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J. Math. Anal. Appl. 337 (2008) 13–20
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# Homomorphisms in $C^*$-ternary algebras and $JB^*$-triples

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Received 8 December 2006
Available online 3 April 2007
Submitted by D. O'Regan

**Abstract**

In this paper, we investigate homomorphisms between $C^*$-ternary algebras and derivations on $C^*$-ternary algebras, and homomorphisms between $JB^*$-triples and derivations on $JB^*$-triples, associated with the following Apollonius type additive functional equation

$$f(z-x)+f(z-y)=-\frac{1}{2}f(x+y)+2f\left(z-\frac{x+y}{4}\right).$$



*Keywords:* Apollonius type additive functional equation; $C^*$-ternary algebra homomorphism; Generalized Hyers–Ulam stability; $C^*$-ternary derivation; $JB^*$-triple homomorphism; $JB^*$-triple derivation

## 1. Introduction and preliminaries

A $C^*$-ternary algebra is a complex Banach space $A$, equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of $A^3$ into $A$, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\leqslant\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see [20]).

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1 The first author was supported by grant No. F01-2006-000-10111-0 from the Korea Science & Engineering Foundation.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.03.073

If a $C^*$-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that $A$, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital $C^*$-algebra. Conversely, if $(A, \circ)$ is a unital $C^*$-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes $A$ into a $C^*$-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *$C^*$-ternary algebra homomorphism* if

$$H\big([x, y, z]\big) = \big[H(x), H(y), H(z)\big]$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *$C^*$-ternary derivation* if

$$\delta\big([x, y, z]\big) = \big[\delta(x), y, z\big] + \big[x, \delta(y), z\big] + \big[x, y, \delta(z)\big]$$

for all $x, y, z \in A$ (see [9]).

Ternary structures and their generalization, the so-called $n$-ary structures, raise certain hopes in view of their applications in physics (see [6]).

Suppose that $\mathcal{J}$ is a complex vector space endowed with a real trilinear composition $\mathcal{J} \times \mathcal{J} \times \mathcal{J} \ni (x, y, z) \mapsto \{xy^*z\} \in \mathcal{J}$ which is complex bilinear in $(x, z)$ and conjugate linear in $y$. Then $\mathcal{J}$ is called a *Jordan triple system* if $\{xy^*z\} = \{zy^*x\}$ and

$$\big\{\{xy^*z\}u^*v\big\} + \big\{\{xy^*v\}u^*z\big\} - \big\{xy^*\{zu^*v\}\big\} = \big\{z\{yx^*u\}^*v\big\}$$

hold.

We are interested in Jordan triple systems having a Banach space structure. A complex Jordan triple system $\mathcal{J}$ with a Banach space norm $\|\cdot\|$ is called a *$J^*$-triple* if, for every $x \in \mathcal{J}$, the operator $x\Box x^*$ is hermitian in the sense of Banach algebra theory. Here the operator $x\Box x^*$ on $\mathcal{J}$ is defined by $(x\Box x^*)y := \{xx^*y\}$. This implies that $x\Box x^*$ has real spectrum $\sigma(x\Box x^*) \subset \mathbb{R}$. A $J^*$-triple $\mathcal{J}$ is called a $JB^*$-triple if every $x \in \mathcal{J}$ satisfies $\sigma(x\Box x^*) \geqslant 0$ and $\|x\Box x^*\| = \|x\|^2$.

A $\mathbb{C}$-linear mapping $H : \mathcal{J} \to \mathcal{L}$ is called a *$JB^*$-triple homomorphism* if

$$H\big(\{xyz\}\big) = \big\{H(x)H(y)H(z)\big\}$$

for all $x, y, z \in \mathcal{J}$. A $\mathbb{C}$-linear mapping $\delta : \mathcal{J} \to \mathcal{J}$ is called a *$JB^*$-triple derivation* if

$$\delta\big(\{xyz\}\big) = \big\{\delta(x)yz\big\} + \big\{x\delta(y)z\big\} + \big\{xy\delta(z)\big\}$$

for all $x, y, z \in \mathcal{J}$ (see [7]).

Ulam [19] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

*We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?*

Hyers [3] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\big\|f(x + y) - f(x) - f(y)\big\| \leqslant \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\big\|f(x) - L(x)\big\| \leqslant \epsilon.$$

Th.M. Rassias [11] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

**Theorem 1.1.** *(See Th.M. Rassias [11].) Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

*exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

*for all $x \in E$. If $p < 0$ then inequality* (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is $\mathbb{R}$-linear.*

Th.M. Rassias [12] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. Gajda [1], following the same approach as in Th.M. Rassias [11], gave an affirmative solution to this question for $p > 1$. For further research developments in stability of functional equations the readers are referred to the works of Găvruta [2], Jung [5], Park [10], Th.M. Rassias [13–16], Th.M. Rassias and Šemrl [17], Skof [18] and references cited therein.

In an inner product space, the equality

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{x+y}{2}\right\|^2$$

holds, and is called the *Apollonius' identity*. The following functional equation, which was motivated by this equation,

$$Q(z - x) + Q(z - y) = \frac{1}{2}Q(x - y) + 2Q\left(z - \frac{x+y}{2}\right), \tag{1.3}$$

is quadratic. For this reason, the function equation (1.3) is called a *quadratic functional equation of Apollonius type*, and each solution of the functional equation (1.3) is said to be a *quadratic mapping of Apollonius type*. Jun and Kim [4] investigated the quadratic functional equation of Apollonius type.

In this paper, employing the above equality (1.3), we introduce a new functional equation, which is called the *Apollonius type additive functional equation* and whose solution of the functional equation is said to be the *Apollonius type additive mapping*:

$$L(z - x) + L(z - y) = -\frac{1}{2}L(x + y) + 2L\left(z - \frac{x+y}{4}\right).$$

In this paper, we investigate homomorphisms and derivations in $C^*$-ternary algebras, and homomorphisms and derivations in $JB^*$-triples.

## 2. Homomorphisms between $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\|\cdot\|_A$ and that $B$ is a $C^*$-ternary algebra with norm $\|\cdot\|_B$.

In this section, we investigate homomorphisms between $C^*$-ternary algebras.

**Lemma 2.1.** *Let $f : A \to B$ be a mapping such that*

$$\left\| f(z-x) + f(z-y) + \frac{1}{2} f(x+y) \right\|_B \leqslant \left\| 2f\left(z - \frac{x+y}{4}\right) \right\|_B \tag{2.1}$$

*for all $x, y, z \in A$. Then $f$ is additive.*

**Proof.** Letting $x = y = z = 0$ in (2.1), we get

$$\left\| \frac{5}{2} f(0) \right\|_B \leqslant \left\| 2f(0) \right\|_B.$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.1), we get

$$\left\| f(-x) + f(x) \right\|_B \leqslant \left\| 2f(0) \right\|_B = 0$$

for all $x \in A$. Hence $f(-x) = -f(x)$ for all $x \in A$.

Letting $x = y = 2z$ in (2.1), we get

$$\left\| 2f(-z) + \frac{1}{2} f(4z) \right\|_B \leqslant \left\| 2f(0) \right\|_B = 0$$

for all $z \in A$. Hence

$$f(4z) = -4f(-z) = 4f(z)$$

for all $z \in A$.

Letting $z = \frac{x+y}{4}$ in (2.1), we get

$$\left\| f\left(\frac{-3x+y}{4}\right) + f\left(\frac{x-3y}{4}\right) + \frac{1}{2} f(x+y) \right\|_B \leqslant \left\| 2f(0) \right\|_B = 0$$

for all $x, y \in A$. So

$$f\left(\frac{-3x+y}{4}\right) + f\left(\frac{x-3y}{4}\right) + \frac{1}{2} f(x+y) = 0 \tag{2.2}$$

for all $x, y \in A$. Let $w_1 = \frac{-3x+y}{4}$ and $w_2 = \frac{x-3y}{4}$ in (2.2). Then

$$f(w_1) + f(w_2) = -\frac{1}{2} f(-2w_1 - 2w_2) = \frac{1}{2} f(2w_1 + 2w_2) = 2f\left(\frac{w_1 + w_2}{2}\right)$$

for all $w_1, w_2 \in A$ and so $f$ is additive. □

**Theorem 2.2.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\left\| f(z-\mu x)+\mu f(z-y)+\frac{1}{2}f(x+y)\right\|_B \leqslant \left\|2f\left(z-\frac{x+y}{4}\right)\right\|_B, \tag{2.3}$$

$$\left\| f([x,y,z]) - [f(x), f(y), f(z)]\right\|_B \leqslant \theta\left(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\right) \tag{2.4}$$

*for all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ *and all* $x, y, z \in A$. *Then the mapping* $f : A \to B$ *is a* $C^*$-*ternary algebra homomorphism.*

**Proof.** Assume $r > 1$.

Let $\mu = 1$ in (2.3). By Lemma 2.1, the mapping $f : A \to B$ is additive.

Letting $y = -x$ and $z = 0$, we get

$$\left\| f(-\mu x)+\mu f(x)\right\|_B \leqslant \left\|2f(0)\right\|_B = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So

$$-f(\mu x)+\mu f(x) = f(-\mu x)+\mu f(x) = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of Theorem 2.1 of [8], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.4) that

$$\begin{aligned}
&\left\| f([x,y,z]) - [f(x), f(y), f(z)]\right\|_B \\
&\quad = \lim_{n\to\infty} 8^n \left\| f\left(\frac{[x,y,z]}{2^n\cdot 2^n\cdot 2^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\|_B \\
&\quad \leqslant \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}\left(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\right) = 0
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$f([x,y,z]) = [f(x), f(y), f(z)]$$

for all $x, y, z \in A$. Hence the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.

Similarly, one obtains the result for the case $r < 1$. □

## 3. Derivations on $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\|\cdot\|_A$.

In this section, we investigate derivations on $C^*$-ternary algebras.

**Theorem 3.1.** *Let* $r \neq 1$ *and* $\theta$ *be nonnegative real numbers, and let* $f : A \to A$ *be a mapping satisfying* (2.3) *such that*

$$\begin{aligned}
&\left\| f([x,y,z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\right\|_A \\
&\quad \leqslant \theta\left(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\right)
\end{aligned} \tag{3.1}$$

*for all* $x, y, z \in A$. *Then the mapping* $f : A \to A$ *is a* $C^*$-*ternary derivation.*

**Proof.** Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.1) that

$$\begin{aligned}
&\big\| f\big([x,y,z]\big) - \big[f(x),y,z\big] - \big[x,f(y),z\big] - \big[x,y,f(z)\big]\big\|_A \\
&\quad = \lim_{n\to\infty} 8^n \bigg\| f\bigg(\frac{[x,y,z]}{8^n}\bigg) - \bigg[f\bigg(\frac{x}{2^n}\bigg),\frac{y}{2^n},\frac{z}{2^n}\bigg] - \bigg[\frac{x}{2^n},f\bigg(\frac{y}{2^n}\bigg),\frac{z}{2^n}\bigg] \\
&\qquad - \bigg[\frac{x}{2^n},\frac{y}{2^n},f\bigg(\frac{z}{2^n}\bigg)\bigg]\bigg\|_A \\
&\quad \leqslant \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}\big(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}\big) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$f\big([x,y,z]\big) = \big[f(x),y,z\big] + \big[x,f(y),z\big] + \big[x,y,f(z)\big]$$

for all $x, y, z \in A$.

Thus the mapping $f : A \to A$ is a $C^*$-ternary derivation.

Similarly, one obtains the result for the case $r < 1$. □

## 4. Homomorphisms between $JB^*$-triples

Throughout this paper, assume that $\mathcal{J}$ is a $JB^*$-triple with norm $\|\cdot\|_{\mathcal{J}}$ and that $\mathcal{L}$ is a $JB^*$-triple with norm $\|\cdot\|_{\mathcal{L}}$.

In this section, we investigate homomorphisms between $JB^*$-triples.

**Theorem 4.1.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : \mathcal{J} \to \mathcal{L}$ be a mapping such that*

$$\bigg\| f(z-\mu x) + \mu f(z-y) + \frac{1}{2} f(x+y)\bigg\|_{\mathcal{L}} \leqslant \bigg\| 2f\bigg(z - \frac{x+y}{4}\bigg)\bigg\|_{\mathcal{L}}, \tag{4.1}$$

$$\big\| f\big(\{xyz\}\big) - \big\{f(x)f(y)f(z)\big\}\big\|_{\mathcal{L}} \leqslant \theta\big(\|x\|_{\mathcal{J}}^{3r} + \|y\|_{\mathcal{J}}^{3r} + \|z\|_{\mathcal{J}}^{3r}\big) \tag{4.2}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{L}$ is a $JB^*$-triple homomorphism.*

**Proof.** Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.2, the mapping $f : \mathcal{J} \to \mathcal{L}$ is $\mathbb{C}$-linear.

It follows from (4.2) that

$$\begin{aligned}
&\big\| f\big(\{xy\}\big) - \big\{f(x)f(y)f(z)\big\}\big\|_{\mathcal{L}} \\
&\quad = \lim_{n\to\infty} 8^n \bigg\| f\bigg(\frac{\{xyz\}}{2^n\cdot 2^n\cdot 2^n}\bigg) - \bigg\{f\bigg(\frac{x}{2^n}\bigg)f\bigg(\frac{y}{2^n}\bigg)f\bigg(\frac{z}{2^n}\bigg)\bigg\}\bigg\|_{\mathcal{L}} \\
&\quad \leqslant \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}\big(\|x\|_{\mathcal{J}}^{3r} + \|y\|_{\mathcal{J}}^{3r} + \|z\|_{\mathcal{J}}^{3r}\big) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. Thus

$$f\big(\{xyz\}\big) = \big\{f(x)f(y)f(z)\big\}$$

for all $x, y, z \in \mathcal{J}$. Hence the mapping $f : \mathcal{J} \to \mathcal{L}$ is a $JB^*$-triple homomorphism.

Similarly, one obtains the result for the case $r < 1$. □

## 5. Derivations on $JB^*$-triples

Throughout this paper, assume that $\mathcal{J}$ is a $JB^*$-triple with norm $\|\cdot\|_{\mathcal{J}}$.

In this section, we investigate derivations on $JB^*$-triples.

**Theorem 5.1.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : \mathcal{J} \to \mathcal{J}$ be a mapping satisfying* (4.1) *such that*

$$\begin{aligned} &\big\| f\big(\{xyz\}\big) - \big\{f(x)yz\big\} - \big\{xf(y)z\big\} - \big\{xyf(z)\big\}\big\|_{\mathcal{J}} \\ &\quad \leqslant \theta\big(\|x\|_{\mathcal{J}}^{3r} + \|y\|_{\mathcal{J}}^{3r} + \|z\|_{\mathcal{J}}^{3r}\big) \end{aligned} \tag{5.1}$$

*for all $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{J}$ is a $JB^*$-triple derivation.*

**Proof.** Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.2, the mapping $f : \mathcal{J} \to \mathcal{J}$ is $\mathbb{C}$-linear.

It follows from (5.1) that

$$\begin{aligned} &\big\| f\big(\{xyz\}\big) - \big\{f(x)yz\big\} - \big\{xf(y)z\big\} - \big\{xyf(z)\big\}\big\|_{\mathcal{J}} \\ &\quad = \lim_{n\to\infty} 8^n \left\| f\left(\frac{\{xyz\}}{8^n}\right) - \left\{ f\left(\frac{x}{2^n}\right)\frac{y}{2^n}\frac{z}{2^n}\right\} - \left\{\frac{x}{2^n} f\left(\frac{y}{2^n}\right)\frac{z}{2^n}\right\} - \left\{\frac{x}{2^n}\frac{y}{2^n} f\left(\frac{z}{2^n}\right)\right\}\right\|_{\mathcal{J}} \\ &\quad \leqslant \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}\big(\|x\|_{\mathcal{J}}^{3r} + \|y\|_{\mathcal{J}}^{3r} + \|z\|_{\mathcal{J}}^{3r}\big) = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So

$$f\big(\{xyz\}\big) = \big\{f(x)yz\big\} + \big\{xf(y)z\big\} + \big\{xyf(z)\big\}$$

for all $x, y, z \in \mathcal{J}$.

Thus the mapping $f : \mathcal{J} \to \mathcal{J}$ is a $JB^*$-triple derivation.

Similarly, one obtains the result for the case $r < 1$. □

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