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# Homomorphisms and derivations in proper $JCQ^*$-triples

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**Abstract**

In this paper, we investigate homomorphisms in proper $JCQ^*$-triples and derivations on proper $JCQ^*$-triples associated with the following functional equation

$$\frac{1}{k} f(kx + ky + kz) = f(x) + f(y) + f(z)$$

for a fixed positive integer $k$. We moreover prove the generalized Hyers–Ulam stability of homomorphisms in proper $JCQ^*$-triples and of derivations on proper $JCQ^*$-triples. This is applied to investigate isomorphisms between proper $JCQ^*$-triples.


*Keywords:* Functional equation; Generalized Hyers–Ulam stability; Proper $JCQ^*$-triple homomorphism; Proper $JCQ^*$-triple derivation

## 1. Introduction and preliminaries

As it is extensively discussed in [47], the full description of a physical system $\mathcal{S}$ implies the knowledge of three basic ingredients: the set of the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given state. Originally the set of the observables was considered to be a $C^*$-algebra [27]. In many applications, however, this was shown not to be the most convenient choice and the $C^*$-algebra was replaced by a von Neumann algebra, because the role of the representation turns out to be crucial mainly when long range interactions are involved (see [10] and references therein). Here we use a different algebraic structure, similar to the one considered in [22], which is suggested by the considerations above: because of the relevance of the unbounded operators in the description of $\mathcal{S}$, we will assume that the observables of the system belong to a quasi $*$-algebra $(A, A_0)$ (see [51] and references therein), while, in order to have a richer mathematical structure, we will use a slightly different algebraic structure: $(A, A_0)$ will be assumed to be a proper $CQ^*$-algebra, which has nicer topological properties. In particular, for instance, $A_0$ is a $C^*$-algebra.

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Let $A$ be a linear space and $A_0$ is a $*$-algebra contained in $A$ as a subspace. We say that $A$ is a quasi $*$-algebra over $A_0$ if

(i) the right and left multiplications of an element of $A$ and an element of $A_0$ are defined and linear;
(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;
(iii) an involution $*$, which extends the involution of $A_0$, is defined in $A$ with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi $*$-algebra $(A, A_0)$ is said to be a locally convex quasi $*$-algebra if in $A$ a locally convex topology $\tau$ is defined such that

(i) the involution is continuous and the multiplications are separately continuous;
(ii) $A_0$ is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra $(A[\tau], A_0)$ is complete. For an overview on partial $*$-algebra and related topics we refer to [2].

In a series of papers [6,14,16,17], many authors have considered a special class of quasi $*$-algebras, called *proper $CQ^*$-algebras*, which arise as completions of $C^*$-algebras. They can be introduced in the following way:

Let $A$ be a right Banach module over the $C^*$-algebra $A_0$ with involution $*$ and $C^*$-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that $(A, A_0)$ is a *proper $CQ^*$-algebra* if

(i) $A_0$ is dense in $A$ with respect to its norm $\|\cdot\|$;
(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;
(iii) $\|y\|_0 = \sup_{a\in A, \|a\|\leqslant 1} \|ay\|$ for all $y \in A_0$.

Several mathematician have contributed works on these subjects (see [1,3–9,11–15,18,19,23,24,31–33,49, 50,52,53]).

Ulam [54] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

*We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?*

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.

Hyers [28] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leqslant \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\big\|f(x) - L(x)\big\| \leqslant \epsilon.$$

Th.M. Rassias [40] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

**Theorem 1.1** *(Th.M. Rassias). Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leqslant \epsilon\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

*exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

*for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is linear.*

Th.M. Rassias [41] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. Gajda [25] gave an affirmative solution to this question for $p > 1$. The inequality (1.1) provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *generalized Hyers–Ulam stability* of functional equations (see [20,21,29]). Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (see [26,30,34–39,42–46,48]).

**Definition 1.2.** A proper $CQ^*$-algebra $(A, A_0)$, endowed with the Jordan triple product

$$\{z, x, w\} = \frac{1}{2}\big(zx^*w + wx^*z\big)$$

for all $x \in A$ and all $z, w \in A_0$, is called a *proper $JCQ^*$-triple*, and denoted by $(A, A_0, \{\cdot,\cdot,\cdot\})$.

**Definition 1.3.** Let $(A, A_0\{\cdot,\cdot,\cdot\})$ and $(B, B_0\{\cdot,\cdot,\cdot\})$ be proper $JCQ^*$-triples.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper $JCQ^*$-triple homomorphism* if $H(z) \in B_0$ and

$$H\big(\{z, x, w\}\big) = \big\{H(z), H(x), H(w)\big\}$$

for all $z, w \in A_0$ and all $x \in A$. If, in addition, the mapping $H : A \to B$ and the mapping $H|_{A_0} : A_0 \to B_0$ are bijective, then the mapping $H : A \to B$ is called a *proper $JCQ^*$-triple isomorphism*.

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *proper $JCQ^*$-triple derivation* if

$$\delta\big(\{w_0, w_1, w_2\}\big) = \big\{\delta(w_0), w_1, w_2\big\} + \big\{w_0, \delta(w_1), w_2\big\} + \big\{w_0, w_1, \delta(w_2)\big\}$$

for all $w_0, w_1, w_2 \in A_0$.

Throughout this paper, assume that $k$ is a fixed positive integer.

This paper is organized as follows: In Sections 2 and 3, we investigate homomorphisms and derivations in proper $JCQ^*$-triples associated with the functional equation

$$\frac{1}{k} f(kx + ky + kz) = f(x) + f(y) + f(z).$$

In Sections 4 and 6, we prove the generalized Hyers–Ulam stability of homomorphisms and of derivations in proper $JCQ^*$-triples. In Section 5, we investigate isomorphisms between proper $JCQ^*$-triples.

## 2. Homomorphisms in proper $JCQ^*$-triples

Throughout this section, assume that $(A, A_0, \{\cdot,\cdot,\cdot\})$ is a proper $JCQ^*$-triple with $C^*$-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that $(B, B_0, \{\cdot,\cdot,\cdot\})$ is a proper $JCQ^*$-triple with $C^*$-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

**Proposition 2.1.** *Let $X$ and $Y$ be normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $f : X \to Y$ be a mapping such that*

$$\big\|f(x) + f(y) + f(z)\big\|_Y \leqslant \left\|\frac{1}{k} f(kx + ky + kz)\right\|_Y \tag{2.1}$$

*for all $x, y, z \in X$. Then $f$ is Cauchy additive.*

**Proof.** Letting $x = y = z = 0$ in (2.1), we get

$$\|3f(0)\|_Y \leqslant \left\|\frac{1}{k}f(0)\right\|_Y.$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.1), we get

$$\|f(x) + f(-x)\|_Y \leqslant \left\|\frac{1}{k}f(0)\right\|_Y = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (2.1), we get

$$\|f(x) + f(y) - f(x+y)\|_Y = \|f(x) + f(y) + f(-x-y)\|_Y \leqslant \left\|\frac{1}{k}f(0)\right\|_Y = 0$$

for all $x, y \in X$. Thus

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

We investigate homomorphisms in proper $JCQ^*$-triples.

**Theorem 2.2.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|\mu f(x) + f(y) + f(z)\|_B \leqslant \left\|\frac{1}{k}f(k\mu x + ky + kz)\right\|_B, \tag{2.2}$$

$$\|f(\{w_0, x, w_1\}) - \{f(w_0), f(x), f(w_1)\}\|_B \leqslant \theta\left(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}\right) \tag{2.3}$$

*for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$, all $w_0, w_1 \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism.*

**Proof.** Let $\mu = 1$ in (2.2). By Proposition 2.1, the mapping $f : A \to B$ is Cauchy additive.

Letting $z = 0$ and $y = -\mu x$ in (2.2), we get

$$\mu f(x) - f(\mu x) = \mu f(x) + f(-\mu x) = 0$$

for all $x \in A$. So $f(\mu x) = \mu f(x)$ for all $x \in A$. By the same reasoning as in the proof of Theorem 2.1 of [35], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.3),

$$\begin{aligned}
&\|f(\{w_0, x, w_1\}) - \{f(w_0), f(x), f(w_1)\}\|_B \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n}\|f(8^n\{w_0, x, w_1\}) - \{f(2^n w_0), f(2^n x), f(2^n w_1)\}\|_B \\
&\quad \leqslant \lim_{n\to\infty} \frac{8^{nr}}{8^n}\theta\left(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}\right) = 0
\end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. So

$$f(\{w_0, x, w_1\}) = \{f(w_0), f(x), f(w_1)\}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(\{w_0, x, w_1\}) = \{f(w_0), f(x), f(w_1)\}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

Since $f(w) \in B_0$ for all $w \in A_0$, the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism, as desired. □

**Theorem 2.3.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.2) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\left\| f\left(\{w_0, x, w_1\}\right) - \left\{f(w_0), f(x), f(w_1)\right\}\right\|_B \leqslant \theta \cdot \|w_0\|_A^r \cdot \|x\|_A^r \cdot \|w_1\|_A^r \tag{2.4}$$

*for all $w_0, w_1 \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism.*

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.4),

$$\begin{aligned}
&\left\| f\left(\{w_0, x, w_1\}\right) - \left\{f(w_0), f(x), f(w_1)\right\}\right\|_B \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n} \left\| f\left(8^n\{w_0, x, w_1\}\right) - \left\{f\left(2^n w_0\right), f\left(2^n x\right), f\left(2^n w_1\right)\right\}\right\|_B \\
&\quad \leqslant \lim_{n\to\infty} \frac{8^{nr}}{8^n}\theta \cdot \|w_0\|_A^r \cdot \|x\|_A^r \cdot \|w_1\|_A^r = 0
\end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. So

$$f\left(\{w_0, x, w_1\}\right) = \left\{f(w_0), f(x), f(w_1)\right\}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f\left(\{w_0, x, w_1\}\right) = \left\{f(w_0), f(x), f(w_1)\right\}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism. □

## 3. Derivations on proper $JCQ^*$-triples

Throughout this section, assume that $(A, A_0, \{\cdot,\cdot,\cdot\})$ is a proper $JCQ^*$-triple with $C^*$-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$. We investigate derivations on proper $JCQ^*$-triples.

**Theorem 3.1.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A_0 \to A$ a mapping such that*

$$\left\|\mu f(x) + f(y) + f(z)\right\|_A \leqslant \left\|\frac{1}{k} f(k\mu x + ky + kz)\right\|_A, \tag{3.1}$$

$$\begin{aligned}
&\left\| f\left(\{w_0, w_1, w_2\}\right) - \left\{f(w_0), w_1, w_2\right\} - \left\{w_0, f(w_1), w_2\right\} - \left\{w_0, w_1, f(w_2)\right\}\right\|_A \\
&\quad \leqslant \theta\left(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}\right)
\end{aligned} \tag{3.2}$$

*for all $\mu \in \mathbb{T}^1$ and all $w_0, w_1, w_2, x, y, z \in A_0$. Then the mapping $f : A_0 \to A$ is a proper $JCQ^*$-triple derivation.*

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A_0 \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.2),

$$\begin{aligned}
&\left\| f\left(\{w_0, w_1, w_2\}\right) - \left\{f(w_0), w_1, w_2\right\} - \left\{w_0, f(w_1), w_2\right\} - \left\{w_0, w_1, f(w_2)\right\}\right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n} \left\| f\left(8^n\{w_0, w_1, w_2\}\right) - \left\{f\left(2^n w_0\right), 2^n w_1, 2^n w_2\right\} - \left\{2^n w_0, f\left(2^n w_1\right), 2^n w_2\right\} \right. \\
&\qquad \left. - \left\{2^n w_0, 2^n w_1, f\left(2^n w_2\right)\right\}\right\|_A \\
&\quad \leqslant \lim_{n\to\infty} \frac{8^{nr}}{8^n}\theta\left(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}\right) = 0
\end{aligned}$$

for all $w_0, w_1, w_2 \in A_0$. So

$$f\left(\{w_0, w_1, w_2\}\right) = \left\{f(w_0), w_1, w_2\right\} + \left\{w_0, f(w_1), w_2\right\} + \left\{w_0, w_1, f(w_2)\right\}$$

for all $w_0, w_1, w_2 \in A_0$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f\big(\{w_0, w_1, w_2\}\big) = \big\{f(w_0), w_1, w_2\big\} + \big\{w_0, f(w_1), w_2\big\} + \big\{w_0, w_1, f(w_2)\big\}$$

for all $w_0, w_1, w_2 \in A_0$.

Therefore, the mapping $f : A_0 \to A$ is a proper $JCQ^*$-triple derivation. □

**Theorem 3.2.** *Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A_0 \to A$ a mapping satisfying* (3.1) *such that*

$$\begin{aligned}&\big\|f\big(\{w_0, w_1, w_2\}\big) - \big\{f(w_0), w_1, w_2\big\} - \big\{w_0, f(w_1), w_2\big\} - \big\{w_0, w_1, f(w_2)\big\}\big\|_A \\ &\quad \leqslant \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r \cdot \|w_2\|_A^r \end{aligned} \tag{3.3}$$

*for all $w_0, w_1, w_2 \in A_0$. Then the mapping $f : A_0 \to A$ is a proper $JCQ^*$-triple derivation.*

**Proof.** The proof is similar to the proofs of Theorems 2.3 and 3.1. □

## 4. Stability of homomorphisms in proper $JCQ^*$-triples

We prove the generalized Hyers–Ulam stability of homomorphisms in proper $JCQ^*$-triples.

**Theorem 4.1.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that $f(w) \in B_0$ for all $w \in A_0$ and*

$$\left\|\frac{1}{k} f(\mu kx + \mu ky + \mu kz) - \mu f(x) - \mu f(y) - \mu f(z)\right\|_B \leqslant \theta\big(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\big), \tag{4.1}$$

$$\left\|\frac{1}{k} f(kw_0 + kw_1 + kw_2) - f(w_0) - f(w_1) - f(w_2)\right\|_B \leqslant \theta\big(\|w_0\|_{A_0}^r + \|w_1\|_{A_0}^r + \|w_2\|_{A_0}^r\big), \tag{4.2}$$

$$\big\|f\big(\{w_0, x, w_1\}\big) - \big\{f(w_0), f(x), f(w_1)\big\}\big\|_B \leqslant \theta\big(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}\big) \tag{4.3}$$

*for all $\mu \in \mathbb{T}^1$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper $JCQ^*$-triple homomorphism $H : A \to B$ such that*

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{3k\theta}{(3k)^r - 3k}\|x\|_A^r \tag{4.4}$$

*for all $x \in A$.*

**Proof.** Let us assume $\mu = 1$ and $x = y = z$ in (4.1). Then we get

$$\left\|\frac{1}{k} f(3kx) - 3f(x)\right\|_B \leqslant 3\theta\|x\|_A^r \tag{4.5}$$

for all $x \in A$. So

$$\left\|f(x) - 3kf\left(\frac{x}{3k}\right)\right\|_B \leqslant \frac{3k\theta}{(3k)^r}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned}\left\|(3k)^l f\left(\frac{x}{(3k)^l}\right) - (3k)^m f\left(\frac{x}{(3k)^m}\right)\right\|_B &\leqslant \sum_{j=l}^{m-1}\left\|(3k)^j f\left(\frac{x}{(3k)^j}\right) - (3k)^{j+1} f\left(\frac{x}{(3k)^{j+1}}\right)\right\|_B \\ &\leqslant \frac{3k\theta}{(3k)^r}\sum_{j=l}^{m-1}\frac{(3k)^j}{(3k)^{rj}}\|x\|_A^r \end{aligned} \tag{4.6}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. From this it follows that the sequence $\{(3k)^n f(\frac{x}{(3k)^n})\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\{(3k)^n f(\frac{x}{(3k)^n})\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} (3k)^n f\left(\frac{x}{(3k)^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.6), we get (4.4).

It follows from (4.1) that

$$\begin{aligned}
&\left\|\frac{1}{k}H(\mu kx + \mu ky + \mu kz) - \mu H(x) - \mu H(y) - \mu H(z)\right\|_B \\
&\quad = \lim_{n\to\infty} (3k)^n \left\|\frac{1}{k}f\left(\frac{\mu kx + \mu ky + \mu kz}{(3k)^n}\right) - \mu f\left(\frac{x}{(3k)^n}\right) - \mu f\left(\frac{y}{(3k)^n}\right) - \mu f\left(\frac{z}{(3k)^n}\right)\right\|_B \\
&\quad \leqslant \lim_{n\to\infty} \frac{(3k)^n\theta}{(3k)^{nr}}\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. So

$$\frac{1}{k}H(\mu kx + \mu ky + \mu kz) = \mu H(x) + \mu H(y) + \mu H(z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By the same reasoning as in the proof of Theorem 2.1 of [35], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Now, let $T : A \to B$ be another additive mapping satisfying (4.4). Then we have

$$\begin{aligned}
\left\|H(x) - T(x)\right\|_B &= (3k)^n \left\|H\left(\frac{x}{(3k)^n}\right) - T\left(\frac{x}{(3k)^n}\right)\right\|_B \\
&\leqslant (3k)^n \left(\left\|H\left(\frac{x}{(3k)^n}\right) - f\left(\frac{x}{(3k)^n}\right)\right\|_B + \left\|T\left(\frac{x}{(3k)^n}\right) - f\left(\frac{x}{(3k)^n}\right)\right\|_B\right) \\
&\leqslant \frac{6k(3k)^n\theta}{(3k)^{nr}((3k)^r - 3k)}\|x\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of $H$.

It follows from (4.2) that $H(w) = \lim_{n\to\infty}(3k)^n f(\frac{w}{(3k)^n}) \in B_0$ for all $w \in A_0$. So it follows from (4.3) that

$$\begin{aligned}
&\left\|H(\{w_0, x, w_1\}) - \{H(w_0), H(x), H(w_1)\}\right\|_B \\
&\quad = \lim_{n\to\infty} (3k)^{3n} \left\|f\left(\frac{\{w_0, x, w_1\}}{(3k)^{3n}}\right) - \left\{f\left(\frac{w_0}{(3k)^n}\right), f\left(\frac{x}{(3k)^n}\right), f\left(\frac{w_1}{(3k)^n}\right)\right\}\right\|_B \\
&\quad \leqslant \lim_{n\to\infty} \frac{(3k)^{3n}}{(3k)^{3nr}}\theta\left(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}\right) = 0
\end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. So

$$H(\{w_0, x, w_1\}) = \{H(w_0), H(x), H(w_1)\}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

Thus the mapping $H : A \to B$ is a unique proper $JCQ^*$-triple homomorphism satisfying (4.4), as desired. □

**Theorem 4.2.** *Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (4.1), (4.2) *and* (4.3) *such that $f(w) \in B_0$ for all $w \in A_0$. Then there exists a unique proper $JCQ^*$-triple homomorphism $H : A \to B$ such that*

$$\left\|f(x) - H(x)\right\|_B \leqslant \frac{3k\theta}{3k - (3k)^r}\|x\|_A^r \tag{4.7}$$

*for all $x \in A$.*

**Proof.** It follows from (4.5) that

$$\left\| f(x) - \frac{1}{3k} f(3kx) \right\|_B \leqslant \theta \|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\| \frac{1}{(3k)^l} f\big((3k)^l x\big) - \frac{1}{(3k)^m} f\big((3k)^m x\big) \right\|_B &\leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{(3k)^j} f\big((3k)^j x\big) - \frac{1}{(3k)^{j+1}} f\big((3k)^{j+1} x\big) \right\|_B \\ &\leqslant \sum_{j=l}^{m-1} \frac{(3k)^{jr}}{(3k)^j} \theta \|x\|_A^r \end{aligned} \tag{4.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. From this it follows that the sequence $\{\frac{1}{(3k)^n} f((3k)^n x)\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\{\frac{1}{(3k)^n} f((3k)^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{(3k)^n} f\big((3k)^n x\big)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

## 5. Isomorphisms between proper *JCQ**-triples

We investigate isomorphisms between proper $JCQ^*$-triples.

**Theorem 5.1.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (4.1) *and* (4.2) *such that*

$$f\big(\{w_0, x, w_1\}\big) = \big\{f(w_0), f(x), f(w_1)\big\} \tag{5.1}$$

*for all $w_0, w_1 \in A_0$ and all $x \in A$. If $\lim_{n\to\infty} (3k)^n f(\frac{e}{(3k)^n}) = e'$ and $f|_{A_0} : A_0 \to B_0$ is bijective, then the mapping $f : A \to B$ is a proper $JCQ^*$-triple isomorphism.*

**Proof.** By the same reasoning as in the proof of Theorem 4.1, there is a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (4.4). The mapping $H : A \to B$ is given by

$$H(x) := \lim_{n\to\infty} (3k)^n f\left(\frac{x}{(3k)^n}\right)$$

for all $x \in A$.

Since $f(\{w_0, x, w_1\}) = \{f(w_0), f(x), f(w_1)\}$ for all $w_0, w_1 \in A_0$ and all $x \in A$,

$$\begin{aligned} H\big(\{w_0, x, w_1\}\big) &= \lim_{n\to\infty} (3k)^{3n} \left\{ f\left(\frac{w_0}{(3k)^n}\right), f\left(\frac{x}{(3k)^n}\right), f\left(\frac{w_1}{(3k)^n}\right) \right\} \\ &= \lim_{n\to\infty} \left\{ (3k)^n f\left(\frac{w_0}{(3k)^n}\right), (3k)^n f\left(\frac{x}{(3k)^n}\right), (3k)^n f\left(\frac{w_1}{(3k)^n}\right) \right\} \\ &= \big\{H(w_0), H(x), H(w_1)\big\} \end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

It follows from (4.2) that $H(w) = \lim_{n\to\infty} (3k)^n f(\frac{w}{(3k)^n}) \in B_0$ for all $w \in A_0$. So the mapping $H : A \to B$ is a proper $JCQ^*$-triple homomorphism.

By the assumption,

$$H(x) = H(ex) = \lim_{n\to\infty}(3k)^n f\left(\frac{ex}{(3k)^n}\right) = \lim_{n\to\infty}(3k)^n f\left(\frac{e}{(3k)^n}x\right) = \lim_{n\to\infty}(3k)^n f\left(\frac{e}{(3k)^n}\right)f(x)$$
$$= e' f(x) = f(x)$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a proper $JCQ^*$-triple isomorphism. □

**Theorem 5.2.** *Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (4.1), (4.2) *and* (5.1). *If* $\lim_{n\to\infty}\frac{1}{(3k)^n}f((3k)^n e) = e'$ *and* $f|_{A_0} : A_0 \to B_0$ *is bijective, then the mapping* $f : A \to B$ *is a proper $JCQ^*$-triple isomorphism.*

**Proof.** By the same reasoning as in the proof of Theorem 4.2, there is a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (4.7). The mapping $H : A \to B$ is given by

$$H(x) := \lim_{n\to\infty}\frac{1}{(3k)^n}f\big((3k)^n x\big)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.1. □

## 6. Stability of derivations on proper *JCQ**-triples

We prove the generalized Hyers–Ulam stability of derivations on proper $JCQ^*$-triples.

**Theorem 6.1.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A_0 \to A$ be a mapping such that*

$$\left\|\frac{1}{k}f(\mu kx + \mu ky + \mu kz) - \mu f(x) - \mu f(y) - \mu f(z)\right\|_A \leqslant \theta\big(\|x\|_{A_0}^r + \|y\|_{A_0}^r + \|z\|_{A_0}^r\big), \tag{6.1}$$

$$\big\|f\big(\{w_0, w_1, w_2\}\big) - \big\{f(w_0), w_1, w_2\big\} - \big\{w_0, f(w_1), w_2\big\} - \big\{w_0, w_1, f(w_2)\big\}\big\|_A$$
$$\leqslant \theta\big(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}\big) \tag{6.2}$$

*for all $\mu \in \mathbb{T}^1$ and all $w_0, w_1, w_2, x, y, z \in A_0$. Then there exists a unique proper $JCQ^*$-triple derivation $\delta : A_0 \to A$ such that*

$$\big\|f(w) - \delta(w)\big\|_A \leqslant \frac{3k\theta}{(3k)^r - 3k}\|w\|_{A_0}^r \tag{6.3}$$

*for all $w \in A_0$.*

**Proof.** By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ satisfying (6.3). The mapping $\delta : A_0 \to A$ is defined by

$$\delta(w) := \lim_{n\to\infty}(3k)^n f\left(\frac{w}{(3k)^n}\right)$$

for all $w \in A_0$.

It follows from (6.2) that

$$\big\|\delta\big(\{w_0, w_1, w_2\}\big) - \big\{\delta(w_0), w_1, w_2\big\} - \big\{w_0, \delta(w_1), w_2\big\} - \big\{w_0, w_1, \delta(w_2)\big\}\big\|_A$$
$$= \lim_{n\to\infty}(3k)^{3n}\left\|f\left(\frac{\{w_0, w_1, w_2\}}{(3k)^{3n}}\right) - \left\{f\left(\frac{w_0}{(3k)^n}\right), \frac{w_1}{(3k)^n}, \frac{w_2}{(3k)^n}\right\} - \left\{\frac{w_0}{(3k)^n}, f\left(\frac{w_1}{(3k)^n}\right), \frac{w_2}{(3k)^n}\right\}\right.$$
$$\left. - \left\{\frac{w_0}{(3k)^n}, \frac{w_1}{(3k)^n}, f\left(\frac{w_2}{(3k)^n}\right)\right\}\right\|_A$$
$$\leqslant \lim_{n\to\infty}\frac{(3k)^{3n}}{(3k)^{3nr}}\theta\big(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}\big) = 0$$

for all $w_0, w_1, w_2 \in A_0$. So

$$\delta(\{w_0, w_1, w_2\}) = \{\delta(w_0), w_1, w_2\} + \{w_0, \delta(w_1), w_2\} + \{w_0, w_1, \delta(w_2)\}$$

for all $w_0, w_1, w_2 \in A_0$.

Thus the mapping $\delta : A_0 \to A$ is a unique proper $JCQ^*$-triple derivation satisfying (6.3), as desired. □

**Theorem 6.2.** *Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A_0 \to A$ be a mapping satisfying* (6.1) *and* (6.2). *Then there exists a unique proper $JCQ^*$-triple derivation $\delta : A_0 \to A$ such that*

$$\left\| f(w) - \delta(w) \right\|_A \leqslant \frac{3k\theta}{3k - (3k)^r} \|w\|_{A_0}^r \tag{6.4}$$

*for all $w \in A_0$.*

**Proof.** The proof is similar to the proofs of Theorems 4.1, 4.2 and 6.1. □

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