others, he showed that there exist a crystal basis $B(\infty)$ for the negative part of a quantum group and a crystal basis $B(\lambda)$ for the irreducible highest weight module $V(\lambda)$ with a dominant integral highest weight λ. During the past 15 years, it has

* Corresponding author.
E-mail addresses: khjeong@math.snu.ac.kr (K. Jeong), sjkang@math.snu.ac.kr (S.-J. Kang), jakim@uos.ac.kr (J.-A. Kim), dushin@hanyang.ac.kr (D.-U. Shin).
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become one of the most exciting themes in combinatorial representation theory, for it has a lot of important and interesting applications both in combinatorics and in representation theory.

In [9,10], Nakajima discovered that the set of monomials appearing in t-analogue of q-characters for finite dimensional representations of quantum affine algebras has a colored oriented graph structure. These monomials are called the *Nakajima monomials*, and in [8] and [11], Kashiwara and Nakajima independently defined a crystal structure on the set of Nakajima monomials. Moreover, it was shown that the connected component containing a maximal vector with a dominant integral weight λ is isomorphic to the crystal $B(\lambda)$.

In [6], Kang, Kim and Shin extended the above idea to the realization of the crystal $B(\infty)$ in terms of Nakajima monomials. That is, by adding a new variable $\mathbf{1}$, they introduced the notion of *modified Nakajima monomials*, defined a crystal structure on the set of modified Nakajima monomials, and showed that the connected component containing $\mathbf{1}$ is isomorphic to the crystal $B(\infty)$.

On the other hand, in [2], Jeong, Kang and Kashiwara developed the crystal basis theory for the quantum generalized Kac–Moody algebras – the quantum groups associated with generalized Kac–Moody algebras. As in the Kac–Moody algebra case, they showed that there exist a crystal basis $B(\infty)$ for the negative part of a quantum generalized Kac–Moody algebra and a crystal basis $B(\lambda)$ for the irreducible highest weight module $V(\lambda)$ with a dominant integral highest weight λ.

In this paper, we introduce the notion of Nakajima monomials for quantum generalized Kac–Moody algebras and construct the crystals $B(\infty)$ and $B(\lambda)$ in terms of Nakajima monomials. We first prove the *recognition theorems* for $B(\infty)$ and $B(\lambda)$ in which they are characterized as the crystals satisfying certain rank 2 conditions. We then introduce two kinds of Nakajima monomials – *Verma type* and *integrable type* – and define a crystal structure on each set of Nakajima monomials.

Using the crystal embedding theorem (see [3]) and the recognition theorems, we show that the connected component of Nakajima monomials of Verma type (respectively integrable type) containing $\mathbf{1}$ (respectively a maximal vector with a dominant integral weight λ) is isomorphic to the crystal $B(\infty)$ (respectively $B(\lambda)$). Finally, we give an explicit description of the Nakajima monomials in the crystals $B(\infty)$ and $B(\lambda)$ for the rank 2 quantum generalized Kac–Moody algebras and for the quantum Monster algebra.

1. Crystals

Let I be a countable index set. A *Borcherds–Cartan matrix* $A = (a_{ij})_{i,j\in I}$ is a real matrix satisfying the following conditions: (i) $a_{ii} = 2$ or $a_{ii} \leqslant 0$ for all $i \in I$, (ii) $a_{ij} \leqslant 0$ if $i \neq j$, (iii) $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$, (iv) $a_{ij} = 0$ if and only if $a_{ji} = 0$. We say that an index $i \in I$ is *real* if $a_{ii} = 2$ and *imaginary* if $a_{ii} \leqslant 0$. We denote by $I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}$ and $I^{\mathrm{im}} = \{i \in I \mid a_{ii} \leqslant 0\}$ the set of real indices and the set of imaginary indices, respectively. In this paper, we assume that $a_{ij} \in \mathbf{Z}$, $a_{ii} \in 2\mathbf{Z}$, and A is *symmetrizable*.

A *Borcherds–Cartan datum* $(A, P^\vee, P, \Pi^\vee, \Pi)$ consists of

(i) A: a Borcherds–Cartan matrix,
(ii) $P^\vee = (\bigoplus_{i\in I} \mathbf{Z}h_i) \oplus (\bigoplus_{i\in I} \mathbf{Z}d_i)$: the *dual weight lattice*,
(iii) $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbf{Z}\}$, where $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$: the *weight lattice*,
(iv) $\Pi^\vee = \{h_i \mid i \in I\}$: the set of *simple coroots*,
(v) $\Pi = \{\alpha_i \mid i \in I\}$: the set of *simple roots*.

In particular, we have $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$.

We denote by $P^+ = \{\lambda \in P \mid \lambda(h_i) \geqslant 0 \text{ for all } i \in I\}$ the set of *dominant integral weights*. For instance, the *fundamental weight* Λ_i $(i \in I)$ defined by

$$\Lambda_i(h_j) = \delta_{ij} \quad \text{and} \quad \Lambda_i(d_j) = 0 \quad (j \in I)$$

is a dominant integral weight. For convenience, we will abuse the notation and write $\lambda = \sum_{i\in I} a_i\Lambda_i$ whenever $\langle h_i, \lambda\rangle = a_i \in \mathbf{Z}$, $\langle d_i, \lambda\rangle = 0$. We also use the notation $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ and $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geqslant 0}\alpha_i$.

Let $U_q(\mathfrak{g})$ be the *quantum generalized Kac–Moody algebra* associated with the Borcherds–Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ (see, for example, [2,4]). We recall the definition of abstract crystals for quantum generalized Kac–Moody algebras introduced in [3].

Definition 1.1. An *abstract $U_q(\mathfrak{g})$-crystal* or simply a *crystal* is a set B together with the maps $\mathrm{wt}: B \to P$, $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}$ and $\varepsilon_i, \varphi_i : B \to \mathbf{Z} \sqcup \{-\infty\}$ $(i \in I)$ satisfying the following conditions:

(i) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}\, b + \alpha_i$ if $\tilde{e}_i b \neq 0$,
(ii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}\, b - \alpha_i$ if $\tilde{f}_i b \neq 0$,
(iii) for any $i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}\, b\rangle$,
(iv) for any $i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
(v) for any $i \in I$ and $b \in B$ such that $\tilde{e}_i b \neq 0$, we have
 (a) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $i \in I^{\mathrm{re}}$,
 (b) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + a_{ii}$ if $i \in I^{\mathrm{im}}$,
(vi) for any $i \in I$ and $b \in B$ such that $\tilde{f}_i b \neq 0$, we have
 (a) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $i \in I^{\mathrm{re}}$,
 (b) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^{\mathrm{im}}$,
(vii) for any $i \in I$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition 1.2. Let B_1 and B_2 be crystals. A map $\psi : B_1 \to B_2$ is called a *morphism of crystals* or a *crystal morphism* if it satisfies the following conditions:

(i) for $b \in B_1$, we have

$$\mathrm{wt}\big(\psi(b)\big) = \mathrm{wt}(b), \qquad \varepsilon_i\big(\psi(b)\big) = \varepsilon_i(b), \qquad \varphi_i\big(\psi(b)\big) = \varphi_i(b) \quad \text{for all } i \in I,$$

(ii) if $b \in B_1$ and $\tilde{f}_i b \in B_1$, then we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

Example 1.3.

(a) The crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ with $\lambda \in P^+$ is a $U_q(\mathfrak{g})$-crystal, where the maps ε_i, φ_i $(i \in I)$ are given by

$$\varepsilon_i(b) = \begin{cases} \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{\mathrm{re}}, \\ 0 & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_i(b)=\begin{cases}\max\{k\geqslant 0 \mid \tilde{f}_i^k b\neq 0\} & \text{for } i\in I^{\mathrm{re}},\\ \langle h_i,\mathrm{wt}(b)\rangle & \text{for } i\in I^{\mathrm{im}}.\end{cases}$$

(b) The crystal basis $B(\infty)$ of $U_q^-(\mathfrak{g})$ is a $U_q(\mathfrak{g})$-crystal, where

$$\varepsilon_i(b)=\begin{cases}\max\{k\geqslant 0 \mid \tilde{e}_i^k b\neq 0\} & \text{for } i\in I^{\mathrm{re}},\\ 0 & \text{for } i\in I^{\mathrm{im}},\end{cases}$$

$$\varphi_i(b)=\varepsilon_i(b)+\big\langle h_i,\mathrm{wt}(b)\big\rangle \quad (i\in I).$$

Example 1.4. For $\lambda\in P$, the singletons $T_\lambda=\{t_\lambda\}$ and $R_\lambda=\{r_\lambda\}$ are $U_q(\mathfrak{g})$-crystals with the maps defined by

$$\mathrm{wt}(t_\lambda)=\lambda,\qquad \varepsilon_i(t_\lambda)=\varphi_i(t_\lambda)=-\infty,\qquad \tilde{e}_i t_\lambda=\tilde{f}_i t_\lambda=0\quad\text{for all } i\in I,$$

and

$$\mathrm{wt}(r_\lambda)=\lambda,\qquad \varepsilon_i(r_\lambda)=-\langle h_i,\lambda\rangle,\qquad \varphi_i(r_\lambda)=0,\qquad \tilde{e}_i r_\lambda=\tilde{f}_i r_\lambda=0\quad\text{for all } i\in I.$$

Example 1.5. For each $i\in I$, let $B_i=\{b_i(-n)\mid n\geqslant 0\}$. Then B_i is a crystal with the maps defined by

$$\begin{gathered}\mathrm{wt}\big(b_i(-n)\big)=-n\alpha_i,\\ \tilde{e}_i b_i(-n)=b_i(-n+1),\qquad \tilde{f}_i b_i(-n)=b_i(-n-1),\\ \tilde{e}_j b_i(-n)=\tilde{f}_j b_i(-n)=0\quad\text{if } j\neq i,\\ \varepsilon_i\big(b_i(-n)\big)=n,\qquad \varphi_i\big(b_i(-n)\big)=-n\quad\text{if } i\in I^{\mathrm{re}},\\ \varepsilon_i\big(b_i(-n)\big)=0,\qquad \varphi_i\big(b_i(-n)\big)=-na_{ii}\quad\text{if } i\in I^{\mathrm{im}},\\ \varepsilon_j\big(b_i(-n)\big)=\varphi_j\big(b_i(-n)\big)=-\infty\quad\text{if } j\neq i.\end{gathered}$$

Here, we understand $b_i(-n)=0$ for $n<0$. The crystal B_i is called an *elementary crystal*.

Example 1.6. For two crystals B_1 and B_2, their tensor product $B_1\otimes B_2$ is a crystal with the maps wt, ε_i,φ_i given by

$$\begin{aligned}&\mathrm{wt}(b\otimes b')=\mathrm{wt}(b)+\mathrm{wt}(b'),\\ &\varepsilon_i(b\otimes b')=\max\big(\varepsilon_i(b),\varepsilon_i(b')-\big\langle h_i,\mathrm{wt}(b)\big\rangle\big),\\ &\varphi_i(b\otimes b')=\max\big(\varphi_i(b)+\big\langle h_i,\mathrm{wt}(b')\big\rangle,\varphi_i(b')\big),\\ &\tilde{f}_i(b\otimes b')=\begin{cases}\tilde{f}_i b\otimes b' & \text{if } \varphi_i(b)>\varepsilon_i(b'),\\ b\otimes\tilde{f}_i b' & \text{if } \varphi_i(b)\leqslant\varepsilon_i(b'),\end{cases}\\ &\tilde{e}_i(b\otimes b')=\begin{cases}\tilde{e}_i b\otimes b' & \text{if } \varphi_i(b)\geqslant\varepsilon_i(b') \text{ and } i\in I^{\mathrm{re}}\\ & \text{or } \varphi_i(b)>\varepsilon_i(b')-a_{ii} \text{ and } i\in I^{\mathrm{im}},\\ 0 & \text{if } \varepsilon_i(b')<\varphi_i(b)\leqslant\varepsilon_i(b')-a_{ii},\\ b\otimes\tilde{e}_i b' & \text{if } \varphi_i(b)<\varepsilon_i(b') \text{ and } i\in I^{\mathrm{re}},\\ & \text{or } \varphi_i(b)\leqslant\varepsilon_i(b') \text{ and } i\in I^{\mathrm{im}}.\end{cases}\end{aligned}$$

Example 1.7. Let $\mathbf{i} = (i_1, i_2, \ldots)$ be an infinite sequence in I such that every $i \in I$ appears infinitely many times in $\mathbf{i}$, and let

$$B(\mathbf{i}) = \big\{\cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \\ \in \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_1};\ x_k \in \mathbf{Z}_{\geqslant 0},\ \text{and } x_k = 0 \text{ for } k \gg 0\big\}.$$

Then $B(\mathbf{i})$ has a crystal structure as follows. Let $b = \cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in B(\mathbf{i})$. Then we have

$$\mathrm{wt}(b) = -\sum_k x_k \alpha_{i_k}.$$

For $i \in I^{\mathrm{re}}$, we have

$$\varepsilon_i(b) = \max\left\{x_k + \sum_{l>k} \langle h_i, \alpha_{i_l}\rangle x_l\ 1 \leqslant k,\ i_k = i\right\},$$

$$\varphi_i(b) = \max\left\{-x_k - \sum_{1 \leqslant l<k} \langle h_i, \alpha_{i_l}\rangle x_l;\ 1 \leqslant k,\ i_k = i\right\},$$

and, for $i \in I^{\mathrm{im}}$, we have

$$\varepsilon_i(b) = 0 \quad \text{and} \quad \varphi_i(b) = \big\langle h_i, \mathrm{wt}(b)\big\rangle.$$

For $i \in I^{\mathrm{re}}$, we have

$$\tilde{e}_i b = \begin{cases} \cdots \otimes b_{i_{n_e}}(-x_{n_e} + 1) \otimes \cdots \otimes b_{i_1}(-x_1) & \text{if } \varepsilon_i(b) > 0, \\ 0 & \text{if } \varepsilon_i(b) \leqslant 0, \end{cases}$$

$$\tilde{f}_i b = \cdots \otimes b_{i_{n_f}}(-x_{n_f} - 1) \otimes \cdots \otimes b_{i_1}(-x_1),$$

where n_e (respectively n_f) is the largest (respectively smallest) $k \geqslant 1$ such that $i_k = i$ and $x_k + \sum_{l>k} \langle h_i, \alpha_{i_l}\rangle x_l = \varepsilon_i(b)$. When $i \in I^{\mathrm{im}}$, let n_f be the smallest k such that

$$i_k = i \quad \text{and} \quad \sum_{l>k} \langle h_i, \alpha_{i_l}\rangle x_l = 0.$$

Then we have

$$\tilde{f}_i b = \cdots \otimes b_{i_{n_f}}(-x_{n_f} - 1) \otimes \cdots \otimes b_{i_1}(-x_1)$$

and

$$\tilde{e}_i b = \begin{cases} \cdots \otimes b_{i_{n_f}}(-x_{n_f} + 1) \otimes \cdots \otimes b_{i_1}(-x_1) \\ \quad \text{if } x_{n_f} > 0 \text{ and } \sum_{k<l\leqslant n_f} \langle h_i, \alpha_{i_l}\rangle x_l < a_{ii} \\ \quad \text{for any } k \text{ such that } 1 \leqslant k < n_f \text{ and } i_k = i, \\ 0 \quad \text{otherwise.} \end{cases}$$

Example 1.8. Let $R_\lambda = \{r_\lambda\}$ be the crystal given in Example 1.4. Then for a crystal B, $B \otimes R_\lambda$ is a crystal with the maps wt, ε_i, φ_i given by

$$\begin{aligned}
&\mathrm{wt}(b \otimes r_\lambda) = \mathrm{wt}(b) + \lambda,\\
&\varepsilon_i(b \otimes r_\lambda) = \max\bigl(\varepsilon_i(b), -\langle h_i, \lambda + \mathrm{wt}(b)\rangle\bigr),\\
&\varphi_i(b \otimes r_\lambda) = \begin{cases} \varphi_i(b) + \langle h_i, \lambda\rangle & \text{for } i \in I^{\mathrm{re}},\\ \max(\varphi_i(b) + \langle h_i, \lambda\rangle, 0) & \text{for } i \in I^{\mathrm{im}},\end{cases}\\
&\tilde{e}_i(b \otimes r_\lambda) = \begin{cases} \tilde{e}_i b \otimes r_\lambda & \text{if } \varphi_i(b) \geqslant -\langle h_i, \lambda\rangle \text{ and } i \in I^{\mathrm{re}},\\ & \text{or } \varphi_i(b) + \langle h_i, \lambda\rangle + a_{ii} > 0 \text{ and } i \in I^{\mathrm{im}},\\ 0 & \text{otherwise},\end{cases}\\
&\tilde{f}_i(b \otimes r_\lambda) = \begin{cases} \tilde{f}_i b \otimes r_\lambda & \text{if } \varphi_i(b) > -\langle h_i, \lambda\rangle,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

2. The recognition theorems

Let B be an abstract crystal and let J be a subset of I. We denote by $U_q(\mathfrak{g}_J)$ the quantum group associated with the Borcherds–Cartan matrix $A_J = (a_{ij})_{i,j \in J}$. Moreover, we denote by $\psi_J(B)$ the $U_q(\mathfrak{g}_J)$-crystal obtained from B by removing all the i-arrows with $i \notin J$.

Theorem 2.1. *Suppose that B is an abstract crystal satisfying the following conditions*:

(i) *there exists a unique element $b_0 \in B$ such that $\tilde{e}_i b_0 = 0$ for all $i \in I$,*
(ii) *for all $b \in B$, there exist $i_1, \ldots, i_r \in I$ $(r \geqslant 0)$ such that $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} b_0$,*
(iii) *for all $J \subset I$ with $|J| \leqslant 2$, $\psi_J(B)$ is a disjoint union of the crystals $B_J(\infty) \otimes T_\mu$ with $\mu \in P_J$.*

Then there is a crystal isomorphism $B \xrightarrow{\sim} B(\infty) \otimes T_\lambda$ with $\lambda = \mathrm{wt}(b_0)$.

Proof. The proof is almost the same as the one for quantum groups associated with Kac–Moody algebras (see [5, Proposition 2.4.4]). □

Theorem 2.2. *Suppose that B is an abstract crystal satisfying the following conditions*:

(i) *there exists a unique element $b_0 \in B$ such that $\tilde{e}_i b_0 = 0$ for all $i \in I$,*
(ii) *for all $b \in B$, there exist $i_1, \ldots, i_r \in I$ $(r \geqslant 0)$ such that $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} b_0$,*
(iii) *for all $J \subset I$ with $|J| \leqslant 2$, $\psi_J(B)$ is a disjoint union of the crystals $B_J(\mu)$ with $\mu \in P_J^+$.*

Then there is a crystal isomorphism $B \xrightarrow{\sim} B(\lambda)$ with $\lambda = \mathrm{wt}(b_0)$.

Proof. By (i) and (ii), $B = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} b_0 \mid r \geqslant 0,\ i_k \in I\}$. Moreover, by (ii), the $U_q(\mathfrak{g}_J)$-crystal generated by b_0 is isomorphic to $B_J(\lambda)$ with $\lambda = \mathrm{wt}_J(b_0) \in P_J^+$. For $\sigma = (\sigma_1, \ldots, \sigma_r) \in I^r$, write $|\sigma| = r$ and $\tilde{f}_\sigma = \tilde{f}_{\sigma_1} \cdots \tilde{f}_{\sigma_r}$. We will show by induction on r that

$$A(r):\quad \tilde{f}_\sigma b_0 = 0 \quad \text{if and only if} \quad \tilde{f}_\sigma u_\lambda = 0 \quad \text{for all } |\sigma| = r,$$

$$B(r):\quad \tilde{e}_i \tilde{f}_\sigma b_0 = 0 \quad \text{if and only if} \quad \tilde{e}_i \tilde{f}_\sigma (u_\lambda) = 0 \quad \text{for all } |\sigma| = r,$$

$$C(r):\quad \tilde{f}_\sigma b_0 = \tilde{f}_\tau b_0 \quad \text{if and only if} \quad \tilde{f}_\sigma (u_\lambda) = \tilde{f}_\tau (u_\lambda) \quad \text{for all } |\sigma| = |\tau| = r.$$

When $r = 0$, our assertions are trivial. Assume that our assertions are true for all sequences σ with $|\sigma| < r$. By the same argument given in [5, Proposition 2.4.4], one can prove $B(r)$ and $C(r)$. So it suffices to show $A(r)$. Write $j = \sigma_1$, $\sigma' = (\sigma_2, \dots, \sigma_r)$ for $\sigma = (\sigma_1, \dots, \sigma_r)$. If $\tilde{f}_{\sigma'} u_\lambda = 0$, then $\tilde{f}_\sigma u_\lambda = 0$, and by the induction hypothesis $A(r-1)$, $\tilde{f}_{\sigma'} b_0 = 0$, which implies $\tilde{f}_\sigma b_0 = 0$. If $\tilde{f}_{\sigma'} u_\lambda \neq 0$, write $\tilde{f}_{\sigma'} u_\lambda = \tilde{f}_j^k \tilde{f}_\tau u_\lambda$, where $k \geqslant 0$ and $\tilde{e}_j \tilde{f}_\tau u_\lambda = 0$. By the induction hypothesis $B(r-1)$, $\tilde{e}_j \tilde{f}_\tau b_0 = 0$ and $\tilde{f}_{\sigma'} b_0 = \tilde{f}_j^k \tilde{f}_\tau b_0$. Now, by our assumption (iii), we have

$$\begin{aligned}
\tilde{f}_\sigma b_0 = 0 \quad &\Leftrightarrow \quad \tilde{f}_j(\tilde{f}_{\sigma'} b_0) = 0 \\
&\Leftrightarrow \quad \varphi_j(\tilde{f}_{\sigma'} b_0) = 0 \\
&\Leftrightarrow \quad 0 = \varepsilon_j(\tilde{f}_{\sigma'} b_0) + \langle h_j, \mathrm{wt}(\tilde{f}_{\sigma'} b_0)\rangle \\
&\qquad = \begin{cases} k + \langle h_j, \mathrm{wt}(\tilde{f}_\tau b_0) - k\alpha_j\rangle & \text{if } j \in I^{\mathrm{re}}, \\ \langle h_j, \mathrm{wt}(\tilde{f}_\tau b_0) - k\alpha_j\rangle & \text{if } j \in I^{\mathrm{im}} \end{cases} \\
&\qquad = \begin{cases} -k + \varphi_j(\tilde{f}_\tau u_\lambda) - \varepsilon_j(\tilde{f}_\tau u_\lambda) & \text{if } j \in I^{\mathrm{re}}, \\ -k a_{jj} + \varphi_j(\tilde{f}_\tau u_\lambda) & \text{if } j \in I^{\mathrm{im}} \end{cases} \\
&\Leftrightarrow \quad \varphi_j(\tilde{f}_\tau u_\lambda) = k \quad \text{if } j \in I^{\mathrm{re}}, \quad \text{and} \quad \varphi_j(\tilde{f}_\tau u_\lambda) = 0 \quad \text{if } j \in I^{\mathrm{im}} \\
&\Leftrightarrow \quad \tilde{f}_j^{k+1} \tilde{f}_\tau u_\lambda = \tilde{f}_\sigma u_\lambda = 0.
\end{aligned}$$

Hence $A(r)$ is proved.

Define a map $\psi : B \to B(\lambda)$ by $\psi(\tilde{f}_\sigma b_0) = \tilde{f}_\sigma u_\lambda$. Then by $A(r)$, it commutes with $\tilde{f}_i$. By $B(r)$ and $C(r)$, we have

$$\tilde{e}_i \tilde{f}_\sigma b_0 = 0 \quad \Leftrightarrow \quad \tilde{e}_i \tilde{f}_\sigma u_\lambda = 0$$

and

$$\begin{aligned}
\tilde{e}_i \tilde{f}_\sigma b_0 = \tilde{f}_\tau b_0 \quad &\Leftrightarrow \quad \tilde{f}_\sigma b_0 = \tilde{f}_i \tilde{f}_\tau b_0 \\
&\Leftrightarrow \quad \tilde{f}_\sigma u_\lambda = \tilde{f}_i \tilde{f}_\tau u_\lambda \\
&\Leftrightarrow \quad \tilde{e}_i \tilde{f}_\sigma u_\lambda = \tilde{f}_\tau u_\lambda,
\end{aligned}$$

which shows that ψ commutes with $\tilde{e}_i$. Hence B is isomorphic to $B(\lambda)$. □

3. Monomial realization of $B(\infty)$

In this section, we introduce the notion of *Nakajima monomials* for quantum generalized Kac–Moody algebras and give a realization of the crystal $B(\infty)$ in terms of Nakajima monomials.

Let $A = (a_{ij})_{i,j\in I}$ be a Borcherds–Cartan matrix. For each $i \in I$, we define an integer $N_i \in \mathbf{Z}$ by

$$N_i = \begin{cases} 1 + a_{ii} & \text{if } a_{ii} < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and set $\mathbf{Z}_i = \{n \in \mathbf{Z} \mid n \geqslant N_i\}$.

Let $Y_i(n)$ $(i \in I, n \in \mathbf{Z})$ and $\mathbf{1}$ be commuting variables, and let $\mathcal{M}$ be the set of monomials in $Y_i(n)$'s and $\mathbf{1}$ of the form

$$\mathbf{1} \cdot \prod_{i\in I, n\in \mathbf{Z}_i} Y_i(n)^{y_i(n)} \tag{3.1}$$

satisfying the following conditions:

(i) $y_i(n) \in \mathbf{Z}$ if $i \in I^{\mathrm{re}}$, and $y_i(n) \in \mathbf{Z}_{\geqslant 0}$ if $i \in I^{\mathrm{im}}$,
(ii) for each $i \in I$, $y_i(n) = 0$ for all but finitely many n,
(iii) for each i with $a_{ii} < 0$, if $y_i(k) > 0$ for some $k = a_{ii} + 1, \ldots, -1$, then $y_i(k+1) > 0$.

Note that this is a product of infinite variables. This can be interpreted as a function $f : I \times \mathbf{Z} \to \mathbf{Z}$ defined by $f(i,n) = y_i(n)$. The multiplication of two such functions is given by $(f \times g)(i,n) = f(i,n) + g(i,n)$. For convenience, we will use the monomial notation.

The monomials in $\mathcal{M}$ are called the *Nakajima monomials of Verma type*. We wish to define a crystal structure on $\mathcal{M}$. For a Nakajima monomial $M \in \mathcal{M}$ of the form (3.1), we define

$$\begin{aligned} \mathrm{wt}(M) &= \sum_{i\in I}\Bigl(\sum_n y_i(n)\Bigr)\Lambda_i, \\ \varphi_i(M) &= \max\Biggl\{\sum_{N_i\leqslant k\leqslant n} y_i(k) \Bigm| n \geqslant N_i\Biggr\}, \\ \varepsilon_i(M) &= \varphi_i(M) - \bigl\langle h_i, \mathrm{wt}(M)\bigr\rangle. \end{aligned} \tag{3.2}$$

To define the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ on $\mathcal{M}$, we choose a set $C = (c_{ij})_{i\neq j}$ of nonnegative integers such that $c_{ij} + c_{ji} = 1$ and define

$$A_i(n) = \begin{cases} Y_i(n)Y_i(n+1)\prod_{j\neq i} Y_j(n+c_{ji})^{a_{ji}} & \text{if } i \in I^{\mathrm{re}}, \\ Y_i(n)^{-1}\cdots Y_i(n+a_{ii}+1)^{-1}\prod_{j\neq i} Y_j(n+c_{ji})^{a_{ji}} & \text{if } a_{ii} < 0, \\ \prod_{j\neq i} Y_j(n+c_{ji})^{a_{ji}} & \text{if } a_{ii} = 0. \end{cases}$$

For each $i \in I$, we set

$$\begin{aligned} n_f = n_f(M) &= \begin{cases} \min\{n \geqslant N_i \mid \varphi_i(M) = \sum_{N_i\leqslant k\leqslant n} y_i(k)\} & \text{if } M \neq \mathbf{1}, \\ 0 & \text{if } M = \mathbf{1}, \end{cases} \\ n_e = n_e(M) &= \begin{cases} \max\{n \geqslant N_i \mid \varphi_i(M) = \sum_{N_i\leqslant k\leqslant n} y_i(k)\} & \text{if } i \in I^{\mathrm{re}}, \\ n_f & \text{if } i \in I^{\mathrm{im}}. \end{cases} \end{aligned} \tag{3.3}$$

Note that $n_f \geqslant 0$ by the conditions (i) and (iii) of $\mathcal{M}$, and for each $i \in I^{\mathrm{im}}$, if $n_f > 0$, then n_f is the largest positive integer k such that $y_i(k) > 0$. Also, for each $i \in I^{\mathrm{im}}$, we define $S_i(n_f)$ as follows:

(i) when $a_{ii} < 0$, $n_f > 0$,

$$S_i(n_f) = Y_i(n_f)^2 Y_i(n_f - 1) \cdots Y_i(n_f + a_{ii} + 1) \prod_{\substack{j \neq i \\ j \in I^{\mathrm{im}}}} Y_j(n_f + c_{ji})^{-a_{ji}},$$

(ii) when $a_{ii} < 0$, $n_f = 0$,

$$S_i(n_f) = Y_i(n_f) \cdots Y_i(n_f + a_{ii} + 1) \prod_{\substack{j \neq i \\ j \in I^{\mathrm{im}}}} Y_j(n_f + c_{ji})^{-a_{ji}},$$

(iii) when $a_{ii} = 0$,

$$S_i(n_f) = \prod_{\substack{j \neq i \\ j \in I^{\mathrm{im}}}} Y_j(n_f + c_{ji})^{-a_{ji}}.$$

We now define the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$ as follows:

$$\begin{aligned} \tilde{f}_i M &= M \cdot A_i(n_f)^{-1}, \\ \tilde{e}_i M &= \begin{cases} M \cdot A_i(n_e) & \text{if } i \in I^{\mathrm{re}} \text{ and } \varepsilon_i(M) > 0, \\ & \text{or } i \in I^{\mathrm{im}} \text{ and } S_i(n_f)^{-1} M \in \mathcal{M}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{3.4}$$

Then it is straightforward to verify that $\mathcal{M}$ becomes a $U_q(\mathfrak{g})$-crystal with the maps wt, ε_i, φ_i, $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$ defined in (3.2) and (3.4). Moreover, we have a realization of the crystal $B(\infty)$ in terms of the monomials in $\mathcal{M}$.

Theorem 3.1. *Fix $p \in \mathbf{Z}_{\geqslant 0}$ and choose a sequence $(\lambda_i(p) \mid i \in I)$ of nonnegative integers. If M is a maximal vector in $\mathcal{M}$ of the form*

$$M = \mathbf{1} \cdot \prod_{i \in I} Y_i(p)^{\lambda_i(p)} \quad (p \geqslant 0),$$

then the connected component $C(M)$ of $\mathcal{M}$ containing M is isomorphic to the $U_q(\mathfrak{g})$-crystal $B(\infty) \otimes T_\lambda$, where $\lambda = \mathrm{wt}(M) = \sum_{i \in I} \lambda_i(p) \Lambda_i$.

In particular, we have $C(\mathbf{1}) \xrightarrow{\sim} B(\infty)$.

Proof. Let $M = \mathbf{1} \cdot \prod_{i \in I} Y_i(p)^{\lambda_i(p)}$ $(p \geqslant 0)$ be a maximal vector in $\mathcal{M}$. Thanks to Theorem 2.1, it suffices to prove that for any subset J of I with $|J| \leqslant 2$, the connected component of $\Psi_J(\mathcal{M})$ containing M is isomorphic to the $U_q(\mathfrak{g}_J)$-crystal $B_J(\infty) \otimes T_\lambda$. If $|J| = 1$, it is easy to see that $\tilde{f}_i^N M \neq 0$ for all $N \geqslant 0$, and so the connected component $C(M)$ of $\mathcal{M}$ containing M is isomorphic to the $U_q(\mathfrak{g}_J)$-crystal $B_J(\infty) \otimes T_{\mathrm{wt}(M)}$.

If $|J| = 2$, we assume $J = \{1, 2\}$ and take $c_{12} = 1$, $c_{21} = 0$. Since the result for Kac–Moody algebra case was already known in [6], we may assume that at least one of the indices is imaginary, say, $a_{11} \leqslant 0$. Then

$$A_1(n) = \begin{cases} Y_1(n)^{-1} \cdots Y_1(n + a_{11} + 1)^{-1} Y_2(n)^{a_{21}} \prod_{j \neq 1,2} Y_j(n + c_{j1})^{a_{j1}} & \text{if } a_{11} < 0, \\ Y_2(n)^{a_{21}} \prod_{j \neq 1,2} Y_j(n + c_{j1})^{a_{j1}} & \text{if } a_{11} = 0, \end{cases}$$

and

$$A_2(n) = \begin{cases} Y_2(n) Y_2(n+1) Y_1(n+1)^{a_{12}} \prod_{j \neq 1,2} Y_j(n + c_{j2})^{a_{j2}} & \text{if } a_{22} = 2, \\ Y_2(n)^{-1} \cdots Y_2(n + a_{22} + 1)^{-1} Y_1(n+1)^{a_{12}} \prod_{j \neq 1,2} Y_j(n + c_{j2})^{a_{j2}} & \text{if } a_{22} < 0, \\ Y_1(n+1)^{a_{12}} \prod_{j \neq 1,2} Y_j(n + c_{j2})^{a_{j2}} & \text{if } a_{22} = 0. \end{cases}$$

Set

$$\begin{aligned} K_p = \Big\{ b &:= \bigotimes_{n \geqslant p} \big(b_2\big(z_2(n)\big) \otimes b_1\big(z_1(n)\big)\big) \otimes t_\lambda \\ &= \cdots \otimes b_2\big(z_2(p+1)\big) \otimes b_1\big(z_1(p+1)\big) \otimes b_2\big(z_2(p)\big) \otimes b_1\big(z_1(p)\big) \otimes t_\lambda \\ &\in \cdots \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1 \otimes T_\lambda \;\Big|\; z_1(n) = z_2(n) = 0 \text{ for } n \gg p \Big\}. \end{aligned} \tag{3.5}$$

We define a map $\Phi : K_p \to \mathcal{M}$ by

$$\begin{aligned} b &= \bigotimes_{n \geqslant p} \big(b_2\big(z_2(n)\big) \otimes b_1\big(z_1(n)\big)\big) \otimes t_\lambda \\ &\mapsto M := \mathbf{1} \cdot \prod_{i \in I} Y_i(p)^{\lambda_i(p)} \prod_{n \geqslant p} A_1(n)^{z_1(n)} A_2(n)^{z_2(n)} \\ &= \mathbf{1} \cdot \prod_{\substack{i \in I \\ n \in \mathbf{Z}_i}} Y_i(n)^{y_i(n)}. \end{aligned}$$

It is easy to see that $\Phi(b)$ belongs to $\mathcal{M}$ and

$$\mathrm{wt}(b) = \sum_{n \geqslant p} \big(z_1(n)\alpha_1 + z_2(n)\alpha_2\big) + \sum_{i \in I} \lambda_i(p) \Lambda_i = \mathrm{wt}\, M.$$

Note that

$$\varphi_1(b) = \lambda_1(p) + a_{11} z_1(p) + \sum_{k > p} \big(a_{11} z_1(k) + a_{12} z_2(k-1)\big),$$

and that

$$y_1(n) = \begin{cases} -(z_1(n) + \cdots + z_1(n - a_{11} - 1)) + z_2(n-1) a_{12} & \text{if } n > p, \\ \lambda_1(p) - (z_1(p) + \cdots + z_1(p - a_{11} - 1)) & \text{if } n = p, \\ -(z_1(p) + \cdots + z_1(n - a_{11} - 1)) & \text{if } p + a_{11} + 1 \leqslant n < p, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\varphi_1(M) = \max\left\{ \sum_{1+a_{11}\leqslant k\leqslant n} y_1(k) \,\middle|\, n \geqslant 1 + a_{11} \right\} = \varphi_1(b),$$

and hence $\varepsilon_1(b) = \varepsilon_1(M)$.

Now, suppose that $\tilde{f}_1$ acts on the kth component of b; i.e.,

$$\tilde{f}_1 b = \cdots \otimes b_1\big(z_1(k) - 1\big) \otimes b_2\big(z_2(k-1)\big) \otimes \cdots \otimes b_2\big(z_2(p)\big) \otimes b_1\big(z_1(p)\big) \otimes t_\lambda.$$

Then $\Phi(\tilde{f}_1 b) = M \cdot A_1(k)^{-1}$. On the other hand, by the definition of Kashiwara operator $\tilde{f}_i$ given in Example 1.7, we have

$$\left\langle h_1, z_2(k)\alpha_2 + \sum_{n>k}\big(z_1(n)\alpha_1 + z_2(n)\alpha_2\big)\right\rangle = 0, \tag{3.6}$$

and

$$\left\langle h_1, z_2(k-1)\alpha_2 + \sum_{n\geqslant k}\big(z_1(n)\alpha_1 + z_2(n)\alpha_2\big)\right\rangle > 0. \tag{3.7}$$

Hence $y_1(n) = 0$ for all $n > k$ and $y_1(k) > 0$. Therefore,

$$\tilde{f}_1 \Phi(b) = \tilde{f}_1 M = M \cdot A_1(k)^{-1} = \Phi(\tilde{f}_1 b).$$

Next, suppose that $\tilde{e}_1$ acts on the kth component of b; i.e.,

$$\tilde{e}_1 b = \cdots \otimes b_1\big(z_1(k) + 1\big) \otimes b_2\big(z_2(k-1)\big) \otimes \cdots \otimes b_2\big(z_2(p)\big) \otimes b_1\big(z_1(p)\big) \otimes t_\lambda.$$

Then $\Phi(\tilde{e}_1 b) = M \cdot A_1(k)$. On the other hand, by the definition of Kashiwara operator $\tilde{e}_i$ given in Example 1.7, we have (3.6), (3.7) and

$$z_1(k) < 0, \quad \text{and if} \quad z_1(k) = -1 \quad \text{and } k > p, \qquad z_2(k-1)\langle h_1, \alpha_2\rangle > 0. \tag{3.8}$$

From (3.8), we know that $S_1(n_f) = S_1(k)$ is a factor of $\Phi(b) = M$. It follows that $\tilde{e}_1\Phi(b) = \tilde{e}_1 M = MA_1(k) = \Phi(\tilde{e}_1 b)$. Moreover, according to the definition of Kashiwara operator $\tilde{e}_i$, it is easy to see that $\tilde{e}_1 b = 0$ if and only if $\tilde{e}_1 M = 0$.

Now, if $a_{22} = 2$, by the same argument given in the proof of Theorem 3.1 of [6], we have $\varphi_2(b) = \varphi_2(M)$, and $\tilde{f}_2$ and $\tilde{e}_2$ commute with Φ. If $a_{22} \leqslant 0$, by the same argument as above, $\varphi_2(b) = \varphi_2(M)$, $\varepsilon_2(b) = \varepsilon_2(M)$ and $\tilde{f}_2$ and $\tilde{e}_2$ commute with Φ. Therefore, we conclude that Φ defines a $U_q(\mathfrak{g}_J)$-crystal morphism $K_p \to \Psi_J(\mathcal{M})$.

By the crystal embedding theorem, it was shown that the connected component of K_p containing $\cdots \otimes b_1(0) \otimes b_2(0) \otimes b_1(0) \otimes t_\lambda$ is isomorphic to the crystal $B_J(\infty) \otimes T_\lambda$ (see [3,12]). Therefore, the connected component of $\Psi_J(\mathcal{M})$ containing $M = \mathbf{1} \cdot \prod_{i\in I} Y_i(p)^{\lambda_i(p)}$ is isomorphic to the $U_q(\mathfrak{g}_J)$-crystal $B_J(\infty) \otimes T_\lambda$. □

Example 3.2. Let

$$A = \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}$$

be a Borcherds–Cartan matrix with positive integers α, β, γ and δ. That is, all the indices $1, 2$ are imaginary. Then we have

$$A_1(k) = Y_1(k)^{-1} Y_1(k-1)^{-1} \cdots Y_1(k-\alpha+1)^{-1} Y_2(k)^{-\gamma},$$
$$A_2(k) = Y_2(k)^{-1} Y_2(k-1)^{-1} \cdots Y_2(k-\delta+1)^{-1} Y_1(k+1)^{-\beta}.$$

By a direct calculation, we have

$$C(\mathbf{1}) = \left\{ \mathbf{1} \cdot \prod_{k=0}^{r} A_1(k)^{-a_1(k)} A_2(k)^{-a_2(k)} \;\middle|\; \begin{array}{l} \text{(i) } r \geqslant 0,\ a_1(k) \geqslant 0, a_2(k) \geqslant 0, \\ \text{(ii) if } a_1(k) \neq 0, \text{ then } a_2(k-1) \neq 0 \end{array} \right\}.$$

Example 3.3. Let

$$A = \begin{pmatrix} 2 & -\alpha \\ -\beta & -\gamma \end{pmatrix}$$

be a Borcherds–Cartan matrix with $\alpha, \beta, \gamma \in \mathbf{Z}_{\geqslant 0}$. Then we have

$$A_1(k) = Y_1(k) Y_1(k+1) Y_2(k)^{-\beta},$$
$$A_2(k) = Y_2(k)^{-1} Y_2(k-1)^{-1} \cdots Y_2(k-\gamma+1)^{-1} Y_1(k+1)^{-\alpha}.$$

We claim that the connected component $C(\mathbf{1})$ of $\mathcal{M}$ containing $\mathbf{1}$ is the set $\mathcal{M}(\infty)$ of monomials of the form

$$\mathbf{1} \cdot \prod_{k=0}^{r} A_1(k)^{-a_1(k)} A_2(k)^{-a_2(k)} \quad \big(r \geqslant 0, a_1(k) \geqslant 0, a_2(k) \geqslant 0\big)$$

satisfying the following conditions:

(i) for each $k \geqslant 0$, $\alpha a_2(k) - a_1(k+1) \geqslant 0$,
(ii) for each $k \geqslant 1$, if $a_2(k) > 0$, then $a_1(k) > 0$ and $\alpha a_2(k) - a_1(k+1) > 0$.

We first show that $\mathcal{M}(\infty)$ is closed under the Kashiwara operators $\tilde{f}_i$. Let M be a monomial in $\mathcal{M}(\infty)$. Suppose that $\tilde{f}_i M$ does not satisfy the condition (i). Then $i = 1$ and $\alpha a_2(k) = a_1(k+1)$ for some $k \geqslant 0$ in M and $\tilde{f}_1 M$ is obtained from M by multiplying $A_1(k+1)^{-1}$. In particular, $n_f = k+1$ and $y_1(k+1) > 0$. However, the multiplicity $y_1(k+1)$ of $Y_1(k+1)$ in M is

$$y_1(k+1) = -a_1(k) - a_1(k+1) + \alpha a_2(k) \leqslant 0,$$

which is a contradiction.

Suppose that $\tilde{f}_i M$ does not satisfy the condition (ii). Then we have the following two possibilities:

(a) $i = 1$, $a_2(k) > 0$, $a_1(k) > 0$ and $\alpha a_2(k) - a_1(k+1) = 1$ in M, and $\tilde{f}_1 M$ is obtained by multiplying $A_1(k+1)^{-1}$.
(b) $i = 2$, $a_1(k) = 0$, $a_2(k) = 0$ in M, and $\tilde{f}_2 M$ is obtained by multiplying $A_2(k)^{-1}$.

For the case (a), by the same argument as above, we get a contradiction. For the case (b), we have $n_f = k$ and $y_2(k) > 0$. On the other hand, the multiplicity $y_2(k)$ of $Y_2(k)$ in M is

$$y_2(k) = \beta a_1(k) + \sum_{t=k}^{k+\gamma-1} a_2(t).$$

In this case, since $a_2(k) = 0$, by (i) and (ii), $a_1(t) = a_2(t) = 0$ for all $t \geqslant k$, and hence $y_2(k) = 0$, which is a contradiction. Therefore, $\mathcal{M}(\infty)$ is closed under the Kashiwara operator $\tilde{f}_i$.

Similarly, we can prove that $\mathcal{M}(\infty)$ is closed under the Kashiwara operator $\tilde{e}_i$.

It remains to show that $\mathcal{M}(\infty)$ is connected. Suppose that $M \neq \mathbf{1}$ and $\tilde{e}_i M = 0$ for all $i \in I$. Let j_1 (respectively j_2) be the greatest integer j such that $a_1(j) > 0$ (respectively $a_2(j) > 0$) in M. If $j_1 > j_2$, then $\varepsilon_1(M) > 0$ and $\tilde{e}_1 M \neq 0$, which implies $j_1 \leqslant j_2$. In this case, $S_2(j_2)$ is a factor of M and $\tilde{e}_2 M = M \cdot A_2(j_2) \neq 0$, which is also a contradiction.

Remark 3.4. It can be shown that $\Phi(B) = \mathcal{M}(\infty)$, where Φ is the map in the proof of Theorem 3.1 and B is the crystal in [3, Example 4.3].

4. Monomial realization of $B(\lambda)$

In this section, we introduce another set of Nakajima monomials and give a realization of the crystal $B(\lambda)$ in terms of these monomials. Let $\mathfrak{M}$ be the set of monomials of the form $\mathfrak{m} = \prod_{i \in I, n \in \mathbf{Z}} Y_i(n)^{y_i(n)}$, where $Y_i(n)$ ($i \in I, n \in \mathbf{Z}$) are commuting variables, $y_i(n) \in \mathbf{Z}$ for $i \in I^{\mathrm{re}}$, $y_i(n) \in \mathbf{Z}_{\geqslant 0}$ for $i \in I^{\mathrm{im}}$ and for each $i \in I$, $y_i(n) = 0$ for all but finitely many n. The monomials in $\mathfrak{M}$ are called the *Nakajima monomials of integrable type*.

For a monomial $\mathfrak{m} \in \mathfrak{M}$, we define

$$\begin{aligned}
\mathrm{wt}(\mathfrak{m}) &= \sum_i \Big(\sum_n y_i(n)\Big) \Lambda_i, \\
\varphi_i(\mathfrak{m}) &= \max\Big\{\sum_{k \leqslant n} y_i(k) \;\Big|\; n \in \mathbf{Z}\Big\}, \\
\varepsilon_i(\mathfrak{m}) &= \max\Big\{-\sum_{k > n} y_i(k) \;\Big|\; n \in \mathbf{Z}\Big\}.
\end{aligned} \tag{4.1}$$

To define the Kashiwara operators, we take c_{ij} and $A_i(n)$ to be the same ones as in Section 3, and define

$$\begin{aligned}
n_f = n_f(\mathfrak{m}) &= \min\Big\{n \in \mathbf{Z} \;\Big|\; \varphi_i(\mathfrak{m}) = \sum_{k \leqslant n} y_i(k)\Big\}, \\
n_e = n_e(\mathfrak{m}) &= \begin{cases} \max\{n \in \mathbf{Z} \mid \varphi_i(\mathfrak{m}) = \sum_{k \leqslant n} y_i(k)\} & \text{if } i \in I^{\mathrm{re}}, \\ n_f & \text{if } i \in I^{\mathrm{im}}, \end{cases}
\end{aligned}$$

$$T_i(n_f)=\begin{cases} Y_i(n_f)^2 Y_i(n_f-1)\cdots Y_i(n_f+a_{ii}+1)\prod_{\substack{j\neq i\\ j\in I^{\mathrm{im}}}} Y_j(n_f+c_{ji})^{-a_{ji}} & \text{if } a_{ii}<0,\\ \prod_{\substack{j\neq i\\ j\in I^{\mathrm{im}}}} Y_j(n_f+c_{ji})^{-a_{ji}} & \text{if } a_{ii}=0. \end{cases}$$

We now define the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i\in I)$ by

$$\tilde{f}_i\mathfrak{m}=\begin{cases} \mathfrak{m}\cdot A_i(n_f)^{-1} & \text{if } \varphi_i(\mathfrak{m})>0,\\ 0 & \text{otherwise,}\end{cases}$$
$$\tilde{e}_i\mathfrak{m}=\begin{cases} \mathfrak{m}\cdot A_i(n_e) & \text{if } i\in I^{\mathrm{re}} \text{ and } \varepsilon_i(\mathfrak{m})>0,\\ & \text{or } i\in I^{\mathrm{im}} \text{ and } T_i(n_f)^{-1}\mathfrak{m}\in\mathfrak{M},\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

Then it is straightforward to verify that $\mathfrak{M}$ becomes a $U_q(\mathfrak{g})$-crystal with the maps wt, ε_i, φ_i, $\tilde{e}_i$, $\tilde{f}_i$ $(i\in I)$ defined in (4.1) and (4.2). Moreover, we have a realization of the crystal $B(\lambda)$ in terms of Nakajima monomials of integrable type.

Theorem 4.1. *Fix $p\in\mathbf{Z}_{\geqslant 0}$ and choose a sequence $(\lambda_i(p)\mid i\in I)$ of nonnegative integers. If $\mathfrak{m}$ is a maximal vector in $\mathfrak{M}$ of the form*

$$\mathfrak{m}=\prod_{i\in I} Y_i(p)^{\lambda_i(p)} \quad (p\in\mathbf{Z}),$$

then the connected component $C(\mathfrak{m})$ of $\mathfrak{M}$ containing $\mathfrak{m}$ is isomorphic to the $U_q(\mathfrak{g})$-crystal $B(\lambda)$ with $\lambda=\mathrm{wt}(\mathfrak{m})=\sum_{i\in I}\lambda_i(p)\Lambda_i$.

Proof. Let $\mathfrak{m}=\prod_{i\in I} Y_i(p)^{\lambda_i(p)}$ be a monomial in $\mathfrak{M}$ such that $\tilde{e}_i\mathfrak{m}=0$ for all $i\in I$. Thanks to Theorem 2.2, it suffices to prove that for any subset J of I with $|J|\leqslant 2$, the connected component of $\Psi_J(\mathfrak{M})$ containing $\mathfrak{m}$ is isomorphic to the $U_q(\mathfrak{g}_J)$-crystal $B_J(\lambda)$. We assume $J=\{1,2\}$ and take $c_{12}=1$, $c_{21}=0$. Since the proof for the Kac–Moody algebras case was already known, we may assume that at least one of the indices is imaginary, say, $a_{11}\leqslant 0$. Set

$$\begin{aligned}\overline{K}_p=\Big\{b:=&\bigotimes_{n\geqslant p}\big(b_2\big(z_2(n)\big)\otimes b_1\big(z_1(n)\big)\big)\otimes r_\lambda\\ &=\cdots\otimes b_2\big(z_2(p+1)\big)\otimes b_1\big(z_1(p+1)\big)\otimes b_2\big(z_2(p)\big)\otimes b_1\big(z_1(p)\big)\otimes r_\lambda\\ &\in\cdots\otimes B_2\otimes B_1\otimes B_2\otimes B_1\otimes R_\lambda \;\Big|\; z_1(n)=z_2(n)=0 \text{ for } n\gg p\Big\}.\end{aligned} \tag{4.3}$$

We define a map $\Phi:\overline{K}_p\to\mathfrak{M}$ by

$$\begin{aligned} b\;&=\bigotimes_{n\geqslant p}\big(b_2\big(z_2(n)\big)\otimes b_1\big(z_1(n)\big)\big)\otimes r_\lambda\\ &\mapsto \mathfrak{m}:=\prod_{i\in I} Y_i(p)^{\lambda_i(p)}\cdot\prod_{n\geqslant p} A_1(n)^{z_1(n)}A_2(n)^{z_2(n)}.\end{aligned}$$

It is easy to see that $\Phi(b)$ belongs to $\mathfrak{M}$, $\mathrm{wt}(b)=\mathrm{wt}(\mathfrak{m})$, $\varphi_1(b)=\varphi_1(\mathfrak{m})$, and $\varepsilon_1(b)=\varepsilon_1(\mathfrak{m})$.

Now, suppose that $\tilde{f}_1$ acts on the kth component of b; i.e.,

$$\tilde{f}_1 b = \cdots \otimes b_1\big(z_1(k)-1\big) \otimes b_2\big(z_2(k-1)\big) \otimes \cdots \otimes b_2\big(z_2(p)\big) \otimes b_1\big(z_1(p)\big) \otimes r_\lambda.$$

Then $\Phi(\tilde{f}_1 b) = \mathfrak{m} \cdot A_1(k)^{-1}$ and by the tensor product rule, we have

$$\big\langle h_1, \mathrm{wt}(b)\big\rangle > 0, \tag{4.4}$$

$$\left\langle h_1, z_2(k)\alpha_2 + \sum_{n>k}\big(z_1(n)\alpha_1 + z_2(n)\alpha_2\big)\right\rangle = 0, \tag{4.5}$$

and

$$\left\langle h_1, z_2(k-1)\alpha_2 + \sum_{n\geqslant k}\big(z_1(n)\alpha_1 + z_2(n)\alpha_2\big)\right\rangle > 0. \tag{4.6}$$

By (4.5) and (4.6), we have $y_1(n) = 0$ for all $n > k$ and $y_1(k) > 0$. Therefore,

$$\tilde{f}_1 \Phi(b) = \tilde{f}_1 \mathfrak{m} = \mathfrak{m} \cdot A_1(k)^{-1} = \Phi(\tilde{f}_1 b).$$

Suppose that $\tilde{e}_1$ acts on the kth component of b; i.e.,

$$\tilde{e}_1 b = \cdots \otimes b_1\big(z_1(k)+1\big) \otimes b_2\big(z_2(k-1)\big) \otimes \cdots \otimes b_2\big(z_2(p)\big) \otimes b_1\big(z_1(p)\big) \otimes t_\lambda.$$

Then $\Phi(\tilde{e}_1 b) = \mathfrak{m} \cdot A_1(k)$. On the other hand, we have (4.4)–(4.6) and

$$z_1(k) < 0, \quad \text{and if} \quad z_1(k) = -1 \quad \text{and } k > p, \qquad z_2(k-1)\langle h_1, \alpha_2\rangle > 0. \tag{4.7}$$

From (4.4) and (4.7), we know that $T_1(n_f) = T_1(k)$ is a factor of $\Phi(b) = \mathfrak{m}$. It follows that $\tilde{e}_1 \Phi(b) = \tilde{e}_1 \mathfrak{m} = \mathfrak{m} \cdot A_1(k) = \Phi(\tilde{e}_1 b)$.

Now, if $a_{22} = 2$, we have

$$\begin{aligned}
\varepsilon_2(\mathfrak{m}) &= \max\left\{ -\sum_{k>n} y_2(k) \;\middle|\; n \in \mathbf{Z} \right\} \\
&= \max\left\{ -\sum_{k>p-1} y_2(k), \max\left\{ -\sum_{k>n} y_2(k) \;\middle|\; n \geqslant p \right\} \right\} \\
&= \max\Bigg\{ -\lambda_2(p) - \sum_{k\geqslant p}\big(2z_2(k) + \langle h_2, \alpha_1\rangle z_1(k)\big), \\
&\qquad - z_2(n) - \sum_{k>n}\big(2z_2(k) + \langle h_2, \alpha_1\rangle z_1(k)\big) \;\Big|\; n \geqslant p \Bigg\} \\
&= \varepsilon_2(b),
\end{aligned}$$

and hence $\varphi_2(b) = \varphi_2(\mathfrak{m})$. Moreover, by the tensor product rule of Kashiwara operators and the definition of Kashiwara operator in $\mathfrak{M}$, it is easy to see that $\tilde{f}_2$ and $\tilde{e}_2$ commute with the map Φ.

If $a_{22} \leqslant 0$, by the same argument as above, $\varphi_2(b) = \varphi_2(\mathfrak{m})$, $\varepsilon_2(b) = \varepsilon_2(\mathfrak{m})$, and $\tilde{f}_2$ and $\tilde{e}_2$ commute with the map Φ.

Therefore, we conclude Φ defines a $U_q(\mathfrak{g}_J)$-crystal morphism $\overline{K}_p \to \Psi_J(\mathfrak{M})$. Note that the connected component of $\overline{K}_p$ containing $\cdots \otimes b_2(0) \otimes b_1(0) \otimes r_\lambda$ is isomorphic to the crystal $B_J(\lambda)$ [3,13]. Therefore, the connected component of $\Psi_J(\mathfrak{M})$ containing $\mathfrak{m} = \prod_{i \in I} Y_i(p)^{\lambda_i(p)}$ is isomorphic to the $U_q(\mathfrak{g}_J)$-crystal $B_J(\lambda)$. $\square$

Example 4.2. Let

$$A = \begin{pmatrix} 2 & -\alpha \\ -\beta & -\gamma \end{pmatrix}$$

be a Borcherds–Cartan matrix with $\alpha, \beta, \gamma \in \mathbf{Z}_{\geqslant 0}$, and let $\mathfrak{m} = Y_1(p)^{\lambda_1(p)} Y_2(p)^{\lambda_2(p)}$ be a maximal vector so that $\lambda = \mathrm{wt}(\mathfrak{m}) = \lambda_1(p)\Lambda_1 + \lambda_2(p)\Lambda_2$. Then $C(\mathfrak{m})$ is the set $\mathfrak{M}(\lambda)$ consisting of monomials of the form

$$\mathfrak{m} \cdot \prod_{k=p}^{r} A_1(k)^{-a_1(k)} A_2(k)^{-a_2(k)} \quad \bigl(r \geqslant p, a_1(k), a_2(k) \geqslant 0\bigr)$$

satisfying the following conditions:

(i) for each $k \geqslant p$, $\alpha a_2(k) - a_1(k+1) \geqslant 0$,
(ii) for each $k \geqslant p+1$, if $a_2(k) > 0$, then $a_1(k) > 0$ and $\alpha a_2(k) - a_1(k+1) > 0$,
(iii) $0 \leqslant a_1(p) \leqslant \lambda_1(p)$,
(iv) if $a_2(p) \neq 0$, then $\beta a_1(p) + \lambda_2(p) > 0$.

Since the rest of our proof is similar to that of Example 3.3, we only prove that the conditions (iii) and (iv) are preserved by the Kashiwara operator $\tilde{f}_i$.

Suppose that $\tilde{f}_1\mathfrak{m}$ does not satisfy the condition (iii). Then $a_1(p) = \lambda_1(p)$ in $\mathfrak{m}$ and $\tilde{f}_1\mathfrak{m}$ is obtained by multiplying $A_1(p)^{-1}$. But, in this case, $n_f > p$, which cannot occur by the definition of Kashiwara operators.

Suppose that $\tilde{f}_i\mathfrak{m}$ does not satisfy the condition (iv). Then $i = 2$, $a_2(p) = \beta a_1(p) + \lambda_2(p) = 0$ in $\mathfrak{m}$, and $\tilde{f}_2\mathfrak{m}$ is obtained by multiplying $A_2(p)^{-1}$. However, this cannot occur because the multiplicity $y_2(p)$ is 0 in $\mathfrak{m}$.

Remark 4.3. It can be shown that $\Phi(B^\lambda) = \mathfrak{M}(\lambda)$, where Φ is the map in the proof of Theorem 4.1 and B^λ is the crystal in [3, Example 5.3].

5. Quantum monster algebra

Let $I = \{(i,t) \mid i \in \mathbf{Z}_{\geqslant -1},\ 1 \leqslant t \leqslant c(i)\}$, where $c(i)$ is the ith coefficient of the elliptic modular function

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots = \sum_{i=-1}^{\infty} c(i)q^i.$$

Consider the Borcherds–Cartan matrix $A = (\alpha_{(i,t),(j,s)})_{(i,t),(j,s)\in I}$ whose entries are given by $\alpha_{(i,t),(j,s)} = -(i+j)$. The associated generalized Kac–Moody algebra $\mathfrak{g}$ is called the *Monster Lie algebra*, and it played a crucial role in Borcherds' proof of the Moonshine conjecture [1]. The corresponding quantum generalized Kac–Moody algebra is called the *quantum Monster algebra*.

For $(p_1, q_1), (p_2, q_2) \in I$, we define $(p_1, q_1) > (p_2, q_2)$ if and only if $p_1 > p_2$, or $p_1 = p_2$ and $q_1 > q_2$. Also, for $(p_1, q_1, r_1), (p_2, q_2, r_2) \in I \times \mathbf{Z}_{\geqslant 0}$, with $(p_1, q_1) \in I$, $r_1 \in \mathbf{Z}_{\geqslant 0}$, we define $(p_1, q_1, r_1) > (p_2, q_2, r_2)$ if and only if

$$r_1 > r_2, \quad \text{or} \quad r_1 = r_2 \quad \text{and} \quad (p_1, q_1) > (p_2, q_2).$$

In the following proposition, we give an explicit description of the Nakajima monomials in $B(\infty)$ for the quantum Monster algebra.

Proposition 5.1. *The connected component $C(\mathbf{1})$ of $\mathcal{M}$ containing $\mathbf{1}$ is the set $\mathcal{M}(\infty)$ consisting of monomials of the form*

$$\mathbf{1} \cdot \prod_{(i,t)\in I} \prod_{k=0}^{r} A_{(i,t)}(k)^{-a_{(i,t)}(k)} \quad \bigl(r \geqslant 0, a_{(i,t)}(k) \geqslant 0\bigr)$$

satisfying the following conditions:

(i) *for each $k \geqslant 0$,*

$$\sum_{i\geqslant 2} \sum_{t=1}^{c(i)} (i-1) a_{(i,t)}(k) - a_{(-1,1)}(k+1) \geqslant 0,$$

(ii) *if $a_{(i,t)}(k) > 0$ $(t = 1, \ldots, c(i),\ k \geqslant 1)$ with $i \neq -1$, then there is a (p, q, r) such that*

$$(i, t, k-1) < (p, q, r) < (i, t, k) \quad \textit{and} \quad (p+i) a_{(p,q)}(r) > 0. \tag{5.1}$$

In addition, if there exists a unique $(p, q, r) = (-1, 1, k)$ satisfying (5.1)*, then*

$$\sum_{i\geqslant 2} \sum_{t=1}^{c(i)} (i-1) a_{(i,t)}(k) - a_{(-1,1)}(k+1) > 0.$$

Proof. We first show that $\mathcal{M}(\infty)$ is closed under the Kashiwara operators $\tilde{f}_{(i,t)}$. Let M be a monomial in $\mathcal{M}(\infty)$. Suppose that $\tilde{f}_{(i,t)} M$ does not satisfy the condition (i). Then

$$(i,t) = (-1,1), \qquad \sum_{i\geqslant 2} \sum_{t=1}^{c(i)} (i-1) a_{(i,t)}(k) - a_{(-1,1)}(k+1) = 0 \quad \text{for some } k \geqslant 0 \text{ in } M,$$

and $\tilde{f}_{(-1,1)}M$ is obtained by multiplying $A_{(-1,1)}(k+1)^{-1}$. In particular, $n_f = k+1$ and $y_{(-1,1)}(k+1) > 0$. However, the multiplicity $y_{(-1,1)}(k+1)$ in M is

$$y_{(-1,1)}(k+1) = -a_{(-1,1)}(k) - a_{(-1,1)}(k+1) + \sum_{i\geqslant 2}\sum_{t=1}^{c(i)}(i-1)a_{(i,t)}(k) \leqslant 0,$$

which is a contradiction.

Suppose that $\tilde{f}_{(i,t)}M$ does not satisfy the condition (ii). Then we have the following two possibilities:

(a) $a_{(i,t)}(k) = 0$, $(p+i)a_{(p,q)}(r) = 0$ for all $(i,t,k-1) < (p,q,r) < (i,t,k)$ in M, and $\tilde{f}_{(i,t)}M$ is obtained from M by multiplying $A_{(i,t)}(k)^{-1}$.

(b) $a_{(i,t)}(k) > 0$, $a_{(-1,1)}(k) > 0$, $a_{(p,q)}(r) = 0$ for all $(i,t,k-1) < (p,q,r) < (i,t,k)$ with $(p,q,r) \neq (-1,1,k)$,

$$\sum_{i\geqslant 2}\sum_{t=1}^{c(i)}(i-1)a_{(i,t)}(k) - a_{(-1,1)}(k+1) = 1 \quad \text{in } M,$$

and $\tilde{f}_{(-1,1)}M$ is obtained from M by multiplying $A_{(-1,1)}(k+1)^{-1}$.

For the case (a), we have $n_f = k$ and the multiplicity $y_{(i,t)}(k)$ of $Y_{(i,t)}(k)$ in M is

$$\begin{aligned} y_{(i,t)}(k) &= \sum_{s=k}^{k+2i-1} a_{(i,t)}(s) + \sum_{(l,m)<(i,t)} (l+i)a_{(l,m)}(k) + \sum_{(l,m)>(i,t)} (l+i)a_{(l,m)}(k-1) \\ &= \sum_{s>k}^{k+2i-1} a_{(i,t)}(s), \end{aligned}$$

which should be positive. However, if $y_{(i,t)}(k) > 0$, then $a_{(i,t)}(s) > 0$ for some $k < s \leqslant k+2i-1$, which implies $y_{(i,t)}(s) > 0$. This is a contradiction to the fact that $n_f = k$.

For the case (b), we have $n_f = k+1$ and $y_{(-1,1)}(k+1) > 0$. However, the multiplicity $y_{(-1,1)}(k+1)$ is

$$\begin{aligned} y_{(-1,1)}(k+1) &= -a_{(-1,1)}(k) - a_{(-1,1)}(k+1) + \sum_{i\geqslant 2}\sum_{t=1}^{c(i)}(i-1)a_{(i,t)}(k) \\ &= -a_{(-1,1)}(k) + 1 \leqslant 0, \end{aligned}$$

which is a contradiction. Therefore, $\mathcal{M}(\infty)$ is closed under the Kashiwara operator $\tilde{f}_{(i,t)}$.

Similarly, we can prove that $\mathcal{M}(\infty)$ is closed under the Kashiwara operator $\tilde{e}_{(i,t)}$.

It remains to show that $\mathcal{M}(\infty)$ is connected. Suppose that $M \neq \mathbf{1}$ and $\tilde{e}_{(i,t)}M = 0$ for all $(i,t) \in I$. Let $(i_0, t_0, k_0) \in I \times \mathbf{Z}_{\geqslant 0}$ be such that

$$a_{(i_0,t_0)}(k_0) > 0 \quad \text{and} \quad a_{(i,t)}(k) = 0 \quad \text{for all } (i,t,k) > (i_0,t_0,k_0).$$

Then we have

$$\varepsilon_{(i,t)}(M) > 0 \quad \text{when } (i,t) = (-1,1) \in I^{\mathrm{re}},$$
$$S_{(i,t)}(k_0) \text{ is a factor of } M \quad \text{when } (i,t) \neq (-1,1).$$

In either case, $\tilde{e}_{(i,t)} M \neq 0$, which is a contradiction. □

Now, we give an explicit description of the Nakajima monomials in $B(\lambda)$ for the quantum Monster algebra.

Proposition 5.2. *Let $(\lambda_{(i,t)}(0) \mid (i,t) \in I)$ be a sequence of nonnegative integers. If $\mathfrak{m}$ is a maximal vector in $\mathfrak{M}$ of the form*

$$\mathfrak{m} = \prod_{(i,t)\in I} Y_{(i,t)}(0)^{\lambda_{(i,t)}(0)},$$

then the connected component $C(\mathfrak{m})$ of $\mathfrak{M}$ containing $\mathfrak{m}$ is the set $\mathfrak{M}(\lambda)$ consisting of monomials of the form

$$\mathfrak{m} \cdot \prod_{(i,t)\in I} \prod_{k=0}^{r} A_{(i,t)}(k)^{-a_{(i,t)}(k)} \quad \bigl(r \geqslant 0, a_{(i,t)}(k) \geqslant 0\bigr)$$

satisfying the conditions (i)–(ii) *in Proposition* 5.1 *and two additional conditions*:

(iii) $0 \leqslant a_{(-1,1)}(0) \leqslant \lambda_{(-1,1)}(0)$,
(iv) *if $a_{(i,t)}(0) > 0$ and $\lambda_{(i,t)}(0) = 0$ with $(i,t,0) \neq (-1,1,0)$, then there is a $(j,s,0)$ such that*

$$(j,s,0) < (i,t,0) \quad \textit{and} \quad (i+j)a_{(j,s)}(0) > 0.$$

Proof. Since the proof is similar to that of Proposition 5.1, we only prove that the conditions (iii) and (iv) are preserved by the Kashiwara operator $\tilde{f}_{(i,t)}$. Suppose that $\tilde{f}_{(-1,1)}\mathfrak{m}$ does not satisfy the condition (iii). Then $a_{(-1,1)}(0) = \lambda_{(-1,1)}(0)$ in $\mathfrak{m}$ and $\tilde{f}_{(-1,1)}\mathfrak{m}$ is obtained by multiplying $A_{(-1,1)}(0)^{-1}$. However, in this case, $n_f > 0$, which cannot occur by the definition of Kashiwara operators.

Suppose that $\tilde{f}_{(i,t)}\mathfrak{m}$ does not satisfy the condition (iv). Then $a_{(i,t)}(0) = \lambda_{(i,t)}(0) = 0$, $(i+j)a_{(j,s)}(0) = 0$ for all $(j,s,0) < (i,t,0)$ in $\mathfrak{m}$, and $\tilde{f}_{(i,t)}\mathfrak{m}$ is obtained by multiplying $A_{(i,t)}(0)^{-1}$. However, this cannot occur, since the multiplicity $y_{(i,t)}(0)$ is 0 in $\mathfrak{m}$. □

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