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NEARLY TERNARY DERIVATIONS

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Abstract. Let A be a normed algebra and X a normed A-bimodule. By a ternary derivation we mean a triple (D_1, D_2, D_3) of linear mappings $D_1, D_2, D_3 : A \to X$ such that $D_1(ab) = D_2(a)b + aD_3(b)$ for all $a, b \in A$. Our aim is to establish the stability of ternary derivations associated with the extended Jensen functional equation

$$qf(\frac{\sum_{k=1}^{q} x_k}{q}) = \sum_{k=1}^{q} f(x_k)$$

for all $x_1, \cdots, x_q \in A$, where $q > 1$ is a fixed positive integer.

1. INTRODUCTION

One of interesting questions in the theory of functional equations is the following:

> When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?

If there exists an affirmative answer we say that the functional equation $\mathcal{E}$ is *stable*. In 1940, the stability problem concerning group homomorphisms was raised by Ulam [15]. The Ulam's problem was partially solved by Hyers in 1941 in the context of Banach spaces. In 1951, Bourgin [4] treated the Ulam problem for additive mappings. In 1978, Th.M. Rassias [13] succeeded to extend the theorem of Hyers by introducing the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \ (\varepsilon > 0, p \in [0, 1)).$$

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The result of Th.M. Rassias has provided a lot of influence in the development of what is known as *Hyers-Ulam-Rassias stability* of functional equations. In 1992, a generalization of the Th.M. Rassias' theorem, the so-called *generalized Hyers-Ulam-Rassias stability*, was obtained by Găvruta [6]. During the last decades several stability problems of functional equations have been investigated by many mathematicians; see [5, 8, 9, 14] and references therein for more detailed information. Some results concerning stability derivations on ternary structures may be found in [11, 12].

Let A be a normed algebra and X a normed A-bimodule. By a ternary derivation we mean a triple (D_1, D_2, D_3) of linear mappings $D_1, D_2, D_3 : A \to X$ such that

$$D_1(ab) = D_2(a)b + aD_3(b) \tag{1.1}$$

for all $a, b \in A$. Following [10], if $D_1 = D_2$ and D_3 is a usual derivation, we say that $D_1 = D_2$ is a D_3-derivation. In particular, if $D_1 = D_2 = D_3$, then D_1 is a usual derivation. Our aim is to establish the generalized Hyers-Ulam-Rassias stability of ternary derivations associated with the extended Jensen functional equation

$$qf\Big(\frac{\sum_{k=1}^{q} x_k}{q}\Big) = \sum_{k=1}^{q} f(x_k) \tag{1.2}$$

for all $x_1, \cdots, x_q \in A$, where $q > 1$ is a fixed positive integer and $f : A \to X$ is a mapping. If $q = 2$, we get the classical Jensen functional equation. There are other generalizations of the Jensen functional equation and some interesting results on their stability; see e.g. [2].

A mapping $f : A \to X$ with $f(0) = 0$ satisfies an extended Jensen functional equation if and only if it is Cauchy additive. To see this, let $x \in A$ be given. Letting $x_1 = x, x_2 = \cdots = x_q = 0$ in (1.2), we get

$$qf\Big(\frac{x}{q}\Big) = f(x).$$

Given $x, y \in A$, replacing x_1 and x_2 by x and y, respectively, and setting $x_k = 0 \quad (3 \leq k \leq q)$ in (1.2), we get

$$f(x+y) = qf\Big(\frac{x+y+0+\cdots+0}{q}\Big) = f(x)+f(y)+f(0)+\cdots+f(0) = f(x)+f(y).$$

Therefore, $f : A \to X$ is Cauchy additive. The converse is trivial.

In addition, we introduce the notion of nearly ternary derivation and prove a superstability of ternary derivations under some conditions.

Throughout this paper, A and X denote a normed algebra and a Banach A-bimodule, respectively. We assume that $q > 1$ is a given positive integer. For the sake of convenience we use the following notation

$$\Delta f(\lambda, x_1, \ldots, x_q) = qf(\frac{\sum_{k=1}^{q} \lambda x_k}{q}) - \sum_{k=1}^{q} \lambda f(x_k)$$

for all $x_1, \cdots, x_q \in A$, where λ is a scalar.

2. Main Results

In this section, we prove the generalized Hyers-Ulam-Rassias stability of ternary derivations associated with the extended Jensen functional equation (1.2) by using the "direct method"; see [7].

Theorem 2.1. *Let* $\varphi : A^q \to [0, \infty)$ *be a function fulfilling*

$$\widetilde{\varphi}(x_1, \cdots, x_q) := \sum_{j=0}^{\infty} q^j \varphi(q^{-j}x_1, \cdots, q^{-j}x_q) < \infty$$

for all $x_1, \cdots, x_q \in A$*, and let* $f_1, f_2, f_3 : A \to X$ *be mappings satisfying*

$$\begin{aligned} &f_i(0) = 0 \quad (1 \le i \le 3) \\ &\|\Delta f_i(\lambda, x_1, \cdots, x_q)\| \le \varphi(x_1, \cdots, x_q) \quad (1 \le i \le 3) \end{aligned} \tag{2.1}$$

and

$$\|f_1(ab) - f_2(a)b - af_3(b)\| \le \varphi(a, b, 0, \cdots, 0) \tag{2.2}$$

for all $a, b, x_1, \cdots, x_q \in A$ *and all* $\lambda \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$*. Then there exists a unique ternary derivation* (D_1, D_2, D_3) *such that*

$$\|f_i(a) - D_i(a)\| \le \widetilde{\varphi}(a, 0, \cdots, 0) \quad (1 \le i \le 3) \tag{2.3}$$

for all $a \in A$.

Proof. We use the Găvruta's method [6]. Fix $1 \le i \le 3$.

Given $a \in A$, putting $\lambda = 1$, $x_1 = a$ and $x_j = 0$ for all $2 \le j \le q$ in **(??)**, we get

$$\|qf_i(\frac{a}{q}) - f_i(a)\| \le \varphi(a, 0, \cdots, 0).$$

One may use the induction to obtain

$$\|q^j f_i(q^{-j}a) - q^\ell f_i(q^{-\ell}a)\| \leq \sum_{m=\ell}^{j-1} q^m \varphi(a, 0, \cdots, 0), \tag{2.4}$$

for all integers $j > \ell \geq 0$ and all $a \in A$. It follows that the sequence $\{q^j f_i(q^{-j}a)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since X is a Banach space, the sequence $\{q^j f_i(q^{-j}a)\}_{j\in\mathbb{N}}$ converges. Define the mapping $D_i : A \to X$ by

$$D_i(a) := \lim_{j\to\infty} q^j f_i(q^{-j}a) \quad (1 \leq i \leq 3)$$

for all $a \in A$.

Putting $\ell = 0$ and passing to the limit as $j \to \infty$ in (2.4), we get (2.3). In addition,

$$\begin{aligned} \|\Delta D_i(\lambda, x_1, \cdots, x_q)\| &= \lim_{j\to\infty} q^j \|\Delta f_i(\lambda, q^{-j}x_1, \cdots, q^{-j}x_q)\| \\ &\leq \lim_{j\to\infty} q^j \varphi(q^{-j}x_1, \cdots, q^{-j}x_q) \\ &= 0 \end{aligned}$$

for all $x_1, \cdots, x_q \in A$ and all $\lambda \in \mathbb{T}$.

It is easy to see that

$$D_i(\lambda x + \lambda y) = \lambda D_i(x) + \lambda D_i(y)$$

for all $x, y \in A$ and all $\lambda \in \mathbb{T}$. Since every complex number μ can be written as the form $\mu = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$, where $\lambda_m \in \mathbb{T} \quad (m = 1, 2, 3)$ and since, by the additivity of D_i, $D_i(\frac{1}{3}x) = \frac{1}{3}D_i(x)$ one may conclude that D_i is $\mathbb{C}$-linear. The uniqueness assertion can be proved in a standard way; see [1].

Next, replacing a, b by $q^{-j}a, q^{-j}b$ in (2.2), we get the following inequality

$$\|q^{-2j} f_1(q^{2j}ab) - q^{-j} f_2(q^j a)b - aq^{-j} f_3(q^j b)\| \leq q^{-2j}\varphi(q^j a, q^j b, 0, \ldots, 0)$$

for all $a, b \in A$. Tending j to infinity we infer that

$$D_1(ab) = D_2(a)b + aD_2(b),$$

i.e., (D_1, D_2, D_3) is a ternary derivation. ■

The following result is similar to Theorem 2.1 and we omit its proof.

Theorem 2.2. *Let $\varphi : A^q \to [0, \infty)$ be a function fulfilling*

$$\widetilde{\varphi}(x_1, \cdots, x_q) := \sum_{j=1}^{\infty} q^{-j}\varphi(q^j x_1, \ldots, q^j_q) < \infty$$

for all $x_1, \cdots, x_q \in A$*, and let* $f_1, f_2, f_3 : A \to X$ *be mappings satisfying*

$$f_i(0) = 0 \quad (1 \le i \le 3)$$
$$\|\Delta f_i(\lambda, x_1, \cdots, x_q)\| \le \varphi(x_1, \cdots, x_q) \quad (1 \le i \le 3)$$

and

$$\|f_1(ab) - f_2(a)b - af_3(b)\| \le \varphi(a, b, 0, \cdots, 0)$$

for all $a, b, x_1, \cdots, x_q \in A$ *and all* $\lambda \in \mathbb{T}$*. Then there exists a unique ternary derivation* (D_1, D_2, D_3)*, defined by* $D_i(a) := \lim_{j\to\infty} \frac{f_i(q^j a)}{q^j}$ $(1 \le i \le 3)$*, such that*

$$\|f_i(a) - D_i(a)\| \le \widetilde{\varphi}(a, 0, \cdots, 0)$$

for all $a \in A$*.*

Corollary 2.3. *Suppose that* $p \neq 1$ *is a real number and* $\varepsilon > 0$*. Let* $f_1, f_2, f_3 : A \to X$ *be mappings satisfying*

$$\|\Delta f_i(\lambda, x_1, \cdots, x_q)\| \le \varepsilon \sum_{k=1}^{q} \|x_k\|^p \quad (1 \le i \le 3),$$

$$\|f_1(ab) - f_2(a)b - af_3(b)\| \le \varepsilon(\|a\|^p + \|b\|^p)$$

for all $a, b, x_1, \cdots, x_q \in A$ *and all* $\lambda \in \mathbb{T}$*. Then there exists a unique ternary derivation* (D_1, D_2, D_3) *such that*

$$\|f_i(a) - D_i(a)\| \le \frac{1}{|1 - q^{1-p}|}\varepsilon\|a\|^p$$

for all $a \in A$*.*

Proof. Use Theorems 2.1 and 2.2 with $\varphi(x_1, \cdots, x_q) = \varepsilon \sum_{k=1}^{q} \|x_k\|^p$. ∎

In the sequel, we aim to prove the superstability of ternary derivations associated with the extended Jensen functional equation (1.2) under some conditions. First, we introduce the notion of nearly ternary derivation.

Definition 2.4. Given $\varepsilon > 0$, the triple (f_1, f_2, f_3) of mappings $f_i : A \to A$ is called a *nearly ternary derivation* if $f_i(0) = 0$ $(1 \le i \le 3)$ and

$$\|\Delta f_i(\lambda, x_1, \cdots, x_q)\| \le \varepsilon \quad (1 \le i \le 3),$$

$$\|f_1(ab) - f_2(a)b - af_3(b)\| \le \varepsilon$$

for all $a, b, x_1, \cdots, x_q \in A$ and all $\lambda \in \mathbb{T}$.

The following theorem is one of our main results.

Theorem 2.5. *Suppose that A is a unital Banach algebra. Let (f_1, f_2, f_3) be a nearly ternary derivation and $f_3(1) = 0$. Then $D_i(a) := \lim_{j\to\infty} \frac{f_i(q^j a)}{q^j}$ exist for all $a \in A \quad (1 \le i \le 3)$, and $D_1 = D_2$ is a D_3-derivation.*

Proof. Use Theorem 2.2 with $\varphi(x_1, \cdots, x_q) = \varepsilon$ to get a ternary derivation (D_1, D_2, D_3), defined by $D_i(a) := \lim_{j\to\infty} \frac{f_i(q^j a)}{q^j} \quad (1 \le i \le 3)$, such that

$$\|D_i(a) - f_i(a)\| \le \frac{\varepsilon}{q-1} \quad (1 \le i \le 3)$$

for all $a \in A$. We have

$$\begin{aligned}
\|q^j f_3(qa) - q^{j+1} f_3(a)\| &\le \|f_2(q^j 1)qa - q^j 1 f_3(qa) - f_1((q^j 1)(qa))\| \\
&\quad + \|f_1((q^j 1)(qa)) - f_2(q^j 1)qa - q^{j+1} 1 f_3(a)\| \\
&\le \varepsilon + \|f_1((q^j 1)(qa)) - f_2(q^j 1)qa - q^{j+1} 1 f_3(a)\| \\
&\le \varepsilon + \|f_1((q^j 1)(qa)) - D_1((q^j 1)(qa))\| \\
&\quad + \|D_1((q^j 1)(qa)) - f_2(q^j 1)qa - q^{j+1} 1 f_3(a)\| \\
&\le \frac{\varepsilon}{q-1} + \|D_1((q^j 1)(qa)) - f_2(q^j 1)qa - q^{j+1} 1 f_3(a)\| \\
&\le \frac{\varepsilon}{q-1} + q\|D_1(q^j 1a) - f_1(q^j 1a)\| \\
&\quad + q\|f_1(q^j 1a) - f_2(q^j 1)a - q^j 1 f_3(a)\| \\
&\le \frac{2\varepsilon}{q-1},
\end{aligned}$$

for all nonnegative integers j and all $a \in A$. Let j tend to infinity in the inequality

$$\|f_3(qa) - qf_3(a)\| \le \frac{2\varepsilon}{q^j(q-1)}.$$

Then $f_3(qa) = qf_3(a)$ for all $a \in A$. Therefore,

$$D_3(a) = \lim_{j\to\infty} \frac{f_3(q^j a)}{q^j} = f_3(a)$$

for all $a \in A$. Similarly, one can show that $D_2(a) = f_2(a)$ for all $a \in A$.

Now, replacing b by 1 in (1.1), we get

$$D_1(a) = D_2(a) + aD_3(1) = D_2(a),$$

since $f_3(1) = 0$. In addition,

$$D_1(a) = D_1(1a) = D_2(1)a + 1D_3(a) = D_1(1)a + 1D_3(a)$$

and so

$$D_3(a) = D_1(a) - D_1(1)a.$$

By the strategy used in the main theorem [10], we show that D_3 is a usual derivation:

$$\begin{aligned} D_3(ab) &= D_1(ab) - D_1(1)ab \\ &= (D_1(a)b + aD_3(b)) - D_1(1)ab \\ &= D_1(a)b + a(D_1(b) - D_1(1)b) - D_1(1)ab \\ &= (D_1(a) - D_1(1)a)b + a(D_1(b) - D_1(1)b) \\ &= D_3(a)b + aD_3(b). \end{aligned}$$

Therefore, $D_1 = D_2$ is a D_3-derivation. ■

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