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Polyhedral realization of crystal bases for generalized Kac–Moody algebras

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Abstract

In this paper, we give a polyhedral realization of the crystal $B(\infty)$ of $U_q^-(\mathfrak{g})$ for the generalized Kac–Moody algebras. As applications, we give explicit descriptions of crystals for the generalized Kac–Moody algebras of rank 2 and 3, and Monster Lie algebras.

Introduction

In his study of Conway and Norton's *Moonshine conjecture* [3] for the infinite-dimensional $\mathbf{Z}$-graded representation $V^\natural$ of the Monster sporadic simple group, Borcherds introduced a new class of infinite-dimensional Lie algebras called the *generalized Kac–Moody algebras* [1, 2]. The structure and representation theories of generalized Kac–Moody algebras are very similar to those of Kac–Moody algebras, and a lot of facts about Kac–Moody algebras can be extended to generalized Kac–Moody algebras. The main difference is that the generalized Kac–Moody algebras may have simple roots with non-positive norms, the multiplicity of which can be greater than one, called imaginary simple roots, and they may have infinitely many simple roots.

The *quantum groups* $U_q(\mathfrak{g})$ introduced independently by Drinfel'd and Jimbo are q-deformations of the universal enveloping algebras $U(\mathfrak{g})$ of Kac–Moody algebras $\mathfrak{g}$; see [4, 7]. The important feature of quantum groups is that the representation theory of $U_q(\mathfrak{g})$ is the same as that of $U(\mathfrak{g})$. Therefore, to understand the structure of representations over $U_q(\mathfrak{g})$, it is enough to understand that of representations over $U_q(\mathfrak{g})$ for some special parameter q which is easy to treat. The *crystal basis theory*, which can be viewed as the representation theory at $q = 0$, was introduced by Kashiwara [13]. Among others, he showed that there exist a crystal basis $B(\infty)$ for the negative part of a quantum group and a crystal basis $B(\lambda)$ for the irreducible highest weight module $V(\lambda)$ with a dominant integral highest weight λ. Crystal bases are given a structure of coloured oriented graphs, called the *crystal graphs*, which reflect the combinatorial structure of integrable modules. Therefore, one of the most fundamental problems in the crystal basis theory is to construct the crystal basis explicitly. In many articles, one can find several kinds of realizations of crystal bases using combinatorial objects (for example, [9–12, 15–18]).

In [8], Kang introduced the *quantum generalized Kac–Moody algebras* $U_q(\mathfrak{g})$ — the quantum groups associated with generalized Kac–Moody algebras $\mathfrak{g}$ — and he also showed that, for a generic q, the Verma modules and the unitarizable highest weight modules over $\mathfrak{g}$ can be deformed to those over $U_q(\mathfrak{g})$. In [5], Jeong, Kang and Kashiwara developed the crystal basis theory for quantum generalized Kac–Moody algebras. As in the Kac–Moody algebra case, they showed that there exist a crystal basis $B(\infty)$ for the negative part of a quantum generalized Kac–Moody algebra and a crystal basis $B(\lambda)$ for the irreducible highest weight

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module $V(\lambda)$ with a dominant integral highest weight λ. However, unfortunately, there is no explicit realization of crystal bases over quantum generalized Kac–Moody algebras using some combinatorial objects.

Recently, in [6], Jeong, Kang, Kashiwara and the author introduced the notion of *abstract crystals* for quantum generalized Kac–Moody algebras, and the embedding of crystals $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}^{\infty}_{\geqslant 0,\iota}$, where ι is an infinite sequence from the index set of simple roots. This embedding Ψ_ι of crystals is an analogue of the crystal embedding (in the Kac–Moody case) introduced by Kashiwara [14]. However, as in the Kac–Moody case, in general it is not easy to find the image $\operatorname{Im}\Psi_\iota$. In this paper, we give an explicit description of $\operatorname{Im}\Psi_\iota$ using a unified method introduced by Nakashima and Zelevinsky [18], called the *polyhedral realization*. The main obstacle in applying this method to the quantum generalized Kac–Moody algebras was a quite different tensor product rule of Kashiwara operators of crystal bases.

As applications, we give explicit descriptions of the crystals $B(\infty)$ over generalized Kac–Moody algebras of ranks 2 and 3. Finally, for the *Monster Lie algebra*, which played an important role in proving the Moonshine conjecture, we give the explicit description of $\operatorname{Im}\Psi_\iota$. Since the root multiplicity of the Monster Lie algebra is closely related to the ith coefficient $c(i)$ of the elliptic modular function $j(q) - 744$, we expect to obtain some properties about the coefficients $c(i)$.

1. *Crystal bases for quantum generalized Kac–Moody algebras*

1.1. *Quantum generalized Kac–Moody algebras*

Let I be a countable index set. A real matrix $A = (a_{ij})_{i,j\in I}$ is called a *Borcherds–Cartan matrix* if it satisfies:

(i) $a_{ii} = 2$ or $a_{ii} \leqslant 0$ for all $i \in I$,
(ii) $a_{ij} \leqslant 0$ if $i \neq j$,
(iii) $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$,
(iv) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let $I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}$ and $I^{\mathrm{im}} = \{i \in I \mid a_{ii} \leqslant 0\}$. Moreover, we say that an index i in I^{re} or I^{im} is *real* or *imaginary*, respectively.

In this paper, we assume that for all $i, j \in I$, $a_{ij} \in \mathbf{Z}$, $a_{ii} \in 2\mathbf{Z}$ and A is *symmetrizable*. That is, there is a diagonal matrix $D = \operatorname{diag}(s_i \in \mathbf{Z}_{>0} \mid i \in I)$ such that DA is symmetric. We set a Borcherds–Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ as follows:

$$\begin{gathered} A : \text{a Borcherds–Cartan matrix,} \\ P^\vee = \left(\bigoplus_{i\in I} \mathbf{Z}h_i\right) \oplus \left(\bigoplus_{i\in I} \mathbf{Z}d_i\right) : \text{a free abelian group,} \\ P = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(P^\vee) \subset \mathbf{Z}\} : \textit{the weight lattice}, \\ \Pi^\vee = \{h_i \,|\, i \in I\} \subset \mathfrak{h} : \text{the set of } \textit{simple coroots}, \\ \Pi = \{\alpha_i \,|\, i \in I\} \subset \mathfrak{h}^* : \text{the set of } \textit{simple roots}. \end{gathered}$$

Here, $\mathfrak{h} = \mathbf{Q} \bigotimes_{\mathbf{Z}} P^\vee$, and the simple roots α_i $(i \in I)$ are defined by

$$\langle h_j, \alpha_i\rangle = a_{ji} \quad \text{and} \quad \langle d_j, \alpha_i\rangle = \delta_{ji}.$$

We denote by $P^+ = \{\lambda \in P \mid \langle h_i, \lambda\rangle \geqslant 0 \text{ for all } i \in I\}$ the set of *dominant integral weights*. We also use the notation $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ and $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geqslant 0}\alpha_i$.

For an indeterminate q, set $q_i = q^{s_i}$ and define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i!\,[m-n]_i!}.$$

The *quantum generalized Kac–Moody algebra* $U_q(\mathfrak{g})$ associated with a Borcherds–Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the elements e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ with the following defining relations:

$$q^0 = 1, \qquad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee,$$

$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \qquad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P^\vee,\ i \in I,$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{s_i h_i} - q^{-s_i h_i}}{q_i - q_i^{-1}} \quad \text{for } i, j \in I,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{for } a_{ii} = 2,\ i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad \text{for } a_{ii} = 2,\ i \neq j,$$

$$e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad \text{if } a_{ij} = 0.$$

Let us denote by $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i and f_i, respectively.

1.2. *Crystal bases*

The *category* $\mathcal{O}_{\mathrm{int}}$ consists of $U_q(\mathfrak{g})$-modules M satisfying the following properties:

(i) $M = \bigoplus_{\mu \in P} M_\mu$, where $M_\mu = \{v \in M \mid q^h v = q^{\langle h, \mu \rangle} v \text{ for all } h \in P^\vee\}$ is finite-dimensional;

(ii) there exist finitely many elements $\lambda_1, \ldots, \lambda_s \in P$ such that $\mathrm{wt}(M) \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+)$, where $\mathrm{wt}(M) = \{\mu \in P \mid M_\mu \neq 0\}$;

(iii) if $a_{ii} = 2$, then the action of f_i on M is *locally nilpotent*; that is, for any $m \in M$ there exists a positive integer N such that $\tilde{f}_i^N m = 0$;

(iv) if $a_{ii} \leqslant 0$, then $\langle h_i, \mu \rangle \in \mathbf{Z}_{\geqslant 0}$ for every $\mu \in \mathrm{wt}(M)$;

(v) if $a_{ii} \leqslant 0$ and $\langle h_i, \mu \rangle = 0$, then $\tilde{f}_i M_\mu = 0$;

(vi) if $a_{ii} \leqslant 0$ and $\langle h_i, \mu \rangle \leqslant -a_{ii}$, then $\tilde{e}_i M_\mu = 0$.

For instance, the *irreducible highest weight module* $V(\lambda) = U_q(\mathfrak{g}) u_\lambda$ with $\lambda \in P^+$, defined by the relations:

(i) u_λ has weight λ,

(ii) $e_i u_\lambda = 0$ for all $i \in I$,

(iii) $f_i^{\langle h_i, \lambda \rangle + 1} u_\lambda = 0$ for any $i \in I^{\mathrm{re}}$,

(iv) $f_i u_\lambda = 0$ if $\langle h_i, \lambda \rangle = 0$,

belongs to $\mathcal{O}_{\mathrm{int}}$. Moreover, the category $\mathcal{O}_{\mathrm{int}}$ is *semisimple* and every simple object in $\mathcal{O}_{\mathrm{int}}$ is isomorphic to the irreducible highest weight module $V(\lambda)$ with $\lambda \in P^+$; see [5].

Fix an index $i \in I$, and for $k \geqslant 0$, set $f_i^{(k)} = f_i^k / [k]_i!$ if i is real, and $f_i^{(k)} = f_i^k$ if i is imaginary. Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\mathrm{int}}$. It was shown in [5] that every weight vector $v \in M_\lambda$ can be written uniquely as

$$v = \sum_{k \geqslant 0} f_i^{(k)} v_k,$$

where: (i) $v_k \in \operatorname{Ker} e_i \cap M_{\lambda + k\alpha_i}$, (ii) if $a_{ii} = 2$ and $\langle h_i, \lambda + n\alpha_i \rangle < n$, then $v_n = 0$, and (iii) if $a_{ii} \leqslant 0$, $n > 0$ and $\langle h_i, \lambda + n\alpha_i \rangle = 0$, then $v_n = 0$. This expression is called the *i-string decomposition* of v. The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ on M are defined by

$$\tilde{e}_i v = \sum_{k \geqslant 1} f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k \geqslant 0} f_i^{(k+1)} v_k.$$

Let $\mathbf{A}_0 = \{f/g \in \mathbf{Q}(q) \mid f, g \in \mathbf{Q}[q],\ g(0) \neq 0\}$ be the localization of $\mathbf{Q}(q)$ at (q). A *crystal basis* of M is a pair (L, B) such that:

(i) L is a free $\mathbf{A}_0$-submodule of M such that $M \cong \mathbf{Q}(q) \bigotimes_{\mathbf{A}_0} L$;
(ii) B is a $\mathbf{Q}$-basis of $L/qL \cong \mathbf{Q} \bigotimes_{\mathbf{A}_0} L$;
(iii) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$;
(iv) $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$;
(v) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all $i \in I$;
(vi) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$;
(vii) for all $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

It was proved in [5] that every $M \in \mathcal{O}_{\text{int}}$ has a crystal basis unique up to an automorphism. For $\lambda \in P^+$, there is a unique crystal basis $(L(\lambda), B(\lambda))$ of $V(\lambda)$, where

$$L(\lambda) = \mathbf{A}_0\text{-span}\,\{\tilde{f}_{i_1} \cdot \ldots \cdot \tilde{f}_{i_r} v_\lambda \mid i_k \in I, r \in \mathbf{Z}_{\geqslant 0}\},$$
$$B(\lambda) = \{\tilde{f}_{i_1} \cdot \ldots \cdot \tilde{f}_{i_r} v_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda)\} \setminus \{0\}.$$

Fix $i \in I$. For any $P \in U_q^-(\mathfrak{g})$, there exist unique $Q, R \in U_q^-(\mathfrak{g})$ such that

$$e_i P - P e_i = \frac{q^{s_i h_i} Q - q^{-s_i h_i} R}{q_i - q_i^{-1}}.$$

We define the endomorphisms $e_i', e_i'' : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ by

$$e_i'(P) = R, \quad e_i''(P) = Q.$$

Then every $u \in U_q^-(\mathfrak{g})$ can be written uniquely as

$$u = \sum_{k \geqslant 0} f_i^{(k)} u_k,$$

where $e_i' u_k = 0$ for all $k \geqslant 0$ and $u_k = 0$ for $k \gg 0$. Moreover, we have $u_k = q_i^{a_{ii}k(k-1)/4} P_i e_i^{(k)} u$, which is called the *i-string decomposition* of u; see [5]. The Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ on $U_q^-(\mathfrak{g})$ are defined by

$$\tilde{e}_i u = \sum_{k \geqslant 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geqslant 0} f_i^{(k+1)} u_k.$$

The crystal basis of $U_q^-(\mathfrak{g})$ is a pair (L, B) such that:

(i) L is a free $\mathbf{A}_0$-submodule of $U_q^-(\mathfrak{g})$ such that $U_q^-(\mathfrak{g}) \cong \mathbf{Q}(q) \bigotimes_{\mathbf{A}_0} L$;
(ii) B is a $\mathbf{Q}$-basis of $L/qL \cong \mathbf{Q} \bigotimes_{\mathbf{A}_0} L$;
(iii) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all $i \in I$;
(iv) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$;
(v) for all $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

It was proved in [5] that there is a unique crystal basis $(L(\infty), B(\infty))$ of $U_q^-(\mathfrak{g})$, where

$$L(\infty) = \mathbf{A}_0\text{-span}\,\{\tilde{f}_{i_1} \cdot \ldots \cdot \tilde{f}_{i_r} \cdot 1 \mid i_k \in I, r \in \mathbf{Z}_{\geqslant 0}\},$$
$$B(\infty) = \{\tilde{f}_{i_1} \cdot \ldots \cdot \tilde{f}_{i_r} \cdot 1 + qL(\infty) \in L(\infty)/qL(\infty)\} \setminus \{0\}.$$

2. *Abstract crystals*

In this section, we recall the notion of abstract crystals and their examples introduced in [6]. Moreover, we introduce a crystal $\mathbf{Z}^\infty_{\geqslant 0, \iota}$ associated with an infinite sequence ι.

2.1. *Abstract crystals*

An *abstract crystal* for $U_q(\mathfrak{g})$ or a $U_q(\mathfrak{g})$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ $(i \in I)$ and $\varepsilon_i, \varphi_i : B \to \mathbf{Z} \cup \{-\infty\}$ $(i \in I)$ such that for all $b \in B$, we have:

(i) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $i \in I$ and $\tilde{e}_i b \neq 0$,
(ii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $i \in I$ and $\tilde{f}_i b \neq 0$,
(iii) for any $i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
(iv) for any $i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
(v) for any $i \in I$ and $b \in B$ such that $\tilde{e}_i b \neq 0$, we have:
  (a) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $i \in I^{\mathrm{re}}$;
  (b) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + a_{ii}$ if $i \in I^{\mathrm{im}}$;
(vi) for any $i \in I$ and $b \in B$ such that $\tilde{f}_i b \neq 0$, we have:
  (a) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $i \in I^{\mathrm{re}}$;
  (b) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^{\mathrm{im}}$;
(vii) for any $i \in I$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

Let B_1 and B_2 be crystals. A *morphism of crystals* or a *crystal morphism* $\psi : B_1 \to B_2$ is a map $\psi : B_1 \to B_2$ such that:

(i) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$ for all $b \in B_1$;
(ii) $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $b \in B_1$, $i \in I$;
(iii) if $b \in B_1$ and $i \in I$ satisfy $\tilde{f}_i b \in B_1$, then we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

For a morphism of crystals $\psi : B_1 \to B_2$, ψ is called a *strict morphism* if

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b), \qquad \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for all } i \in I \text{ and } b \in B_1.$$

Here we understand that $\psi(0) = 0$. Moreover, ψ is called an *embedding* if the underlying map $\psi : B_1 \to B_2$ is injective. In this case, we say that B_1 is a *subcrystal* of B_2. If ψ is a strict embedding, then we say that B_1 is a *full subcrystal* of B_2.

Example 2.1. (a) The crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ is an abstract crystal, where the maps ε_i, φ_i $(i \in I)$ are given by

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \qquad \varphi_i(b) = \max\{k \geqslant 0 \mid \tilde{f}_i^k b \neq 0\} \quad \text{for } i \in I^{\mathrm{re}},$$
$$\varepsilon_i(b) = 0, \qquad \varphi_i(b) = \langle h_i, \mathrm{wt}(b)\rangle \quad \text{for } i \in I^{\mathrm{im}}.$$

(b) The crystal basis $B(\infty)$ of $U_q^-(\mathfrak{g})$ is an abstract crystal, where the maps ε_i, φ_i $(i \in I)$ are given by

$$\varepsilon_i(b) = \begin{cases} \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{\mathrm{re}}, \\ 0 & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$
$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle \quad \text{for } i \in I.$$

Example 2.2. For $i \in I$, let $B_i = \{b_i(-n) \mid n \geqslant 0\}$ and define

$$\begin{gathered} \mathrm{wt}(b_i(-n)) = -n\alpha_i, \\ \tilde{e}_i b_i(-n) = b_i(-n+1), \quad \tilde{f}_i b_i(-n) = b_i(-n-1), \\ \tilde{e}_j b_i(-n) = \tilde{f}_j b_i(-n) = 0 \quad \text{if } j \neq i, \\ \varepsilon_i(b_i(-n)) = n, \ \ \varphi_i(b_i(-n)) = -n \quad \text{if } i \in I^{\mathrm{re}}, \\ \varepsilon_i(b_i(-n)) = 0, \ \ \varphi_i(b_i(-n)) = \langle h_i, \mathrm{wt}(b_i(-n))\rangle = -na_{ii} \quad \text{if } i \in I^{\mathrm{im}}, \\ \varepsilon_j(b_i(-n)) = \varphi_j(b_i(-n)) = -\infty \quad \text{if } j \neq i. \end{gathered}$$

Here, we understand that $b_i(-n) = 0$ for $n < 0$. Then B_i is an abstract crystal, and it is called an *elementary crystal* [6].

We define the tensor product of a pair of crystals as follows: for two crystals B_1 and B_2, their tensor product $B_1 \otimes B_2$ is $\{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ with the following crystal structure. The maps $\mathrm{wt}, \varepsilon_i, \varphi_i$ are given by

$$\begin{aligned} \mathrm{wt}(b \otimes b') &= \mathrm{wt}(b) + \mathrm{wt}(b'), \\ \varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \langle h_i, \mathrm{wt}(b) \rangle), \\ \varphi_i(b \otimes b') &= \max(\varphi_i(b) + \langle h_i, \mathrm{wt}(b') \rangle, \varphi_i(b')). \end{aligned}$$

For $i \in I$, we define

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leqslant \varepsilon_i(b'). \end{cases}$$

For $i \in I^{\mathrm{re}}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geqslant \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

and for $i \in I^{\mathrm{im}}$ we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') - a_{ii}, \\ 0 & \text{if } \varepsilon_i(b') < \varphi_i(b) \leqslant \varepsilon_i(b') - a_{ii}, \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) \leqslant \varepsilon_i(b'). \end{cases}$$

This tensor product rule is different from the one given in [**5**]. However, when $B_1 = B(\lambda)$ and $B_2 = B(\mu)$ for $\lambda, \mu \in P^+$, the two rules coincide. Note that by the definition above, $B_1 \otimes B_2$ is a crystal. Moreover, it is not difficult to see that the associativity law for the tensor product holds [**6**].

2.2. *Crystal structure of* $\mathbf{Z}^{\infty}_{\geqslant 0, \iota}$

Let $\iota = (\ldots, i_k, \ldots, i_1)$ be an infinite sequence such that

$$i_k \neq i_{k+1} \quad \text{and} \quad \#\{k \mid i_k = i\} = \infty \qquad \text{for any } i \in I. \tag{2.1}$$

Now, we give a crystal structure $\mathbf{Z}^{\infty}_{\geqslant 0, \iota}$ on the set of infinite sequences of non-negative integers

$$\mathbf{Z}^{\infty}_{\geqslant 0} := \{(\ldots, x_k, \ldots, x_1) \mid x_k \in \mathbf{Z}_{\geqslant 0} \text{ and } x_k = 0 \text{ for } k \gg 0\}$$

associated with ι as follows. Let $\overrightarrow{x} = (\ldots, x_k, \ldots, x_1)$ be an element of $\mathbf{Z}^{\infty}_{\geqslant 0}$. For $k \geqslant 1$, we define

$$\sigma_k(\overrightarrow{x}) = \begin{cases} x_k + \displaystyle\sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j & \text{if } i_k \in I^{\mathrm{re}}, \\ \displaystyle\sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j & \text{if } i_k \in I^{\mathrm{im}}. \end{cases} \tag{2.2}$$

Let

$$\begin{aligned} \sigma^{(i)}(\overrightarrow{x}) &= \max\nolimits_{k : i_k = i}\{\sigma_k(\overrightarrow{x})\}, \\ n_f &= \min\{k \mid i_k = i,\ \sigma_k(\overrightarrow{x}) = \sigma^{(i)}(\overrightarrow{x})\}, \\ n_e &= \begin{cases} \max\{k \mid i_k = i,\ \sigma_k(\overrightarrow{x}) = \sigma^{(i)}(\overrightarrow{x})\} & \text{if } i \in I^{\mathrm{re}}, \\ n_f & \text{if } i \in I^{\mathrm{im}}. \end{cases} \end{aligned}$$

Now, we define

$$\tilde{f}_i \overrightarrow{x} = (x_k + \delta_{k, n_f})_{k \geqslant 1},$$

and

$$\tilde{e}_i \overrightarrow{x} = \begin{cases} (x_k - \delta_{k,n_e})_{k \geqslant 1} & \text{if } \overrightarrow{x} \text{ satisfies the condition (EC),} \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where the condition (EC) is as follows:

$$\text{(EC)} \quad \text{(i) } i \in I^{\mathrm{re}} : \sigma^{(i)}(\overrightarrow{x}) > 0,$$
$$\text{(ii) } i \in I^{\mathrm{im}} : \text{for } k = n_e \text{ with } k^{(-)} \neq 0,$$
$$x_k > 1, \quad \text{or } x_k = 1 \quad \text{and} \sum_{k^{(-)} < j < k} \langle h_i, \alpha_{i_j} \rangle x_j < 0.$$

Here, $k^{(-)}$ is the maximal index $j < k$ such that $i_j = i_k$. We also define

$$\mathrm{wt}(\overrightarrow{x}) = -\sum_{j=1}^{\infty} x_j \alpha_{i_j}, \quad \varepsilon_i(\overrightarrow{x}) = \sigma^{(i)}(\overrightarrow{x}), \quad \varphi_i(\overrightarrow{x}) = \langle h_i, \mathrm{wt}(\overrightarrow{x}) \rangle + \varepsilon_i(\overrightarrow{x}).$$

It is easy to see that $\mathbf{Z}_{\geqslant 0}^{\infty}$ is a crystal. We denote this crystal by $\mathbf{Z}_{\geqslant 0,\iota}^{\infty}$.

REMARK 2.3. Since $x_k = 0$ for $k \gg 0$, it is clear that $\varepsilon_i(\overrightarrow{x}) = 0$ for each $i \in I^{\mathrm{im}}$, and so $\varphi_i(\overrightarrow{x}) = \langle h_i, \mathrm{wt}(\overrightarrow{x}) \rangle$.

2.3. *Embedding of crystals*

PROPOSITION 2.4 [6]. *For all $i \in I$, there exists a unique strict embedding*

$$\Psi_i : B(\infty) \longrightarrow B(\infty) \otimes B_i \quad \textit{such that } u_\infty \longmapsto u_\infty \otimes b_i(0),$$

where u_∞ is the highest weight vector in $B(\infty)$.

Proposition 2.4 yields a procedure to determine the structure of the crystal $B(\infty)$ in terms of elementary crystals. Take an infinite sequence $\iota = (\ldots, i_2, i_1)$ in I such that every $i \in I$ appears infinitely many times. For each $N \geqslant 1$, taking the composition of crystal embeddings repeatedly, we obtain a strict crystal embedding

$$\begin{aligned} \Psi^{(N)} &:= (\Psi_{i_N} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \circ \ldots \circ (\Psi_{i_2} \otimes \mathrm{id}) \circ \Psi_{i_1} : \\ & B(\infty) \hookrightarrow B(\infty) \otimes B_{i_1} \hookrightarrow B(\infty) \otimes B_{i_2} \otimes B_{i_1} \hookrightarrow \cdots \hookrightarrow B(\infty) \otimes B_{i_N} \otimes \ldots \otimes B_{i_1}. \end{aligned} \tag{2.4}$$

It is easily seen that, for any $b \in B$, there exists an $N > 0$ such that

$$\Psi^{(N)}(b) = u_\infty \otimes b_{i_N}(-x_N) \otimes \ldots \otimes b_{i_1}(-x_1)$$

for some $x_1, \ldots, x_N \in \mathbf{Z}_{\geqslant 0}$ and $x_k = 0$ for $k > N$. Thus the sequence $(\ldots, 0, x_N, \ldots, x_1)$ belongs to $\mathbf{Z}_{\geqslant 0,\iota}^{\infty}$, and so we obtain a map

$$\begin{array}{rccc} \Psi_\iota : & B(\infty) & \longrightarrow & \mathbf{Z}_{\geqslant 0,\iota}^{\infty} \\ & b & \longmapsto & (\ldots, 0, x_N, \ldots, x_1). \end{array}$$

We can easily see that it is a strict embedding (see also [6]).

3. *Polyhedral realizations of $B(\infty)$*

In [18], Nakashima and Zelevinsky gave the polyhedral realizations of the crystal bases $B(\infty)$ of the negative parts $U_q^-(\mathfrak{g})$ of the quantum groups $U_q(\mathfrak{g})$ associated with Kac–Moody algebras. In this section, we extend their theory to the case of quantum generalized Kac–Moody algebras.

3.1. *Polyhedral realizations of* $B(\infty)$

Let $\iota = (i_k)_{k \geqslant 1}$ be a sequence of indices satisfying (2.1). Let $\mathbf{Q}^\infty$ be an infinite-dimensional vector space

$$\mathbf{Q}^\infty = \{\overrightarrow{x} = (\ldots, x_k, \ldots, x_1) \mid x_k \in \mathbf{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\}.$$

For a linear functional $\psi \in (\mathbf{Q}^\infty)^*$, we write $\psi(\overrightarrow{x}) = \sum_{k \geqslant 1} \psi_k x_k$ ($\psi_k \in \mathbf{Q}$). For each $k \geqslant 1$, we denote by $k^{(+)}$ (resp. $k^{(-)}$) the minimal (resp. maximal) index $j > k$ (resp. $j < k$) such that $i_j = i_k$. Let $\beta_k \in (\mathbf{Q}^\infty)^*$ be a linear form

$$\begin{aligned}\beta_k(\overrightarrow{x}) &= \sigma_k(\overrightarrow{x}) - \sigma_{k^{(+)}}(\overrightarrow{x}) \\ &= \begin{cases} x_k + \displaystyle\sum_{k<j<k^{(+)}} \langle h_{i_k}, \alpha_{i_j}\rangle x_j + x_{k^{(+)}} & \text{if } i_k \in I^{\mathrm{re}}, \\ \displaystyle\sum_{k<j\leqslant k^{(+)}} \langle h_{i_k}, \alpha_{i_j}\rangle x_j & \text{if } i_k \in I^{\mathrm{im}}, \end{cases}\end{aligned} \tag{3.1}$$

and we set $\beta_0(\overrightarrow{x}) = 0$. Then, we define a piecewise-linear operator $S_k = S_{k,\iota}$ on $(\mathbf{Q}^\infty)^*$ by

$$S_k(\psi) = \begin{cases} \psi - \psi_k \beta_k & \text{if } \psi_k > 0,\ i_k \in I^{\mathrm{re}}, \\ \psi - \psi_k\Big(x_k + \displaystyle\sum_{k<j<k^{(+)}} \langle h_{i_k}, \alpha_{i_j}\rangle x_j - x_{k^{(+)}}\Big) & \text{if } \psi_k > 0,\ i_k \in I^{\mathrm{im}}, \\ \psi - \psi_k \beta_{k^{(-)}} & \text{if } \psi_k \leqslant 0. \end{cases}$$

Let

$$\Theta_\iota = \{S_{j_l} \cdot \ldots \cdot S_{j_1} x_{j_0} \mid l \geqslant 0, j_0, \ldots, j_l \geqslant 1\}$$

be the set of linear forms obtained from the coordinate forms x_j by applying transformations S_k. Moreover, for a given $s, t \geqslant 1$ ($t > s$), let $\Theta_\iota^{s\backslash t}$ be the subset of Θ_ι of linear forms obtained from the coordinate forms x_s by applying transformations S_k with $k \neq t$; that is,

$$\Theta_\iota^{s\backslash t} = \{S_{j_l} \cdot \ldots \cdot S_{j_1} x_s \mid l \geqslant 0,\ s, j_1, \ldots, j_l \geqslant 1\},$$

where $j_1, \ldots, j_l \neq t$. We impose on ι the positivity assumption given in [**18**]. That is,

$$\text{if } k^{(-)} = 0, \text{ then } \psi_k \geqslant 0 \quad \text{for any } \psi = \sum \psi_j x_j \in \Theta_\iota. \tag{3.2}$$

Then we have the following main theorem.

THEOREM 3.1. *Let ι be a sequence of indices satisfying* (2.1) *and* (3.2). *Let $\Psi_\iota : B(\infty) \hookrightarrow \boldsymbol{Z}^\infty_{\geqslant 0,\iota}$ be the crystal embedding. Then $\operatorname{Im}\Psi_\iota$ is the set Γ_ι consisting of $\overrightarrow{x} \in \boldsymbol{Z}^\infty_{\geqslant 0,\iota}$ satisfying the following conditions*:

(i) *$\psi(\overrightarrow{x}) \geqslant 0$ for any $\psi \in \Theta_\iota$;*

(ii) *for each t with $i_t \in I^{\mathrm{im}}$, if $x_t \neq 0$ and $t^{(-)} \neq 0$, then*

$$\sum_{t^{(-)}<j<t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0. \tag{3.3}$$

In addition, if $\langle h_{i_t}, \alpha_{i_j}\rangle x_j = 0$ ($t^{(-)} < j < t$) for all j with $i_j \in I^{\mathrm{im}}$, then there exists an integer p ($t^{(-)} < p < t$) such that $i_p \in I^{\mathrm{re}}$,

$$\langle h_{i_t}, \alpha_{i_p}\rangle x_p < 0 \quad \text{and} \quad \psi(\overrightarrow{x}) > 0 \text{ for any } \psi \in \Theta_\iota^{p\backslash t}. \tag{3.4}$$

We prove the theorem in Subsection 3.2.

Corollary 3.2. *Assume that all elements of I are imaginary; that is, $I = I^{\mathrm{im}}$. Then the image of the crystal embedding $\operatorname{Im}\Psi_\iota$ is equal to the set of $\vec{x} \in \mathbf{Z}^{\infty}_{\geqslant 0,\iota}$ satisfying (3.3) of Theorem 3.1.*

Proof. By a simple calculation, it is easy to see that the set Θ_ι consists of the linear combinations of the coordinate forms x_j with non-negative coefficients, which completes the proof. □

Now, we consider the case where the cardinality of I^{re} is 1. Then it is easy to see that $S_j x_j$ is a linear combination of the coordinate forms x_k with non-negative coefficients except for $i_j \in I^{\mathrm{re}}$. If $i_j \in I^{\mathrm{re}}$, then

$$S_j x_j = -\sum_{j<t<j^{(+)}} \langle h_{i_j}, \alpha_{i_t}\rangle x_t - x_{j^{(+)}},$$

and

(i) if $k = j^{(+)}$, then $S_k S_j x_j$ is x_j;
(ii) if $j < k < j^{(+)}$ and $\langle h_{i_k}, \alpha_{i_j}\rangle < 0$, then $S_k S_j x_j$ is a linear combination of the coordinate forms x_t with non-negative coefficients;
(iii) if k does not belong to cases (i) and (ii), then $S_k S_j x_j$ is $S_j x_j$ itself.

Therefore, it is easy to see that condition (i) of Theorem 3.1 is changed to

$$S_j x_j \geqslant 0 \quad \text{for all } j \text{ with } i_j \in I^{\mathrm{re}}. \tag{3.5}$$

Moreover, for given p, t in Theorem 3.1 (ii), since $\psi_t > 0$ for any $\psi \in \Theta_\iota^{p\setminus t}$, the above (i)–(iii) imply that the condition $S_p x_p > 0$ is the same as the condition that $\psi(\vec{x}) > 0$ for all $\psi \in \Theta_\iota^{p\setminus t}$. Finally, by the above (i)–(iii), it is clear that any sequence ι satisfies the positivity assumption (3.2). Therefore, we have the following simple but important corollary.

Corollary 3.3. *Let I be an index set such that the cardinality of I^{re} is 1, and let ι be a sequence of indices in I satisfying (2.1). Then the image $\operatorname{Im}\Psi_\iota$ of the crystal embedding is the set Γ_ι of $\vec{x} \in \mathbf{Z}^{\infty}_{\geqslant 0,\iota}$ satisfying the following conditions.*

(i) *$S_j x_j \geqslant 0$ for all j with $i_j \in I^{\mathrm{re}}$.*
(ii) *For each t with $i_t \in I^{\mathrm{im}}$, if $x_t \neq 0$ and $t^{(-)} \neq 0$, then*

$$\sum_{t^{(-)}<j<t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0.$$

In addition, if $\langle h_{i_t}, \alpha_{i_j}\rangle x_j = 0$ $(t^{(-)} < j < t)$ for all $i_j \in I^{\mathrm{im}}$, then there exists an integer p $(t^{(-)} < p < t)$ such that $i_p \in I^{\mathrm{re}}$,

$$\langle h_{i_t}, \alpha_{i_p}\rangle x_p < 0 \quad \text{and} \quad S_p x_p > 0.$$

Example 3.4. Assume that $I = \{1, 2\}$ and $\iota = (\ldots, 2, 1, 2, 1)$. Set

$$\alpha_1(h_1) = -a, \quad \alpha_1(h_2) = -c, \quad \alpha_2(h_1) = -b \quad \text{and} \quad \alpha_2(h_2) = 2,$$

where $a, b, c \in \mathbf{Z}_{\geqslant 0}$. Then $I^{\mathrm{re}} = \{2\}$, $I^{\mathrm{im}} = \{1\}$, and if $k \geqslant 3$, then $k^{(-)} \neq 0$. Therefore, for each $k \geqslant 1$, if $x_{2k+1} \neq 0$, then we have

$$\sum_{2k-1<j<2k+1} \langle h_1, \alpha_{i_j}\rangle x_j = -b x_{2k} < 0.$$

Moreover, since $x_{2k} > 0$ and $i_{2k} = 2 \in I^{\mathrm{re}}$, we have $S_{2k}x_{2k} = x_{2k} - \beta_{2k} = cx_{2k+1} - x_{2k+2} > 0$. Therefore, by Corollary 3.3 the image of the crystal embedding $\operatorname{Im}\Psi_\iota$ is given by the subset Γ_ι of $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geqslant 0,\iota}$ as follows.

(a) When $b = c = 0$,

$$x_k = 0 \text{ for } k \geqslant 3.$$

(b) When neither b nor c is 0,

(i) for each $k \geqslant 1$, we have $cx_{2k+1} - x_{2k+2} > 0$ unless $x_{2k+1} = x_{2k+2} = 0$,

(ii) for each $k \geqslant 1$, if $x_{2k+1} \neq 0$, then $x_{2k} > 0$.

3.2. *The proof of Theorem* 3.1

We know that $\operatorname{Im}\Psi_\iota$ is a subcrystal of $\mathbf{Z}^{\infty}_{\geqslant 0,\iota}$ obtained by applying the Kashiwara operators $\tilde{f}_i$ to $\Psi_\iota(u_\infty) = \overrightarrow{0} = (\ldots, 0, 0, 0)$ and $\overrightarrow{0}$ belongs to Γ_ι. Therefore, in order to prove that $\operatorname{Im}\Psi_\iota \subset \Gamma_\iota$, it suffices to show that Γ_ι is closed under all $\tilde{f}_i$. Let $\overrightarrow{x} \in \Gamma_\iota$ and $i \in I$. Suppose that $\tilde{f}_i\overrightarrow{x} = (\ldots, x_k + 1, \ldots, x_1)$. Since

$$\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k \geqslant \psi_k \quad \text{for any } \psi \in \Theta_\iota,$$

in order to prove (i), it is enough to consider the case when $\psi_k < 0$. By the positivity condition (3.2) of ι, we have $k^{(-)} \geqslant 1$. By (2.2) and the definition of Kashiwara operator $\tilde{f}_i$ on $\mathbf{Z}^{\infty}_{\geqslant 0,\iota}$, we have $\sigma_k(\overrightarrow{x}) > \sigma_{k^{(-)}}(\overrightarrow{x})$ (indeed, when $i_k \in I^{\mathrm{im}}$, we have $\sigma_k(\overrightarrow{x}) = 0$ and $\sigma_{k^{(-)}}(\overrightarrow{x}) < 0$), and so

$$\beta_{k^{(-)}}(\overrightarrow{x}) = \sigma_{k^{(-)}}(\overrightarrow{x}) - \sigma_k(\overrightarrow{x}) \leqslant -1.$$

Therefore,

$$\begin{aligned} \psi(\tilde{f}_i\overrightarrow{x}) &= \psi(\overrightarrow{x}) + \psi_k \\ &\geqslant \psi(\overrightarrow{x}) - \psi_k\beta_{k^{(-)}}(\overrightarrow{x}) \\ &= (S_k\psi)(\overrightarrow{x}) \geqslant 0. \end{aligned} \tag{3.6}$$

Now, suppose that $\tilde{f}_i\overrightarrow{x}$ does not satisfy the condition (3.3). Then $k = t$, $i_t \in I^{\mathrm{im}}$ and

$$x_t = 0, \qquad \sum_{t^{(-)}<j<t} \langle h_{i_t}, \alpha_{i_j}\rangle x_j \geqslant 0 \quad \text{in } \overrightarrow{x}.$$

However, since $\langle h_{i_t}, \alpha_{i_j}\rangle x_j \leqslant 0$ for all $t^{(-)} < j < t$, we have $\sum_{t^{(-)}<j<t}\langle h_{i_t}, \alpha_{i_j}\rangle x_j = 0$, and this cannot occur, by the definition of Kashiwara operator $\tilde{f}_i$.

Now, we show that $\tilde{f}_i\overrightarrow{x}$ satisfies the condition (3.4). First, suppose that there exist p and t satisfying (3.4) in $\overrightarrow{x}$. Since $\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k$, it is enough to consider the case $\psi_k < 0$. Note that by the condition of p such that $\langle h_{i_t}, \alpha_{i_p}\rangle < 0$, and the definition of the set $\Theta_\iota^{p\setminus t}$, we have $\psi_t \geqslant 0$ for all $\psi \in \Theta_\iota^{p\setminus t}$. Therefore, it suffices to consider the case $k \neq t$. If $k \neq t$, then $S_k\psi \in \Theta_\iota^{p\setminus t}$, and by (3.6) $\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_k \geqslant (S_k\psi)(\overrightarrow{x}) > 0$.

Second, suppose that:

(a) $k = t$, $x_t = 0$,

(b) for any j such that $t^{(-)} < j < t$, $i_j \in I^{\mathrm{re}}$ and $\langle h_{i_t}, \alpha_{i_j}\rangle x_j < 0$,
there is a $\psi \in \Theta_\iota^{j\setminus t}$ such that $\psi(\overrightarrow{x}) = 0$ in $\overrightarrow{x}$.

Note that since j is an index such that $\langle h_{i_t}, \alpha_{i_j}\rangle < 0$, by the definition of $\psi \in \Theta_\iota^{j\setminus t}$ we have $\psi_t > 0$ for all $\psi \in \Theta_\iota^{j\setminus t} \setminus \{x_j\}$. Here, $x_j \in \Theta_\iota^{j\setminus t}$ cannot satisfy condition (b). Therefore,

$$\psi(\tilde{f}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) + \psi_t \geqslant \psi_t > 0 \quad \text{for all } \psi \in \Theta_\iota^{j\setminus t} \setminus \{x_j\}.$$

Therefore, $\operatorname{Im}\Psi_\iota \subset \Gamma_\iota$.

For the proof of the reverse inclusion $\Gamma_\iota \subset \operatorname{Im}\Psi_i$, note that for any $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geqslant 0,\iota} \setminus \{\overrightarrow{0}\}$ satisfying condition (ii), there is an $i \in I$ such that $\tilde{e}_i\overrightarrow{x} \neq 0$. Indeed, for the largest number

k such that $x_k > 0$ in $\overrightarrow{x}$, if $i_k \in I^{\mathrm{re}}$, then $\sigma_k(\overrightarrow{x}) = x_k > 0$ and so $\sigma^{(i_k)}(\overrightarrow{x}) \geqslant \sigma_k(\overrightarrow{x}) > 0$, which implies that $\tilde{e}_{i_k}\overrightarrow{x} \neq 0$. If $i_k \in I^{\mathrm{im}}$, then $n_f = n_e = k$ by condition (3.3), and so we have $\tilde{e}_{i_k}\overrightarrow{x} \neq 0$.

Since $\Gamma_\iota \subset \mathbf{Z}^{\infty}_{\geqslant 0,\iota}$, if Γ_ι is closed under the Kashiwara operators $\tilde{e}_i$ for all $i \in I$, then for any $\overrightarrow{x} \in \Gamma_\iota$, there are $i_1, \ldots, i_t \in I$ such that

$$\tilde{e}_{i_t} \cdot \ldots \cdot \tilde{e}_{i_1}\overrightarrow{x} = \overrightarrow{0}.$$

Moreover, this means that

$$\tilde{f}_{i_1} \cdot \ldots \cdot \tilde{f}_{i_t}\overrightarrow{0} = \overrightarrow{x},$$

which implies that $\Gamma_\iota \subset \operatorname{Im}\Psi_i$. Hence, it is enough to show that $\tilde{e}_i\Gamma_\iota \subset \Gamma_\iota \cup \{0\}$ for all $i \in I$. Let $\overrightarrow{x} \in \Gamma_\iota$ and $i \in I$. Suppose that $\tilde{e}_i\overrightarrow{x} = (\ldots, x_k - 1, \ldots, x_1)$. Since

$$\psi(\tilde{e}_i\overrightarrow{x}) = \psi(\overrightarrow{x}) - \psi_k \geqslant -\psi_k \quad \text{for any } \psi \in \Theta_\iota,$$

to prove (i) it suffices to consider the case when $\psi_k > 0$. By (2.3), we have

$$\beta_k(\overrightarrow{x}) = \sigma_k(\overrightarrow{x}) - \sigma_{k^{(+)}}(\overrightarrow{x}) \geqslant 1 \quad (i \in I^{\mathrm{re}})$$

and

$$x_k + \sum_{k<j<k^{(+)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{k^{(+)}} \geqslant 1 \quad (i \in I^{\mathrm{im}}).$$

Therefore,

$$\begin{aligned}
\psi(\tilde{e}_i\overrightarrow{x}) &= \psi(\overrightarrow{x}) - \psi_k \\
&\geqslant \begin{cases} \psi(\overrightarrow{x}) - \psi_k\beta_k(\overrightarrow{x}) & \text{if } i \in I^{\mathrm{re}}, \\ \psi(\overrightarrow{x}) - \psi_k\left(x_k + \displaystyle\sum_{k<j<k^{(+)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{k^{(+)}}\right) & \text{if } i \in I^{\mathrm{im}}, \end{cases} \\
&= (S_k\psi)(\overrightarrow{x}) \geqslant 0.
\end{aligned} \tag{3.7}$$

Now, suppose that $\tilde{e}_i\overrightarrow{x}$ does not satisfy condition (ii). First, suppose that $\tilde{e}_i\overrightarrow{x}$ does not satisfy (3.3). If $i_k = i \in I^{\mathrm{im}}$ and $t^{(-)} < k < t$, then by the definition of Kashiwara operator $\tilde{e}_i$, we have $\langle h_i, \alpha_{i_t}\rangle = 0$, and so $\langle h_{i_t}, \alpha_i\rangle = 0$. However, in this case, it is clear that (3.3) holds in $\tilde{e}_i\overrightarrow{x}$. Second, suppose that k is a unique index such that $t^{(-)} < k < t$ with $i_k \in I^{\mathrm{re}}$ and $\langle h_{i_t}, \alpha_{i_k}\rangle x_k < 0$ in $\overrightarrow{x}$. In this case, $S_k x_k(\overrightarrow{x}) = x_k - \beta_k > 0$ by (3.4), and by the definition of Kashiwara operator $\tilde{e}_i$, we have $\beta_k > 0$. Hence, $x_k > \beta_k > 0$ and so $x_k > 1$. Therefore, $\tilde{e}_i\overrightarrow{x}$ satisfies (3.3). Hence it suffices to consider the case where $\tilde{e}_i\overrightarrow{x}$ does not satisfy condition (3.4). First, suppose that $k = p$ and $x_p = 1$ in $\overrightarrow{x}$. However, since $S_p x_p(\overrightarrow{x}) = x_p - \beta_p > 0$, we have $\beta_p \leqslant 0$. This contradicts the definition of Kashiwara operator $\tilde{e}_i$.

Second, suppose that $k \neq p$. If $k \neq t$, then by the same argument as in (3.7), we have $\psi(\tilde{e}_i\overrightarrow{x}) > 0$ for all $\psi \in \Theta_\iota^{p\setminus t}$. Therefore, it suffices to consider the case that $k = t$ and $x_t > 1$. However, in this case, by the definition of $\tilde{e}_i$ on $\mathbf{Z}^{\infty}_{\geqslant 0,\iota}$,

$$x_t + \sum_{t<j<t^{(+)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{t^{(+)}} = x_t > 1$$

and so

$$\begin{aligned}
\psi(\tilde{e}_i\overrightarrow{x}) &= \psi(\overrightarrow{x}) - \psi_t \\
&> \psi(\overrightarrow{x}) - \psi_t\left(x_t + \sum_{t<j<t^{(+)}} \langle h_i, \alpha_{i_j}\rangle x_j - x_{t^{(+)}}\right) \\
&= (S_t\psi)(\overrightarrow{x}) \geqslant 0.
\end{aligned} \tag{3.8}$$

Therefore, Γ_ι is closed under all $\tilde{e}_i$.

4. *Applications: rank 3 case and Monster Lie algebra*

In this section, we will give an explicit description of the image of the Kashiwara embedding for the generalized Kac–Moody algebras of rank 3 and Monster Lie algebras.

4.1. *Rank 3 case*

Assume that $I = \{1, 2, 3\}$ and $\iota = (\ldots, 1, 3, 2, 1)$. Consider the case when $1, 2 \in I^{\mathrm{im}}$ and $3 \in I^{\mathrm{re}}$. Let A be a Borcherds–Cartan matrix

$$A = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & 2 \end{pmatrix},$$

where $a, b, c, d, e, f, g, h \in \mathbf{Z}_{\geqslant 0}$. For each $k \geqslant 1$, we have

$$S_{3k} x_{3k} = g x_{k+1} + h x_{k+2} - x_{k+3},$$

Moreover, since $I^{\mathrm{im}} = \{1, 2\}$, for each k with $i_k = 1, 2$,

$$\sum_{k^{(-)} < j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j = \begin{cases} -b x_{k-2} - c x_{k-1} & \text{if } i_k = 1, \\ -f x_{k-2} - d x_{k-1} & \text{if } i_k = 2. \end{cases}$$

Therefore, by Corollary 3.3, we have the following.

COROLLARY 4.1. *Assume that $1, 2 \in I^{\mathrm{im}}$ and $3 \in I^{\mathrm{re}}$. The image of the crystal embedding $\operatorname{Im} \Psi_\iota$ is given by the subset Γ_ι of $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geqslant 0, \iota}$ satisfying the following conditions.*

(i) *$g x_{3k+1} + h x_{3k+2} - x_{3k+3} \geqslant 0$ for $k \geqslant 1$.*

(ii) *For each $k \geqslant 1$, if $x_{3k+1} > 0$ and $x_{3k+2} > 0$, then*

$$b x_{3k-1} + c x_{3k} > 0 \quad \text{and} \quad f x_{3k} + d x_{3k+1} > 0,$$

respectively. Moreover, if $b x_{3k-1} = 0$ and $d x_{3k+1} = 0$, then

$$g x_{3k+1} + h x_{3k+2} - x_{3k+3} > 0.$$

4.2. *Monster Lie algebras*

Let $I = \{-1\} \cup \mathbf{N}$, and let $A = (-(i+j))_{i,j \in I}$ be a Borcherds–Cartan matrix of charge $\underline{m} = (c(i) \mid i \in I)$. Here, $c(i)$ is the coefficient of the elliptic modular function

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots = \sum_{i=-1}^{\infty} c(i) q^i.$$

Then we have the associated generalized Kac–Moody algebra called the Monster Lie algebra.

On the other hand, let

$$I = \{-1 = -1_1\} \cup \{i_t \mid i \in \mathbf{N}, t = 1, \ldots, c(i)\} \quad \text{and} \quad A = (-(i+j))_{p,q \in I},$$

where $p = i_l$ and $q = j_m$ for some $1 \leqslant l \leqslant c(i)$ and $1 \leqslant m \leqslant c(j)$. Then the associated generalized Kac–Moody algebra is also a Monster Lie algebra. From now on, we adopt the latter exposition of the Monster Lie algebra. Assume that

$$\iota = (\ldots, -1, 3_{c(3)}, \ldots, 3_1, 2_{c(2)}, \ldots, 2_1, 1_{c(1)}, \ldots, 1_1, -1, \\ 2_{c(2)}, \ldots, 2_1, 1_{c(1)}, \ldots, 1_1, -1, 1_{c(1)}, \ldots, 1_1, -1).$$

Let $I_{(-1)}$ be the set of positive integers t such that $i_t = -1$, that is,

$$I_{(-1)} = \{1\} \cup \{b(n) = nc(1) + (n-1)c(2) + \ldots + c(n) + n + 1 \mid n \in \mathbf{N}\},$$

and for any $n \geqslant 1$, we set

$$\sigma(n) = c(1) + \ldots + c(n).$$

THEOREM 4.2. *The image of the Kashiwara embedding* $\operatorname{Im}\Psi_\iota$ *is given by the subset* Γ_ι *of* $\overrightarrow{x} \in \mathbf{Z}^{\infty}_{\geqslant 0,\iota}$ *such that:*

(i) $x_{c(1)+2} = 0$, *and for each* $n \geqslant 1$,

$$\sum_{k=1}^{n} k(x_{b(n)+\sigma(k)+1} + \ldots + x_{b(n)+\sigma(k+1)}) - x_{b(n)+\sigma(n+1)+1} \geqslant 0;$$

(ii) *for each* $k \notin I_{(-1)}$, *if* $x_k > 0$ *and* $k^{(-)} \neq 0$, *then*

$$\sum_{k^{(-)}<j<k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j < 0.$$

Moreover, if $\langle h_{i_k}, \alpha_{i_j}\rangle x_j = 0$ *for all* $k^{(-)} < j < k$ *with* $j \notin I_{-1}$, *then there exists* $m \geqslant 1$ *such that* $k^{(-)} < b(m) < k$ *and*

$$\sum_{k=1}^{m} k(x_{b(m)+\sigma(k)+1} + \ldots + x_{b(m)+\sigma(k+1)}) - x_{b(m)+\sigma(m+1)+1} > 0.$$

Proof. By simple calculation, we have

$$S_1 x_1 = x_1 - (x_1 + \langle h_{-1}, \alpha_1\rangle(x_2 + \ldots + x_{c(1)+1}) + x_{c(1)+2}) = -x_{c(1)+2},$$

and for each $n \geqslant 1$

$$\begin{aligned} S_{b(n)}x_{b(n)} &= x_{b(n)} - \Bigg(x_{b(n)} + \sum_{k=1}^{n+1} \langle h_{-1}, \alpha_k\rangle (x_{b(n)+\sigma(k-1)+1} + \ldots + x_{b(n)+\sigma(k)}) \\ &\qquad + x_{b(n)+\sigma(n+1)+1}\Bigg) \\ &= \sum_{k=1}^{n} k(x_{b(n)+\sigma(k)+1} + \ldots + x_{b(n)+\sigma(k+1)}) - x_{b(n)+\sigma(n+1)+1}. \end{aligned}$$

Moreover, it is also easy to see that $S_j S_k x_k$ for all j is a linear combination of the coordinate forms x_j with non-negative coefficients. Therefore, we obtain the results. □

Finally, by Theorem 4.2, we have the following character formula of the negative part $U_q^-(\mathfrak{g})$ of the quantum Monster Lie algebra $U_q(\mathfrak{g})$.

COROLLARY 4.3. *We have*

$$\operatorname{ch} U_q^-(\mathfrak{g}) = \sum_{\overrightarrow{x} \in \Gamma_\iota} e^{\mathrm{wt}(\overrightarrow{x})} = \sum_{\overrightarrow{x} \in \Gamma_\iota} e^{-\sum_{j=1}^{\infty} x_j \alpha_{i_j}}.$$

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