Journal of High Energy Physics

Diffusion in an expanding plasma using AdS/CFT

To cite this article: Keun-Young Kim *et al* JHEP04(2008)047

Published by Institute of Physics Publishing for SISSA

Received: *July 18, 2007*
Revised: *October 24, 2008*
Accepted: *April 1, 2008*
Published: *April 14, 2008*

Diffusion in an expanding plasma using AdS/CFT

Keun-Young Kim,[a] Sang-Jin Sin[ab] and Ismail Zahed[a]

[a] *Department of Physics and Astronomy, SUNY Stony-Brook, NY 11794, U.S.A.*
[b] *Department of Physics, Hanyang University, Seoul 133-791, Korea*
E-mail: keykim@ic.sunysb.edu, sjsin@hanyang.ac.kr, zahed@zahed.physics.sunysb.edu

Abstract: We consider the diffusion of a non-relativistic heavy quark of fixed mass M, in a one-dimensionally expanding and strongly coupled plasma using the AdS/CFT duality. The Green's function constructed around a static string embedded in a background with a moving horizon, is identified with the noise correlation function in a Langevin approach. The (electric) noise decorrelation is of order $1/T(\tau)$ while the velocity de-correlation is of order $MD(\tau)/T(\tau)$. For $MD > 1$, the diffusion regime is segregated and the energy loss is Langevin-like. The time dependent diffusion constant $D(\tau)$ asymptotes its adiabatic limit $2/\pi\sqrt{\lambda}T(\tau)$ when $\tau/\tau_0 = (1/3\eta_0\tau_0)^3$ where η_0 is the drag coefficient at the initial proper time τ_0.

Keywords: AdS-CFT Correspondence, Hadronic Colliders.

JHEP04(2008)047

Contents

1. Introduction

The quark-gluon plasma (QGP) created in Relativistic Heavy Ion Collisions at RHIC is believed to be strongly coupled [1]. The AdS/CFT correspondence [2] has proven to be a useful tool for addressing issues of a strongly coupled plasma albeit in the limit of a large number of colors N_c and strong gauge coupling $\lambda = g_{\rm YM}^2 N_c$. A number of nonperturbative properties in linear response have been recently addressed via the gravity dual calculation in AdS_5 black hole geometry [3]. The transport results bear some relevance to the QGP plasma at RHIC. Much less is known about the time-dependent evolution of the strongly coupled plasma. There have been suggestions that the fireball in Relativistic Heavy Ion Collision (RHIC) can be explained from a dual gravity point of view [4, 5].

In order to model the expanding plasma in a gravity set up, the use of a moving black hole was suggested in [4]. In [6, 7] (hereon refer to as JP), it was shown that the moving horizon black hole geometry can be generated by assuming an asymptotycally expanding perfect fluid in a boost invariant setting. Using conformal invariance and energy-momentum conservation together with holographic renormalization [8], JP constructed the bulk geometry from the perfect fluid boundary data. This metric was extended to the case of shear viscosity [11, 12], R-charge [13] and to a 3 dimensional non-isotropic setting [14]. An exact background with isotropic expansion was worked out in [15].

In this paper we study the diffusion of a non-relativistic heavy quarks in one dimensionally expanding plasma using the JP metric. Heavy quark diffusion have been studied in a static black hole background in various ways [16 − 22]. Here, we follow the suggestion in [20] in the static case, by analyzing the momentum fluctuations of a heavy quark in an expanding plasma, and use it to estimate the drag force and diffusion rate. We will use a generalized Langevin equation to assess the diffusion rate. Unlike [21] we suggest that the green function of string fluctuation should be identified with the correlator of the fluctuation force rather than the total force in a Langevin approach.

The basic object of this procedure is the gauge-invariant electric-electric force decorrelation, which we will calculate using the AdS/CFT duality. The field theory dual (operator) to a quark displacement($\xi(t)|_{u=0}$) is the (gauge invariant) colored force acting on a heavy quark [20 − 22]. According to the AdS/CFT correspondence the generating function in field theory should be related with the classical action by

$$\langle \exp(i \int F(t)\xi(t)) \rangle = \exp(iS_{cl}[\xi])\,, \tag{1.1}$$

whose second derivative gives us the symmetrized Wightman function

$$G(t_1, t_2) \equiv \frac{1}{2}\langle F(t_1)F(t_2) + F(t_2)F(t_1)\rangle\ . \tag{1.2}$$

It is related to the retarded Green's function [24]

$$G(\omega) = -\coth\frac{\omega}{2T_0}\mathrm{Im}G_R(\omega)\,, \tag{1.3}$$

in momentum space. We will calculate $G_R(\omega)$ from the Nambu-Goto action of the fluctuating string in the JP metric following the way proposed in [20] (section 2). We note that this decorrelator applies both to heavy and light fundamental quarks, therefore the decorrelation time is a measure of how the gluon rescattering against external fundamental probes decorrelate.

In section 2 we start with the JP metric in [6] and derive equations of motion of the transverse string fluctuation of a heavy quark. We also calculate the retarded Green's function, which is translated to an electric force-force correlator (or more precisely decorrelator) in section 3. In section 4, we compute the momentum fluctuation(broadening) of a heavy quark. By arguing that a generalized Langevin equation applies for the case,we deduce the heavy quark diffusion property in the expanding and cooling plasma. Our conclusions are in section 5.

2. String fluctuation in Janik-Peschanski metric

We consider a heavy particle moving in an expanding plasma. Let $t_F = 1/T$ be the colored electric force decorrelation time (see below) and let $t_D = MD/T$ be the diffusion time of a particle with diffusion constant D and mass M. For times $t_F < t < t_D$ o r $MD > 1$, the force decorrelation can be treated as instantaneous. Thus, the velocity decorrelation of a heavy quark in a strongly coupled plasma can be followed generically by a Langevin description. For an expanding plasma the description involves a generalized Langevin of the form

$$\frac{dp}{dt} + \eta(t)p(t) = F(t), \tag{2.1}$$

with both the drag coefficient $\eta(t)$ and the diffusion constant $D(t)$ time dependent. The idea here is that the force-force decorrelator can be calculated from the microscopic physics and the drag coefficient $\eta(t)$ is related to the force-force decorrelator by a non-equilibrium relation [27, 28].

For the drag coefficient, we need to calculate the correlators in a frame which moves with the particle. So it is enough to consider a particle moving with the expanding plasma. For a one dimensional expansion, such a comoving frame was introduced by Hwa [9] and by Bjorken [10] in terms of the rapidity y and the proper time τ of the comoving coordinate system. τ and y are related to the time coordinate x^0, and logitudinal coordinate x^3, through $x^0 = \tau \cosh y$ and $x^3 = \tau \sinh y$. The Minkowski metric in terms of τ, y can be written as

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2, \tag{2.2}$$

where $x_\perp = \{x^1, x^2\}$ are the transverse coordinates.

The gravity dual of the Hwa-Bjorken system was worked out in [6, 7] and the metric can be written in the following suggestive form

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\left[-\frac{\left(1-v^4\right)^2}{\left(1+v^4\right)}\mathrm{d}\tau^2 + \left(1+v^4\right)\left(\tau^2\mathrm{d}y^2 + \mathrm{d}x_\perp^2\right) + \mathrm{d}z^2\right], \tag{2.3}$$

where z is the fifth coordinate in AdS_5 and v is a scaling variable defined by

$$v \equiv \frac{z}{(\tau/\tau_0)^{\frac{1}{3}}}\,\varepsilon^{\frac{1}{4}}\,, \qquad \varepsilon \equiv \frac{1}{4}(\pi T_0)^4\,, \tag{2.4}$$

Since the horizon is located at $v = 1$ or $z \sim \tau^{1/3}$, the black hole horizon may be considered as moving away from the boundary. We remark that the solution is valid only for late times (asymptotic solution).

The metric (2.3) is written in Fefferman-Graham coordinate z. It is useful to express it in a more canonical form, that will prove usefull for a paralell calculation with the static case. Indeed, by introducing the coordinate change

$$u(z,\tau) \equiv \frac{2v^2}{1+v^4}\,, \tag{2.5}$$

the metric (2.3) is now

$$\begin{aligned}\mathrm{d}s^2 = \;& \frac{\pi^2 T_0^2 R^2}{u(\tau/\tau_0)^{2/3}}\left[-f(u)\mathrm{d}\tau^2 + \tau^2\mathrm{d}y^2 + \mathrm{d}x_\perp^2\right] + \frac{R^2}{4f(u)}\frac{\mathrm{d}u^2}{u^2}\\ &+\frac{R^2}{9}\tau^{-2}\mathrm{d}\tau^2 - \frac{R^2}{3}\frac{\tau^{-1}}{u\sqrt{f(u)}}\mathrm{d}\tau\mathrm{d}u\,,\end{aligned} \tag{2.6}$$

with $f = 1 - u^2$. We may ingnore the last two terms since the perfect fluid geometry is valid only in the scaling limit $\tau \to \infty$ and $v, u \to$ constant. A further transformation through,

$$t/t_0 \equiv \frac{3}{2}(\tau/\tau_0)^{\frac{2}{3}}\,, \tag{2.7}$$

yields

$$ds^2 = \frac{\pi^2 T_0^2 R^2}{u}\left[-f(u)dt^2 + \frac{4}{9}t^2 dy^2 + \frac{3}{2}\frac{t_0}{t}dx_\perp^2)\right] + \frac{R^2}{4f(u)}\frac{du^2}{u^2} . \tag{2.8}$$

This form is similar to the canonical black hole metic except for the time dependence of the spacial parts. Also, t and u are not the same variables as in the static case. They are related to the original variables, x^0 and z through (2.4), (2.5) and (2.7). In this transformed metric (2.8), the black hole horizon is no longer moving away in the fifth direction but is expanding in the y direction and contracting in the transverse direction as time goes on.

In the background (2.8), let us consider the small string fluctuations in the transverse direction x_1,

$$\delta X^1 = \xi(t, u) . \tag{2.9}$$

The relevant Nambu-Goto action is

$$S = \frac{T_0\sqrt{\lambda}}{8}\int_0^\infty dt \int_{-\infty}^\infty du \left(\frac{3t_0}{2t}\right)\left[\frac{(\partial_t \xi)^2}{u^{\frac{2}{3}}f(u)} - \frac{4f(u)\pi^2 T_0^2}{u^{\frac{1}{2}}}(\partial_u \xi)^2\right] , \tag{2.10}$$

where $\sqrt{\lambda} = \frac{R^2}{\alpha'}$ after subtracting the unperturbed string action. Notice that the action (2.10) is the same as the one in the static black hole metric [20] except for an overall factor of $\left(\frac{3t_0}{2t}\right)$. The latter stems from the metric component $g_{x^1x^1}$ of (2.10), when we evaluate the induced metric in the Nambu-Goto action. The equation of motion for $\xi(t, u)$ is

$$\partial_t^2 \xi - \frac{1}{t}\partial_t \xi + 2\pi^4 T_0^4 f(u)(1 + 3u^2)\partial_u \xi - 4\pi^4 T_0^4 u f(u)^2 \partial_u^2 \xi = 0 . \tag{2.11}$$

To solve (2.11) we define a Fourier-like transform

$$\xi(t, u) \equiv \int_{-\infty}^\infty \frac{d\omega}{2\pi}\sqrt{\frac{i\pi\omega}{2}}\, tH_1^{(2)}(\omega t)\Psi_\omega(u)\tilde{\xi}_0(\omega) , \tag{2.12}$$

where $H_1^{(2)}(\omega t)$ is a Hankel function of the second kind, and $\Psi_\omega(u)$ is normalized such that $\Psi_\omega(0) = 1$. $tH_1^{(2)}(\omega t)$ is chosen to satisfy the time part of the equation of motion (2.11) with the correct boundary condition (see below). Notice that we have extended the region of t from $(0, \infty)$ to $(-\infty, \infty)$.

To proceed, we assume the following 'completeness relation'

$$-\frac{1}{4}\int_{-\infty}^\infty dt\; tH_1^{(2)}(\omega t)H_1^{(2)}(-\omega' t) \simeq \frac{1}{\omega}\delta(\omega - \omega'). \tag{2.13}$$

which will be understood for small ω [1]. While this approximation blurs the rigor of the arguments to follow, it should nevertheless provide us with an insightful understanding

[1] The usual completeness of the Hankel transform is in terms of Bessel functions: $\int_0^\infty dt\; tJ_\nu(\omega t)J_\nu(\omega' t) = \frac{1}{\omega}\delta(\omega - \omega')$, whose origin is the asymptotic form of a Bessel function as an exponential function over $\sqrt{t}$. There is not true completeness for the Hankel function $H_{(1,2)}$ due to the singularity near zero. The use the completeness of the Hankel function is justified for small ω or large times since: 1)the dominant integral contribution is coming from the large time region; 2)the JP background is justified asymptotically. The use of the Hankel function instead of a Bessel function is needed to match the incoming bioundary condition below.

JHEP04(2008)047

of the time scale involved in the relaxation of the diffusion process. These time scales are paramount to our understanding of the approach to equilibrium of a strongly coupled quark-gluon plasma such as the one at RHIC.

After separating the time part, the equation for $\Psi_\omega(u)$ now reads

$$\partial_u^2\Psi_\omega(u) - \frac{3u^2+1}{2uf(u)}\partial_u\Psi_\omega(u) + \frac{\mathfrak{w}^2}{4uf(u)^2}\Psi_\omega(u) = 0\ . \tag{2.14}$$

where $\mathfrak{w} \equiv \frac{\omega}{\pi T_0}$. Notice that (2.14) is of the same form as the one in the static black hole metric [20]. Near the horizon the solution behaves as

$$\Psi_\omega \sim (1-u)^{\pm i\mathfrak{w}/4}\,, \tag{2.15}$$

and the *minus* choice corresponds to the infalling boundary condition. Inserting (2.12) into the action (2.10) yields the reduced boundary action

$$\begin{aligned} S_{\text{boundary}} &= \frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{4}\int \mathrm{d}t\ \frac{f(u)}{\sqrt{ut}}\xi(t,u)\partial_u\xi(t,u)|_{u=0}^{u=1} \\ &= \int\frac{\mathrm{d}\omega}{2\pi}\ \tilde{\xi}_0(-\omega)\left[\left(\frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{4}\right)\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0}^{u=1}\tilde{\xi}_0(\omega)\,, \end{aligned} \tag{2.16}$$

where we used (2.13). Following [23] we identify the retarded Green's function, $G_R(\omega)$, as

$$G_R(\omega) \equiv \left[-\frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{2}\right]\left[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0}. \tag{2.17}$$

3. Electric force-force decorrelation

$G_R(\omega)$ can be calculated analytically only in two limits: $\omega \to 0$ and $\omega \to \infty$. In the small ω limit, we may expand the solution in terms of $\mathfrak{w}$ and solve (2.14) order by order with the incoming boundary condition,

$$\Psi_\omega = (1-u)^{-i\mathfrak{w}/4}\left[1 + \frac{i\mathfrak{w}}{2}\left(-\tan^{-1}\sqrt{u} + \ln(1+\sqrt{u})\right)\right] + \mathcal{O}(\mathfrak{w}^2)\,, \tag{3.1}$$

which gives

$$\lim_{\omega\to 0}\left(\pi\coth\frac{\pi\mathfrak{w}}{2}\right)\mathrm{Im}\left[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0} \to 1\ . \tag{3.2}$$

In the large ω limit, we can use the WKB approximation, which yields

$$\lim_{\omega\to 0}\left(\pi\coth\frac{\pi\mathfrak{w}}{2}\right)\mathrm{Im}\left[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0} \to \frac{\pi|\mathfrak{w}|^3}{2}\,, \tag{3.3}$$

in agreement with the zero temperature result [21]. See appendix A for more details.

For general ω, we have to resort to numerical methods. We should perform the numerical computation and compare it with the analytic result in the small and large ω limits

JHEP04(2008)047

obtained above. The strategy is as follows. [2] First we find two independent solutions near the horizon ($u \sim 1$)

$$\Psi^H_{\omega,in} \equiv (1-u)^{-i\mathfrak{w}/4}\left[1-(1-u)\left(\frac{i\mathfrak{w}^2}{8i+4\mathfrak{w}}\right)\right]+\cdots\,, \tag{3.4}$$

$$\Psi^H_{\omega,out} \equiv (\Psi^H_{\omega,i})^*. \tag{3.5}$$

Here $\Psi^H_{\omega,in}$ is the infalling solution and its complex conjugate is the outgoing solution. Notice that these solutions are valid for all $\mathfrak{w}$. Near the boundary ($u \sim 0$), there are two independent solutions

$$\Psi^B_{\omega,0} \equiv u^{3/2} - \frac{\mathfrak{w}^2}{10}u^{5/2} + \left(\frac{3}{7}+\frac{\mathfrak{w}^4}{280}\right)u^{7/2}+\cdots\,, \tag{3.6}$$

$$\Psi^B_{\omega,1} \equiv 1 + \frac{\mathfrak{w}^2}{2}u - \frac{\mathfrak{w}^4}{8}u^2 + \left(\frac{\mathfrak{w}^2}{9}+\frac{\mathfrak{w}^6}{144}\right)u^3+\cdots\ . \tag{3.7}$$

Notice that $\Psi^B_{\omega,0}$ vanishes, while $\Psi^B_{\omega,1}$ goes to unity near the boundary and both solutions are real. For the retarded Green's function, we need the wave function near zero *satisfying infalling boundary condition* at the horizon. For this, we take the near-horizon wave-function (3.4) with the correct boundary condition as the initial data and numerically integrate it from the horizon to the boundary using (2.14). The solution is expressed as a linear sum of boundary basis $\Psi^B_{\omega,0}$ and $\Psi^B_{\omega,1}$

$$\Psi^H_{\omega,in}(u) \xrightarrow{(2.14)} \mathcal{A}\Psi^B_{\omega,1}(u) + \mathcal{B}\Psi^B_{\omega,0}(u)\ . \tag{3.8}$$

where $\mathcal{A}$ and $\mathcal{B}$ are complex numbers determined numerically. Notice that the right hand side goes to $\mathcal{A}$ at the boundary while we have to normalize Ψ such that it goes to 1 at $u=0$. Therefore the correctly normalized wave function with correct boundary conditions is $\Psi_\omega = \mathcal{A}^{-1}\Psi^H_{\omega,in}(u)$:

$$\Psi_\omega(u) = \Psi^B_{\omega,1}(u) + \frac{\mathcal{B}}{\mathcal{A}}\Psi^B_{\omega,0}(u), \tag{3.9}$$

which readily yields

$$\mathrm{Im}\left[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0} = \frac{3}{2}\ \mathrm{Im}\tilde{\mathcal{B}}\,, \tag{3.10}$$

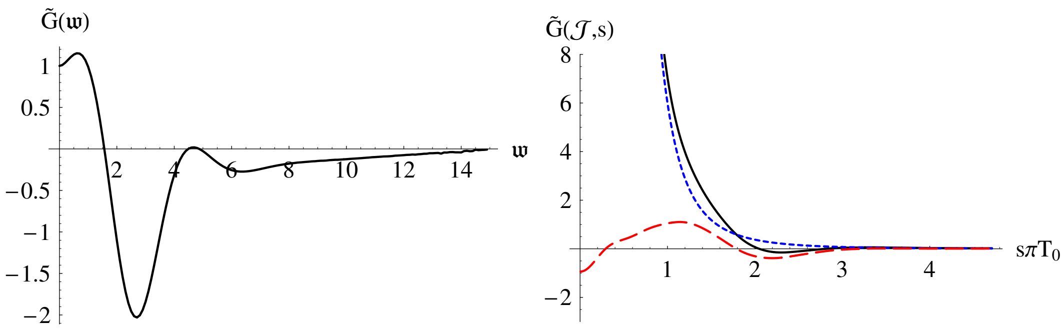
with $\tilde{\mathcal{B}} = \frac{\mathcal{B}}{\mathcal{A}}$. Now the Wightman function $G(\omega)$ (1.3) is given by

$$G(\omega) = \left[\frac{3\pi\sqrt{\lambda}T_0^3\tau_0}{2}\right]\left(\pi\coth\frac{\omega}{2T_0}\right)\left(\frac{3}{2}\ \mathrm{Im}\tilde{\mathcal{B}}(\omega)\right)\ . \tag{3.11}$$

To complete the numerical calculation, we note that while $\mathcal{A}$ is easily accessible numerically in (3.8), $\mathcal{B}$ is not. To resolve this problem, we use the following method. First,

[2] Similar calculations have been done in [25, 26, 21] in other models. Our numerical integration is different.

JHEP04(2008)047



(a) (b)

Figure 1: Force-Force decorrelator: (a) $\widetilde{G}(\mathfrak{w}) = \frac{G(\mathfrak{w}) - \frac{\pi}{2}|\mathfrak{w}^3|}{\left[\frac{3\pi\sqrt{\lambda}T_0^3\tau_0}{2}\right]}$ (b) $\widetilde{G}(\mathcal{T},s) = \frac{G(\mathcal{T},s)}{\left[\frac{3\pi^2\sqrt{\lambda}T_0^4\tau_0}{2\mathcal{T}}\right]}$ [The long dashed red line: discrete Fourier transform of (a). The short dashed blue line: the divergent contribution alone. The solid line: the total result.]

by taking the imaginary part of (3.9) we get

$$\mathrm{Im}\tilde{\mathcal{B}} = \left[\frac{\mathcal{A}^{-1}\Psi^H_{\omega,in}(u)}{\Psi^B_{\omega,0}(u)}\right], \tag{3.12}$$

and then we evaluate it at any point, say, $u = 1$. The only remaining part is the value of $\Psi^B(u)$ at $u = 1$, for which we need to numerically integrate from the boundary to the horizon. We denote the value determined by this procedure by $\Psi^B_{\omega,0}(u \xrightarrow{(2.14)} 1)$. Notice $\mathcal{A}$ is given before by $\Psi^H_{\omega,in}(u \xrightarrow{(2.14)} 0)$. Therefore we get the numerical recipe:

$$\mathrm{Im}\tilde{\mathcal{B}} = \mathrm{Im}\left[\frac{\Psi^H_{\omega,in}(u = 1-\epsilon)}{\Psi^H_{\omega,in}(u \xrightarrow{(2.14)} 0)\cdot\Psi^B_{\omega,0}(u \xrightarrow{(2.14)} 1)}\right], \tag{3.13}$$

where we take $\epsilon = 10^{-6}$.

The result for (3.11) is plotted in figure 1a, with the large $\mathfrak{w}$ asymptotic $\frac{\pi|\mathfrak{w}|^3}{2}$ subtracted. There is a good agreement asymptotically. The numerical results in time can be obtained by using the inverse transformation of (2.13) [3]

$$G(t_1,t_2) = -\frac{1}{4}\int_{-\infty}^{\infty} d\omega\omega H_1^{(2)}(\omega t_1)H_1^{(2)}(-\omega t_2)G(\omega)\ . \tag{3.14}$$

The time-dependence of the problem excludes time-translation invariance. In terms of the relative and CM coordinates

$$s \equiv t_1 - t_2\,, \quad \mathcal{T} \equiv \frac{t_1+t_2}{2}\,, \tag{3.15}$$

the force-force decorrelator is

$$G(\mathcal{T},s) \approx \frac{1}{\mathcal{T}}\frac{1}{\sqrt{1-\frac{s^2}{4\mathcal{T}^2}}}\int\frac{d\omega}{2\pi}\,e^{-i\omega s}G(\omega)\,, \tag{3.16}$$

[3] The same comments footnoted after (2.13) apply.

for $t_1, t_2 \gg 1$, which is an ordinary Fourier transform in the relative time. For $s \ll \mathcal{T}$ the asymptote $G(\omega) \sim |\mathfrak{w}|^3$, yields

$$\int e^{-i\omega s}|\omega|^3 \sim \frac{1}{s^4} \sim \frac{1}{|t_1 - t_2|^4}. \tag{3.17}$$

We note that this is a good approximation of the Hankel transformation (2.12) *for large* t_i. The result (3.17) is expected from the conformal dimension which is 4 of the force-force decorrelator. Similary for $s \sim \mathcal{T}$

$$G(\mathcal{T}, s) \sim \frac{1}{\mathcal{T}^4\sqrt{\mathcal{T}^2 - s^2/4}}\,, \tag{3.18}$$

but it is not reliable due to the nature of our approximation ($t_1, t_2 \gg 1$). In figure 1a, we show the regular part of $G(\omega)$. In figure 1b, we show its discrete Fourier transform in dashed red versus $s\pi T_0$. The short dashed blue line is the divergent contribution alone which is dominant at small relative times. The solid line is the total result. The decorrelation time follows readily from the long dashed red curve as

$$t_F \sim \frac{2}{\pi T_0}\ . \tag{3.19}$$

This result is important as it indicates that from dual AdS/CFT all electric-electric forces applied to either *heavy* or *light* fundamental probes decorrelate on a *short* time scale of the order of $2/\pi T_0$ in the static but strongly coupled QGP. We also note that the decorrelation curve in long dashed-red is stronger than exponential. This time compares favorably with the time read from the lowest quasi-normal mode $\mathfrak{w}_1^{qn}$ associated to string fluctuations (see appendix B)

$$\mathfrak{w}_1^{qn} \approx 2.69 - 2.29i. \tag{3.20}$$

This yields a decorrelation time of order $0.44/T_0$ which is comparable to our $0.64/T_0$ numerical estimate [4]

Using the relation (2.7),

$$t_F = \delta t = (\tau_0/\tau)^{1/3}\delta\tau\,, \tag{3.21}$$

this 'static' time translates to a 'moving' time

$$\delta\tau \sim \frac{(\tau/\tau_0)^{1/3}}{T_0} \equiv \frac{1}{T(\tau)}, \tag{3.22}$$

which is the natural time dependent temperature, $T(\tau)$, in agreement with Bjorken hydrodynamics [10, 6].

[4] The same value was reported in [21] for the fluctuation of the trailing string.

4. Momentum fluctuation and diffusion of a heavy quark

The short force-force decorrelation time assessed above justifies the arguments presented in section 2. Again in the time window $t_F < \Delta t < t_D$, the fluctuations in the momentum of a massive quark with $MD > 1$ read

$$\langle \Delta p(t)^2 \rangle \equiv \langle (p(t+\Delta t) - p(t))^2 \rangle \ . \tag{4.1}$$

For a quark at rest this accounts for its momentum broadening by thermal quicks. Using (3.2), then

$$\begin{aligned}
\langle \Delta p(t)^2 \rangle &= \int_t^{t+\Delta t} \mathrm{d}t_1 \int_t^{t+\Delta t} \mathrm{d}t_2 \langle F(t_1) F(t_2) \rangle \approx \int_t^{t+\Delta t} \mathrm{d}\mathcal{T} \int_{-\infty}^{\infty} \mathrm{d}s G(\mathcal{T}, s) \\
&= \ln\left(1 + \frac{\Delta t}{t}\right) \left[\frac{3\pi\sqrt{\lambda} T_0^3 t_0}{2}\right] \lim_{\mathfrak{w}\to 0} \left(\pi \coth\frac{\pi\mathfrak{w}}{2}\right) \mathrm{Im}\left[\frac{f(u)}{\sqrt{u}} \Psi_{-\omega}(u) \partial_u \Psi_{\omega}, (u)\right]_{u=0} \\
&\approx \frac{3}{2}\pi\sqrt{\lambda} T_0^3 t_0 \frac{\Delta t}{t} \ ,
\end{aligned} \tag{4.2}$$

where we have used the fact that $G(\mathcal{T}, s)$ is well localized. From (2.7), the time dependent momentum transfer is

$$\langle \Delta p(\tau)^2 \rangle = \pi\sqrt{\lambda} T_0^3 t_0 \frac{\Delta\tau}{\tau} := \kappa(\tau)\Delta\tau \ . \tag{4.3}$$

For Brownian diffusion which is the case here given the short decorrelation time of the electric force, this amounts to

$$\kappa(\tau) = \frac{\pi\sqrt{\lambda} T_0^3}{\tau/\tau_0} = \pi\sqrt{\lambda} T^3(\tau) \ . \tag{4.4}$$

with $T(\tau)$ defined in (3.22). $\kappa(\tau)$ is the time-dependent momentum diffusion constant.

Now consider the diffusion of a nonrelativistic heavy quark in the medium of which temperature is cooling down adiabatically. From above it follows that the Langevin equation captures the essentials of the equilibration in the diffusion regime. Since the medium is expanding, the appropriate description is given by

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} = -\eta_D(\tau) p(\tau) + F(\tau) \ , \quad \langle F(\tau) \rangle = 0 \ . \tag{4.5}$$

where $\eta(\tau)$ is a time-dependent drag coefficient which is related to $F(\tau)$ by

$$\eta(\tau) = \frac{1}{2MT(\tau)} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{\tau} \mathrm{d}\tau_1 \int_0^{\tau} \mathrm{d}\tau_2 \langle F(\tau_1) F(\tau_2) \rangle = \frac{\kappa(\tau)}{2MT(\tau)} \ . \tag{4.6}$$

One way to derive this is to follow [28], where all arguments are done for small time steps Δt whereby the expanding medium is frozen.

Multiplying both sides of (4.5) by $x(\tau)$ and taking the ensemble average of the product one gets an ordinary differential equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \langle x^2 \rangle + \eta(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} \langle x^2 \rangle - 2\langle v(\tau)^2 \rangle = 0 \ . \tag{4.7}$$

where the only property of $F(\tau)$ we need is $\langle x(\tau)F(\tau)\rangle = \langle x(\tau)\rangle\langle F(\tau)\rangle = 0$, which is a basic assumption of the Langevin equation. The time-dependent diffusion rate $D(\tau)$ reads

$$D(\tau) \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle x^2\rangle \ . \tag{4.8}$$

so that (4.7) is

$$\dot{D}(\tau) + \eta(\tau)D(\tau) - \langle v(\tau)^2\rangle = 0 \ . \tag{4.9}$$

To solve (4.9) we need two inputs: $\eta(\tau)$ and $\langle v(\tau)^2\rangle$. $\eta(\tau)$ is given by (4.4) and (4.6),

$$\eta(\tau) = \frac{\pi\sqrt{\lambda}T(\tau)^2}{2M} \ , \tag{4.10}$$

which is similar to the static case [17, 19, 20] except for the time dependent temperature. This confirms the adiabatic nature of the expansion in the case of short force-force decorrelations. Using the adiabatic form of the equipartition theorem, we have

$$< v(\tau)^2 >= \frac{T(\tau)}{M} \ . \tag{4.11}$$

Using (4.10) and (4.11), we now have

$$\dot{D}(\tau) + a\ \tau^{-2/3}D(\tau) - b\ \tau^{-1/3} = 0, \tag{4.12}$$

with $a = \eta_0\tau_0^{2/3}$ and $b = T_0\tau_0^{1/3}/M$. The solution is

$$D(\tau) \ = \ \frac{b}{a}\tau^{1/3} + D(0)e^{-3a\tau^{1/3}} \ .$$

This result is important as it shows how the diffusion rate for a quark changes in an expanding medium. At short times it is $D(0)$ while at large times it asymptotes

$$D(\tau) = \frac{2}{\pi\sqrt{\lambda}T(\tau)}. \tag{4.13}$$

which is the result in [20] with an adiabatically changing temperature. In a way, this justifies a posteriori the use of the 'completeness relation' (2.13) for large times $\mathcal{T}$. The cross over between short and long times is exponential and of order $3a\tau^{1/3} = 1$ in time. so (4.13) is reached for $\tau/\tau_0 = (1/3\eta_0\tau_0)^3$. At RHIC $\tau_0 \approx 1\,\mathrm{fm}$ so that $\tau/\tau_0 \approx 1/\eta_0^3$ in the Bjorken phase. Although our arguments rely on large times within the diffusion window (see above) the cross over regime can still be approached albeit from above.

5. Conclusion

We have analyzed the diffusion of a heavy quark in an expanding and strongly coupled QGP using the AdS/CFT construction. Our arguments provide some insights to a truly non-equilibrium phenomenon at strong coupling. Our analysis was restricted to an asymptotically Bjorken expanding fluid which is dual to the JP metric.

JHEP04(2008)047

In the comoving frame the time-dependent diffusion problem is mapped onto a time-independent one, whereby the diffusion is captured in a retarded Green's function with proper boundary conditions in the gravity dual space. The Green's function reflects on the electric nature of the noise in the rest frame of a massive quark. It is important to note that the noise (force-force) decorrelation is short with $t_F \sim 2/\pi T_0$ for both ligh and heavy fundamental quarks. A strongly coupled QGP randomize the electric correlations very efficiently.

If we denote by $T_D = MD/T$ the diffusion time, for times $t_F < t < t_D$ a diffusion regime for massive quarks open up whereby a generic and memoriless Langevin description holds. This desciption requires the knowledge of only two underlying moments of the phase space distribution: the average shift captured by the drag and the diffusion constant. Our construction allows the generalization of these concepts to a time-dependent Langevin description pertinent for an expanding fluid. We have found that asymptotic diffusion sets in on a time scale $\tau \sim 1/\eta_0^3$. This estimate is reached from the generic character of the Langevin description.

Finally and while we have used the construct in [3] for the retarded Green's function, it will be of some interest to check explicitly this using the arguments presented in [30].

Acknowledgments

The work of KYK and IZ was supported in part by US-DOE grants DE-FG02-88ER40388 and DE-FG03-97ER4014. The work of SJS was supported by KOSEF Grant R01-2007-000-10214-0 and also by the CQUEST with grant number R11-2005-021.

A. WKB for large $\mathfrak{w}$

In this appendix we solve (2.14) for large $\mathfrak{w}$. To use the WKB approximation in [23, 25], we need to transform (2.14) to a Schrödinger type equation. By a change of variable

$$\phi(u) \equiv \frac{\sqrt{1-u^2}}{u^{1/4}} \Psi_\omega, (u)\,, \tag{A.1}$$

(2.14) transforms to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - V\phi = 0\,, \quad V \equiv -\frac{-5 + 18u^2 + 3u^4 + 4u\mathfrak{w}^2}{16u^2(1-u^2)^2}\,, \tag{A.2}$$

where V plays the role of a potential. V will be approximated as follows,

$$\begin{aligned} u \approx 1: &\qquad -\frac{4+\mathfrak{w}^2}{16(1-u)^2}\,, \\ \text{WKB}: &\qquad -\frac{\mathfrak{w}^2}{4u(1-u^2)^2}\,, \\ u \approx 0: &\qquad \frac{5}{16u^2} - \frac{\mathfrak{w}^2}{4u}\,, \end{aligned} \tag{A.3}$$

and the corresponding solutions are

$$\begin{aligned} u\approx 1: \quad & (1-u)^{\frac{1}{2}-\frac{i\mathfrak{w}}{4}}\,, \quad (1-u)^{\frac{1}{2}+\frac{i\mathfrak{w}}{4}}\,, \\ \text{WKB}: \quad & \sqrt{\frac{2\sqrt{u}(1-u^2)}{\mathfrak{w}}}\, e^{i\frac{1}{2}(\mathfrak{w}\tan^{-1}(\sqrt{u})+\mathfrak{w}\tanh^{-1}(\sqrt{u}))}\,, \\ & \sqrt{\frac{2\sqrt{u}(1-u^2)}{\mathfrak{w}}}\, e^{-i\frac{1}{2}(\mathfrak{w}\tan^{-1}(\sqrt{u})+\mathfrak{w}\tanh^{-1}(\sqrt{u}))}\,, \\ u\approx 0: \quad & \frac{1}{u^{1/4}}e^{i\sqrt{u}\mathfrak{w}}(1-i\sqrt{u}\mathfrak{w})\,, \quad \frac{1}{u^{1/4}\mathfrak{w}^3}e^{-i\sqrt{u}\mathfrak{w}}(1+i\sqrt{u}\mathfrak{w})\,. \end{aligned} \tag{A.4}$$

Since the solutions are valid only in the limited region we need to tie them to fulfill the expected boundary conditions. There are two boundary conditions: One is the incoming boundary condition near the horizon and the other is the normalization ($\Psi_{\omega,}(0)=1$). The incoming solution near the horizon is $\Psi_{\omega,}\sim(1-u)^{-\frac{i\mathfrak{w}}{4}}$ and it corresponds to $\phi=(1-u)^{\frac{1}{2}-\frac{i\mathfrak{w}}{4}}$. It can be shown that all the first parts of the solutions are connected to each other so that the physical solution near $u=0$ is

$$\Psi_{\omega,} = e^{i\sqrt{u}\mathfrak{w}}(1-i\sqrt{u}\mathfrak{w})\,, \tag{A.5}$$

which gives us

$$\lim_{\mathfrak{w}\to 0}\text{Im}\left[\frac{f(u)}{\sqrt{u}}\Psi_{-\mathfrak{w}}(u)\partial_u\Psi_{\omega,}(u)\right]_{u=0} \to \frac{\pi\mathfrak{w}^3}{2}\,. \tag{A.6}$$

B. Quasi-normal modes

In this section we compute the lowest quasi-normal mode following the method presented in [29]. By a change of variable

$$y \equiv 1-u\,, \tag{B.1}$$

eq. (2.14) can be reduced to the Heun equation:

$$\partial_y^2\psi(y) + \frac{3(1-y)^2+1}{2y(1-y)(2-y)}\partial_y\psi(y) + \frac{\mathfrak{w}^2}{4y^2(1-y)(2-y)^2}\psi(y) = 0\,. \tag{B.2}$$

where y=0(at horizen) is a regular singular point with characteristic exponent $\{-i\frac{\mathfrak{w}}{4}, i\frac{\mathfrak{w}}{4}\}$. The quasinormal mode is the solution of (B.2) obeying the incoming boundary condition at the horizen $y=0$ and the vanishing Dirichlet boundary condition at $y=1$. So we choose the exponent $-i\frac{\mathfrak{w}}{4}$ at $y=0$. To match the boundary condition at $y=1$, it is convenient to transform (B.2) once more by,

$$\psi(y) \equiv y^{-i\frac{\mathfrak{w}}{4}}(y-2)^{-\frac{\mathfrak{w}}{4}}\chi(y)\,. \tag{B.3}$$

Then (B.2) is reduced to the standard form of the Heun equation.

$$\partial_y^2\chi(y) + \left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-2}\right)\partial_y\chi(y) + \frac{\alpha\beta z-q}{z(z-1)(z-2)}\chi(y) = 0\,, \tag{B.4}$$

where

$$\begin{aligned}
\alpha\beta &:= \frac{i\mathfrak{w}^2}{8} - \frac{\mathfrak{w}}{8}(1+i)\,, \qquad \gamma := 1 - i\frac{\mathfrak{w}}{2}\,, \\
\delta &= -\frac{1}{2}\,, \quad \epsilon = 1 - \frac{\mathfrak{w}}{2}\,, \quad q := -\frac{\mathfrak{w}^2}{8}(2-i) - \frac{\mathfrak{w}}{4}\ .
\end{aligned} \tag{B.5}$$

At $y = 0$, the local series solution corresponding to the zero characteristic exponent and normalized to 1 is given by

$$\chi_0(y) = \sum_{n=0}^{\infty} a_n(\mathfrak{w}) y^n\,, \tag{B.6}$$

where

$$\begin{aligned}
a_0 &= 1\,, \\
a_1 &= \frac{q}{2\gamma}\,, \\
a_{n+2} + A_n(\mathfrak{w})a_{n+1} + B_n(\mathfrak{w})a_n &= 0 \quad (n \geq 2)\,, \\
A_n(\mathfrak{w}) &:= -\frac{(n+1)(2\delta + \epsilon + 3(n+\gamma) + q}{2(n+2)(n+1+\gamma)}\,, \\
B_n(\mathfrak{w}) &:= \frac{n^2 + n(\gamma + \delta + \epsilon - 1) + \alpha\beta}{2(n+2)(n+1+\gamma)}\ .
\end{aligned} \tag{B.7}$$

At y=1, we get the boundary value using the series (B.6)

$$\chi_0(1) = \sum_{n=0}^{\infty} a_n(\mathfrak{w})\,, \tag{B.8}$$

To find the quasinormal modes, we need to find the zeroes of (B.8) in the complex $\mathfrak{w}$ plane. This is done by truncating the series after a large number of terms

$$|\chi_0(1)^N|^2 := |\sum_{n=0}^{N} a_n(\mathfrak{w})|^2 = 0\ . \tag{B.9}$$

We search for the minimum of $|\chi_0(1)^N|^2$, and check that the minimum value is zero.

References

[1] E.V. Shuryak, *What RHIC experiments and theory tell us about properties of quark-gluon plasma?*, *Nucl. Phys.* **A 750** (2005) 64 [hep-ph/0405066];
M.J. Tannenbaum, *Recent results in relativistic heavy ion collisions: from 'a new state of matter' to 'the perfect fluid'*, *Rev. Mod. Phys.* **69** (2006) 2005 [nucl-ex/0603003].

[2] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

JHEP04(2008)047

[3] To list some of them: G. Policastro, D.T. Son and A.O. Starinets, *The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [hep-th/0104066];
From AdS/CFT correspondence to hydrodynamics, *JHEP* **09** (2002) 043 [hep-th/0205052];
From AdS/CFT correspondence to hydrodynamics. II: sound waves, *JHEP* **12** (2002) 054 [hep-th/0210220];
H. Liu, K. Rajagopal and U.A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT*, *Phys. Rev. Lett.* **97** (2006) 182301 [hep-ph/0605178].

[4] E. Shuryak, S.J. Sin and I. Zahed, *A gravity dual of RHIC collisions*, *J. Korean Phys. Soc.* **50** (2007) 384 [hep-th/0511199].

[5] H. Nastase, *The RHIC fireball as a dual black hole*, hep-th/0501068;
O. Aharony, S. Minwalla and T. Wiseman, *Plasma-balls in large N gauge theories and localized black holes*, *Class. and Quant. Grav.* **23** (2006) 2171 [hep-th/0507219];
H. Nastase, *DBI skyrmion, high energy (large s) scattering and fireball production*, hep-th/0512171; *More on the RHIC fireball and dual black holes*, hep-th/0603176;
S. Lin and E. Shuryak, *Toward the AdS/CFT gravity dual for high energy heavy ion collisions*, hep-ph/0610168.

[6] R.A. Janik and R. Peschanski, *Asymptotic perfect fluid dynamics as a consequence of AdS/CFT*, *Phys. Rev.* **D 73** (2006) 045013 [hep-th/0512162].

[7] R.A. Janik and R. Peschanski, *Gauge/gravity duality and thermalization of a boost-invariant perfect fluid*, *Phys. Rev.* **D 74** (2006) 046007 [hep-th/0606149].

[8] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230];
K. Skenderis, *Lecture notes on holographic renormalization*, *Class. and Quant. Grav.* **19** (2002) 5849 [hep-th/0209067].

[9] R.C. Hwa, *Statistical description of hadron constituents as a basis for the fluid model of high-energy collisions*, *Phys. Rev.* **D 10** (1974) 2260.

[10] J.D. Bjorken, *Highly relativistic nucleus-nucleus collisions: the central rapidity region*, *Phys. Rev.* **D 27** (1983) 140.

[11] S. Nakamura and S.-J. Sin, *A holographic dual of hydrodynamics*, *JHEP* **09** (2006) 020 [hep-th/0607123].

[12] R.A. Janik, *Viscous plasma evolution from gravity using AdS/CFT*, *Phys. Rev. Lett.* **98** (2007) 022302 [hep-th/0610144].

[13] D. Bak and R.A. Janik, *From static to evolving geometries: R-charged hydrodynamics from supergravity*, *Phys. Lett.* **B 645** (2007) 303 [hep-th/0611304].

[14] S.-J. Sin, S. Nakamura and S.P. Kim, *Elliptic flow, Kasner universe and holographic dual of RHIC fireball*, *JHEP* **12** (2006) 075 [hep-th/0610113].

[15] J.J. Friess, S.S. Gubser, G. Michalogiorgakis and S.S. Pufu, *Expanding plasmas and quasinormal modes of Anti-de Sitter black holes*, *JHEP* **04** (2007) 080 [hep-th/0611005].

[16] S.-J. Sin and I. Zahed, *Holography of radiation and jet quenching*, *Phys. Lett.* **B 608** (2005) 265 [hep-th/0407215].

JHEP04(2008)047

[17] C.P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L.G. Yaffe, *Energy loss of a heavy quark moving through $N = 4$ supersymmetric Yang-Mills plasma*, *JHEP* **07** (2006) 013 [hep-th/0605158].

[18] H. Liu, K. Rajagopal and U.A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT*, *Phys. Rev. Lett.* **97** (2006) 182301 [hep-ph/0605178].

[19] S.S. Gubser, *Drag force in AdS/CFT*, *Phys. Rev.* **D 74** (2006) 126005 [hep-th/0605182].

[20] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled $N = 4$ Yang-Mills*, *Phys. Rev.* **D 74** (2006) 085012 [hep-ph/0605199].

[21] S.S. Gubser, *Momentum fluctuations of heavy quarks in the gauge-string duality*, *Nucl. Phys.* **B 790** (2008) 175 [hep-th/0612143].

[22] J. Casalderrey-Solana and D. Teaney, *Transverse momentum broadening of a fast quark in a $N = 4$ Yang-Mills plasma*, *JHEP* **04** (2007) 039 [hep-th/0701123].

[23] D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence: recipe and applications*, *JHEP* **09** (2002) 042 [hep-th/0205051].

[24] C.P. Herzog and D.T. Son, *Schwinger-Keldysh propagators from AdS/CFT correspondence*, *JHEP* **03** (2003) 046 [hep-th/0212072].

[25] D. Teaney, *Finite temperature spectral densities of momentum and R- charge correlators in $N = 4$ Yang-Mills theory*, *Phys. Rev.* **D 74** (2006) 045025 [hep-ph/0602044].

[26] P. Kovtun and A. Starinets, *Thermal spectral functions of strongly coupled $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **96** (2006) 131601 [hep-th/0602059].

[27] R. Kubo, *The fluctuation-dissipation theorem*, *Rept. Prog. Phys.* **29** (1966) 255.

[28] F. Reif, *Fundamentals of statistical physics*, McGraw-Hill, U.S.A. (1965), see chapter 15.

[29] G.T. Horowitz and V.E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev.* **D 62** (2000) 024027 [hep-th/9909056];
A.O. Starinets, *Quasinormal modes of near extremal black branes*, *Phys. Rev.* **D 66** (2002) 124013 [hep-th/0207133].

[30] C.P. Herzog and D.T. Son, *Schwinger-Keldysh propagators from AdS/CFT correspondence*, *JHEP* **03** (2003) 046 [hep-th/0212072].

JHEP04(2008)047