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Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation

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Abstract

In this paper, we prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation. This is applied to investigate homomorphisms between quasi-Banach algebras. The concept of Hyers–Ulam–Rassias stability originated from Th.M. Rassias' stability theorem that appeared in his paper [Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300].

Keywords: Cauchy functional equation; Homomorphism in quasi-Banach algebra; Hyers–Ulam–Rassias stability; p-Banach algebra

1. Introduction

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a map-

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ping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x * y), h(x) \diamond h(y)\big) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [7] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. In 1978, Th.M. Rassias [21] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 *(Th.M. Rassias). Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\big\| f(x+y) - f(x) - f(y) \big\| \leqslant \epsilon\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\big\| f(x) - L(x) \big\| \leqslant \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if the function $t \to f(tx)$ from $\mathbb{R}$ to E' is continuous for each fixed $x \in E$, then L is linear.*

In 1990, Th.M. Rassias [22] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [5] following the same approach as in [21], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [5], as well as by Th.M. Rassias and P. Šemrl [26], that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [5], as well as of Th.M. Rassias and P. Šemrl [26], have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [6] and S. Jung [11], who among others studied the Hyers–Ulam–Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [21] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [4] and D.H. Hyers, G. Isac and Th.M. Rassias [8]).

J.M. Rassias [17] following the spirit of the innovative approach of Th.M. Rassias [21] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

P. Găvruta [6] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [10] applied the Hyers–Ulam–Rassias stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. In [9], D.H. Hyers, G. Isac and

Th.M. Rassias studied the asymptoticity aspect of Hyers–Ulam stability of mappings. During the past few years several mathematicians have published on various generalizations and applications of Hyers–Ulam stability and Hyers–Ulam–Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings—superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation, Navier–Stokes equations. Several mathematicians have contributed works on these subjects; we mention a few: C. Park [13–15], Th.M. Rassias [23–25], F. Skof [28].

In the period 1982–1994 further generalizations were obtained by J.M. Rassias [16–19].

J.M. Rassias and M.J. Rassias [20] considered and investigated quadratic equations involving a product of powers of norms following the innovative approach of Th.M. Rassias who had introduced the concept of the unbounded Cauchy difference in the year 1978 and he had treated the subject for the sum of powers of norms. They studied the problem in which an approximate quadratic mapping degenerates to a genuine quadratic mapping. Analogous results could be investigated with additive type equations involving a product of powers of norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2,6,13,14,24]).

Theorem 1.2. *(See [16,17,19].) Let X be a real normed linear space and Y be a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geqslant 0$ and $p \in \mathbb{R} \setminus \{1\}$ such that f satisfies inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \theta \|x\|^{p/2} \|y\|^{p/2}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\left\| f(x) - L(x) \right\| \leqslant \frac{\theta}{|2^p - 2|} \|x\|^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.3. (See [3,27].) Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(i) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geqslant 1$ such that $\|x + y\| \leqslant K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a *quasi-normed space* if $\|.\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a *p-norm* $(0 < p \leqslant 1)$ if

$$\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

By the Aoki–Rolewicz theorem [27] (see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.4. (See [1].) Let $(A, \|.\|)$ be a quasi-normed space. The quasi-normed space $(A, \|.\|)$ is called a *quasi-normed algebra* if A is an algebra and there is a constant $K > 0$ such that $\|xy\| \leqslant K\|x\|\|y\|$ for all $x, y \in A$.

A *quasi-Banach algebra* is a complete quasi-normed algebra. If the quasi-norm $\|.\|$ is a p-norm then the quasi-Banach algebra is called a *p-Banach algebra.*

2. Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation

Throughout this section, assume that A is a quasi-normed algebra with quasi-norm $\|.\|_A$ and that B is a p-Banach algebra with p-norm $\|.\|_B$. For convenience, let $K = 1$ be the modulus of concavity of $\|.\|_B$. The stability of homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation, has been investigated in [15]. We prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Pexiderized Cauchy functional equation.

Theorem 2.1. *(See [15].) Let $r > 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that*

$$\left\| f(x+y) - f(x) - f(y) \right\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r, \tag{2.1}$$

$$\left\| f(xy) - f(x)f(y) \right\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r. \tag{2.2}$$

If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $T : A \to B$ such that

$$T(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \qquad \left\| f(x) - T(x) \right\|_B \leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}}\|x\|_A^{2r}$$

for all $x \in A$.

Theorem 2.2. *(See [15].) Let $r < \frac{1}{2}$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying* (2.1) *and* (2.2). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $T : A \to B$ such that*

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right), \qquad \left\| f(x) - T(x) \right\|_B \leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}}\|x\|_A^{2r}$$

for all $x \in A$.

The proofs of the following results are similar to the proofs of Theorems 2.1 and 2.2 and we refer to [15].

Theorem 2.3. *Let θ, r, s be positive real numbers with $r > \frac{1}{2}$ and $s > 1$. Assume that $f : A \to B$ is a mapping such that*

$$\left\| f(x+y) - f(x) - f(y) \right\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r, \tag{2.3}$$

$$\left\| f(xy) - f(x)f(y) \right\|_B \leqslant \theta\|x\|_A^s\|y\|_A^s \tag{2.4}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $T : A \to B$ such that

$$T(x)=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right),\qquad \left\|f(x)-T(x)\right\|_B\leqslant\frac{\theta}{(4^{pr}-2^p)^{1/p}}\|x\|_A^{2r}$$

for all $x\in A$.

Theorem 2.4. *Let* θ, r, s *be positive real numbers with* $0\leqslant r<\frac{1}{2}$ *and* $0\leqslant s<1$. *Assume that* $f:A\to B$ *is a mapping satisfying* (2.3) *and* (2.4) *for all* $x, y\in A$. *If* $f(tx)$ *is continuous in* $t\in\mathbb{R}$ *for each fixed* $x\in A$, *then there exists a unique homomorphism* $T:A\to B$ *such that*

$$T(x)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right),\qquad \left\|f(x)-T(x)\right\|_B\leqslant\frac{\theta}{(2^p-4^{pr})^{1/p}}\|x\|_A^{2r}$$

for all $x\in A$.

Theorem 2.5. *Let* θ, r, s *be positive real numbers with* $r>\frac{1}{2}$ *and* $s>1$. *Assume that* $f, g, h:A\to B$ *are mappings such that*

$$\left\|f(x+y)-g(x)-h(y)\right\|_B\leqslant\theta\|x\|_A^r\|y\|_A^r, \tag{2.5}$$

$$\left\|f(xy)-g(x)h(y)\right\|_B\leqslant\theta\|x\|_A^s\|y\|_A^s \tag{2.6}$$

for all $x, y\in A$. *If at least one of the mappings* $t\mapsto f(tx)$, $t\mapsto g(tx)$ *and* $t\mapsto h(tx)$ *is continuous in* $t\in\mathbb{R}$ *for each fixed* $x\in A$, *then there exists a unique homomorphism* $T:A\to B$ *such that*

$$\begin{aligned}T(x)&=\lim_{n\to\infty}2^n\left[f\left(\frac{x}{2^n}\right)-f(0)\right]=\lim_{n\to\infty}2^n\left[g\left(\frac{x}{2^n}\right)-g(0)\right]\\&=\lim_{n\to\infty}2^n\left[h\left(\frac{x}{2^n}\right)-h(0)\right]\end{aligned}$$

and

$$\begin{aligned}\left\|f(x)-f(0)-T(x)\right\|_B&\leqslant\frac{\theta}{(4^{pr}-2^p)^{1/p}}\|x\|_A^{2r},\\ \left\|g(x)-g(0)-T(x)\right\|_B&\leqslant\frac{\theta}{(4^{pr}-2^p)^{1/p}}\|x\|_A^{2r},\\ \left\|h(x)-h(0)-T(x)\right\|_B&\leqslant\frac{\theta}{(4^{pr}-2^p)^{1/p}}\|x\|_A^{2r}\end{aligned}$$

for all $x\in A$.

Proof. Letting $y=0$ in (2.5) and (2.6), we get that

$$f(x)=g(x)+h(0),\qquad f(0)=g(x)h(0) \tag{2.7}$$

for all $x\in A$. Once again putting $x=0$ in (2.5) and (2.6), we get that

$$f(y)=g(0)+h(y),\qquad f(0)=g(0)h(y) \tag{2.8}$$

for all $y\in A$. So

$$f(x)-f(0)=g(x)-g(0)=h(x)-h(0)$$

for all $x\in A$. Let $H:A\to B$ be a mapping defined by

$$H(x)=f(x)-f(0)$$

for all $x \in A$. It follows from (2.7) and (2.8) that

$$\begin{aligned} H(x+y) - H(x) - H(y) &= \big(f(x+y) - f(0)\big) - \big(g(x) - g(0)\big) - \big(h(y) - h(0)\big) \\ &= f(x+y) - g(x) - h(y) \end{aligned}$$

and

$$\begin{aligned} H(xy) - H(x)H(y) &= \big(f(xy) - f(0)\big) - \big(g(x) - g(0)\big)\big(h(y) - h(0)\big) \\ &= f(xy) - g(x)h(y) \end{aligned}$$

for all $x, y \in A$. Therefore, H satisfies the inequalities (2.3) and (2.4). By the assumption, the mapping $t \mapsto H(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$. By Theorem 2.3, there exists a unique homomorphism $T : A \to B$ such that

$$\big\| H(x) - T(x) \big\|_B \leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$. This implies the requested inequalities. □

Theorem 2.6. *Let r, s and θ be positive real numbers with $r < \frac{1}{2}$ and $s < 1$. Assume that $f, g, h : A \to B$ are mappings satisfying* (2.5) *and* (2.6). *If at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ and $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $T : A \to B$ such that*

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f\big(2^n x\big) = \lim_{n\to\infty} \frac{1}{2^n} g\big(2^n x\big) = \lim_{n\to\infty} \frac{1}{2^n} h\big(2^n x\big)$$

and

$$\begin{aligned} \big\| f(x) - f(0) - T(x) \big\|_B &\leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}} \|x\|_A^{2r}, \\ \big\| g(x) - g(0) - T(x) \big\|_B &\leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}} \|x\|_A^{2r}, \\ \big\| h(x) - h(0) - T(x) \big\|_B &\leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}} \|x\|_A^{2r} \end{aligned}$$

for all $x \in A$.

Proof. Using the proof of Theorem 2.5 and applying Theorem 2.4, we get the result. □

For $r = s = 0$, we have the following theorem.

Theorem 2.7. *Let θ be a positive real number and let $f, g, h : A \to B$ be mappings satisfying*

$$\big\| f(x+y) - g(x) - h(y) \big\|_B \leqslant \theta, \tag{2.9}$$

$$\big\| f(xy) - g(x)h(y) \big\|_B \leqslant \theta \tag{2.10}$$

for all $x, y \in A$. If at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ and $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $T : A \to B$ such that

$$\begin{aligned}&\left\|f(x)-f(0)-T(x)\right\|_B \leqslant 3\theta+M,\\&\left\|g(x)-g(0)-T(x)\right\|_B \leqslant 3\theta+M,\\&\left\|h(x)-h(0)-T(x)\right\|_B \leqslant 3\theta+M\end{aligned} \tag{2.11}$$

for all $x \in A$, *where* $M = \|f(0)-g(0)-h(0)\|_B$.

Proof. By Theorem 2.2 of [12], there exists a unique additive mapping $T : A \to B$ satisfying (2.11) and

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right) = \lim_{n\to\infty} \frac{1}{2^n} g\left(2^n x\right) = \lim_{n\to\infty} \frac{1}{2^n} h\left(2^n x\right) \tag{2.12}$$

for all $x \in A$. By the same reasoning as in the proof of Theorem of [21], the mapping $T : A \to B$ is $\mathbb{R}$-linear. It follows from (2.10) and (2.12) that

$$\begin{aligned}\left\|T(xy)-T(x)T(y)\right\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \left\|f\left(4^n xy\right)-g\left(2^n x\right)-h\left(2^n y\right)\right\|_B\\&\leqslant \lim_{n\to\infty} \frac{\theta}{4^n} = 0\end{aligned}$$

for all $x, y \in A$. Hence $T(xy) = T(x)T(y)$ for all $x, y \in A$.

Therefore, T is a homomorphism. □

Theorem 2.8. *Let* r, t *and* θ *be positive real numbers and let* $q, s < 0$ *be real numbers. Assume that* $f, g, h : A \to B$ *are mappings satisfying*

$$\left\|f(x+y)-g(x)-h(y)\right\|_B \leqslant \theta \|x\|_A^r \|y\|_A^s, \tag{2.13}$$

$$\left\|f(xy)-g(x)h(y)\right\|_B \leqslant \theta \|x\|_A^t \|y\|_A^q \tag{2.14}$$

for all $x \in A$ *and all* $y \in A \setminus \{0\}$. *If* $g(0) = 0$ *and the mappings* $t \mapsto g(tx)$, $t \mapsto f(tx)$ *and* $t \mapsto h(tx)$ *are continuous in* $0 \in \mathbb{R}$ *for each fixed* $x \in A$, *then*

(i) $f = h$;
(ii) *the mapping* $g : A \to B$ *is a homomorphism*;
(iii) *if* $\lambda = r + s \neq 1$, *then*

$$\left\|f(x)-g(x)\right\|_B \leqslant C\|x\|_A^\lambda \tag{2.15}$$

for all $x \in A$, *where* $C = \min\{\theta, \frac{2\theta}{|2^{\lambda p}-2^p|^{1/p}}\}$. *Moreover,* $g : A \to B$ *is a unique homomorphism satisfying* (2.15).

Proof. Letting $x = 0$ in (2.13) and (2.14), we get that

$$f(y) = h(y), \qquad f(0) = 0 \tag{2.16}$$

for all $y \in A \setminus \{0\}$. Replacing y by y/n in (2.16) and letting $n \to \infty$, we get that $f(0) = h(0)$. So $f = h$, and it proves (i).

To prove (ii), replacing y by ny in (2.13), we get that

$$\left\|f(x+ny)-g(x)-f(ny)\right\|_B \leqslant \theta n^s \|x\|_A^r \|y\|_A^s \tag{2.17}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. Therefore, (2.17) implies that

$$\lim_{n\to\infty} \left[f(x+ny)-f(ny)\right] = g(x)$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. So (i) and (2.13) imply that

$$\begin{aligned}
&\|g(x+y)-g(x)-g(y)\|_B \\
&\quad = \lim_{n\to\infty}\big\|\big[f(x+y+ny)-f(ny)\big]-g(x)-\big[f(y+ny)-f(ny)\big]\big\|_B \\
&\quad = \lim_{n\to\infty}\big\|f(x+y+ny)-g(x)-h(y+ny)\big\|_B \\
&\quad \leqslant \lim_{n\to\infty}\theta(n+1)^s\|x\|_A^r\|y\|_A^s = 0
\end{aligned}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. Since $g(0)=0$, $g(x+y)=g(x)+g(y)$ for all $x, y \in A$. So g is $\mathbb{Q}$-linear and the mapping $t \mapsto g(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$. Therefore g is $\mathbb{R}$-linear. Also, we have

$$\begin{aligned}
&\|g(xy)-g(x)g(y)\|_B \\
&\quad = \lim_{n\to\infty}\big\|\big[f(xy+nxy)-f(nxy)\big]-g(x)\big[f(y+ny)-f(ny)\big]\big\|_B \\
&\quad \leqslant \limsup_{n\to\infty}\big\|f(xy+nxy)-g(x)h(y+ny)\big\|_B \\
&\qquad + \limsup_{n\to\infty}\big\|f(nxy)-g(x)h(ny)\big\|_B \\
&\quad \leqslant \lim_{n\to\infty}\theta(n+1)^q\|x\|_A^t\|y\|_A^q + \lim_{n\to\infty}\theta n^q\|x\|_A^t\|y\|_A^q = 0
\end{aligned}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. Since $g(0)=0$, $g(xy)=g(x)g(y)$ for all $x, y \in A$. Thus (ii) is proved.

To prove (iii), we have two cases.

Case I. Let $\lambda > 1$. Letting $y = x$ in (2.13) and using (i), we get that

$$\|f(2x)-g(x)-f(x)\|_B \leqslant \theta\|x\|_A^\lambda \tag{2.18}$$

for all $x \in A \setminus \{0\}$. It is clear that (2.18) holds for all $x \in A$. Once again, letting $x = -y$ in (2.13) and using (i) and (ii) we get that

$$\|f(y)-g(y)\|_B \leqslant \theta\|y\|_A^\lambda \tag{2.19}$$

for all $y \in A \setminus \{0\}$. It is clear that (2.19) holds for all $y \in A$. Hence it follows from (2.18) and (2.19) that

$$\|f(2x)-2f(x)\|_B \leqslant 2\theta\|x\|_A^\lambda \tag{2.20}$$

for all $x \in A$. If we replace x in (2.20) by $\frac{x}{2^{n+1}}$ and multiply both sides of (2.20) to 2^n, then we have

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)\right\|_B \leqslant \theta\left(\frac{2}{2^\lambda}\right)^{n+1}\|x\|_A^\lambda \tag{2.21}$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned}
\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|_B^p &\leqslant \sum_{i=m}^{n}\left\|2^{i+1}f\left(\frac{x}{2^{i+1}}\right)-2^i f\left(\frac{x}{2^i}\right)\right\|_B^p \\
&\leqslant \theta^p\sum_{i=m}^{n}\left(\frac{2}{2^\lambda}\right)^{(i+1)p}\|x\|_A^{\lambda p}
\end{aligned} \tag{2.22}$$

for all non-negative integers m and n with $n \geqslant m$ and all $x \in A$. It follows from (2.22) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define a mapping $T : A \to B$ by

$$T(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. It follows from (ii) and (2.19) that

$$\begin{aligned}\|g(x) - T(x)\|_B &= \lim_{n\to\infty} 2^n \left\| g\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B \\ &\leqslant \theta \lim_{n\to\infty}\left(\frac{2}{2^\lambda}\right)^n \|x\|_A^\lambda = 0\end{aligned}$$

for all $x \in A$. So $T = g$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.22), we get

$$\|f(x) - g(x)\|_B \leqslant \frac{2\theta}{(2^{\lambda p} - 2^p)^{1/p}} \|x\|_A^\lambda \tag{2.23}$$

for all $x \in A$. Therefore (2.15) follows from (2.19) and (2.23). To prove the uniqueness of g, let $Q : A \to B$ be another homomorphism satisfying (2.15). We have

$$\begin{aligned}\|g(x) - Q(x)\|_B &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) - Q\left(\frac{x}{2^n}\right)\right\|_B \\ &\leqslant C \lim_{n\to\infty}\left(\frac{2}{2^\lambda}\right)^n \|x\|_A^\lambda = 0\end{aligned}$$

for all $x \in A$. So $g = Q$.

Case II. Let $\lambda < 1$. If we replace x in (2.20) by $2^n x$ and divide both sides of (2.20) by 2^{n+1}, then we have

$$\left\| \frac{1}{2^{n+1}} f\left(2^{n+1}x\right) - \frac{1}{2^n} f\left(2^n x\right)\right\|_B \leqslant \theta \left(\frac{2^\lambda}{2}\right)^n \|x\|_A^\lambda \tag{2.24}$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned}\left\| \frac{1}{2^{n+1}} f\left(2^{n+1}x\right) - \frac{1}{2^m} f\left(2^m x\right)\right\|_B^p &\leqslant \sum_{i=m}^{n} \left\| \frac{1}{2^{i+1}} f\left(2^{i+1}x\right) - \frac{1}{2^i} f\left(2^i x\right)\right\|_B^p \\ &\leqslant \theta^p \sum_{i=m}^{n} \left(\frac{2^\lambda}{2}\right)^{ip} \|x\|_A^{\lambda p}\end{aligned} \tag{2.25}$$

for all non-negative integers m and n with $n \geqslant m$ and all $x \in A$. It follows from (2.25) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define a mapping $T : A \to B$ by

$$T(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. The rest of the proof is similar to the proof of Case I. This proves (iii). □

Corollary 2.9. *Let r, t and θ be positive real numbers and let $q, s < 0$ be real numbers. Assume that $f : A \to B$ is a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\|_B \leqslant \theta \|x\|_A^r \|y\|_A^s, \tag{2.26}$$

$$\|f(xy) - f(x)f(y)\|_B \leqslant \theta \|x\|_A^t \|y\|_A^q \tag{2.27}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. If the mapping $t \mapsto f(tx)$ is continuous in $0 \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is a homomorphism.

In Theorem 2.8, let $0 < t < 1$ and $\lambda < 1$. If we replace x by nx and divide both sides of (2.14) by n, then we have

$$\left\|\frac{1}{n} f(nxy) - g(x)h(y)\right\|_B \leqslant \theta n^{t-1} \|x\|_A^t \|y\|_A^q$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. Therefore

$$\lim_{n\to\infty} \frac{1}{n} f(nxy) = g(x)h(y)$$

for all $x, y \in A$. It follows from the proof of Theorem 2.8 (part (iii)), $g(xy) = g(x)h(y)$ for all $x, y \in A$. Since the mapping $g : A \to B$ is a homomorphism, then we have

$$g(x)\big[g(y) - h(y)\big] = 0 \tag{2.28}$$

for all $x, y \in A$. Similarly, one can obtain (2.28) if $t > 1$ and $\lambda > 1$. Therefore we have the following results:

Corollary 2.10. *In Theorem 2.8, let $B = \mathbb{C}$ with $p = 1$. Then $f, g, h : A \to B$ are homomorphisms. Moreover, $f = g = h$.*

Corollary 2.11. *In Theorem 2.8, let A and B be unital with units e_A and e_B, respectively. If $g(e_A) = e_B$, then $f, g, h : A \to B$ are homomorphisms. Moreover, $f = g = h$.*

Theorem 2.12. *Let θ be a positive real number and let $r, s < 0$ be real numbers. Assume that $f, g : A \to B$ are mappings satisfying*

$$\|f(x+y) - f(x) - g(y)\|_B \leqslant \theta \|x\|_A^r \|y\|_A^r, \tag{2.29}$$

$$\|f(xy) - f(x)g(y)\|_B \leqslant \theta \|x\|_A^s \|y\|_A^s \tag{2.30}$$

for all $x, y \in A \setminus \{0\}$. If the mapping $t \mapsto g(tx)$ from $\mathbb{R}$ to B is continuous at zero for each fixed $x \in A$, then

(i) *the mapping $g : A \to B$ is a homomorphism*;
(ii) *$f = g$.*

Proof. Replacing x by nx in (2.29), we get that

$$\|f(nx+y) - f(nx) - g(y)\|_B \leqslant \theta n^r \|x\|_A^r \|y\|_A^r \tag{2.31}$$

for all $x, y \in A \setminus \{0\}$ and all positive integers n. Letting $n \to \infty$ in (2.31), we get that

$$\lim_{n\to\infty} \big[f(nx+y) - f(nx)\big] = g(y)$$

for all $x, y \in A \setminus \{0\}$. Once again replacing y by ny in (2.29) and letting $n \to \infty$, we get that

$$\lim_{n\to\infty}\big[f(x+ny)-g(ny)\big]=f(x)$$

for all $x, y \in A \setminus \{0\}$. Let $x \in A \setminus \{0\}$. We have

$$\lim_{n\to\infty}\big[f(nx)-g(nx)\big]=f(y)-g(y) \tag{2.32}$$

for all $y \in A \setminus \{0\}$. Hence we have

$$\begin{aligned}
&\big\|g(x+y)-g(x)-g(y)\big\|_B\\
&\quad=\lim_{n\to\infty}\big\|\big[f(x+y+ny)-f(ny)\big]-g(x)-\big[f(y+ny)-f(ny)\big]\big\|_B\\
&\quad=\lim_{n\to\infty}\big\|f(x+y+ny)-f(y+ny)-g(x)\big\|_B\\
&\quad=\lim_{n\to\infty}\theta(n+1)^r\|x\|_A^r\|y\|_A^r=0
\end{aligned}$$

for all $x, y \in A \setminus \{0\}$. Therefore, $g(x+y)=g(x)+g(y)$ for all $x, y \in A \setminus \{0\}$. Let $x \in A \setminus \{0\}$. Then $g(2x)=2g(x)$. Replacing x by x/n in the last equation, we get that

$$g\left(\frac{2x}{n}\right)=2g\left(\frac{x}{n}\right) \tag{2.33}$$

for all positive integers n. Since the mapping $t \mapsto g(tx)$ is continuous at zero for each fixed $x \in A$, letting $n \to \infty$ in (2.33), we have $g(0)=0$. Therefore, g is $\mathbb{Q}$-linear. The continuity of the mapping $t \mapsto g(tx)$ at zero for each fixed $x \in A$ implies its continuity in $t \in \mathbb{R}$ for each fixed $x \in A$. So g is $\mathbb{R}$-linear. Also, we have

$$\begin{aligned}
&\big\|g(xy)-g(x)g(y)\big\|_B\\
&\quad=\lim_{n\to\infty}\big\|\big[f(xy+nxy)-f(nxy)\big]-\big[f(x+nx)-f(nx)\big]g(y)\big\|_B\\
&\quad\leqslant\limsup_{n\to\infty}\big\|f(xy+nxy)-f(x+nx)g(y)\big\|_B\\
&\qquad+\limsup_{n\to\infty}\big\|f(nxy)-f(nx)g(y)\big\|_B\\
&\quad\leqslant\lim_{n\to\infty}\theta(n+1)^r\|x\|_A^r\|y\|_A^r+\lim_{n\to\infty}\theta n^r\|x\|_A^r\|y\|_A^r=0
\end{aligned}$$

for all $x, y \in A \setminus \{0\}$. Since $g(0)=0$, $g(xy)=g(x)g(y)$ for all $x, y \in A$. Thus the mapping $g: A \to B$ is a homomorphism.

To prove (ii), fix $y_0 \in A \setminus \{0\}$ and let $b=f(y_0)-g(y_0)$. It follows from (2.32) that

$$f(x)=g(x)+b \tag{2.34}$$

for all $x \in A \setminus \{0\}$. Let $x_0, y_0 \in A \setminus \{0\}$. We have two cases:

Case I. $x_0y_0 \neq 0$. In this case, it follows from (i) and (2.34) that

$$\begin{aligned}
f\left(\frac{n}{m}x_0y_0\right)-\frac{1}{m}f(nx_0)g(y_0)&=\left[g\left(\frac{n}{m}x_0y_0\right)+b\right]-\frac{1}{m}\big[g(nx_0)+b\big]g(y_0)\\
&=\frac{n}{m}g(x_0)g(y_0)+b-\frac{n}{m}g(x_0)g(y_0)-\frac{b}{m}g(y_0)\\
&=b-\frac{b}{m}g(y_0)
\end{aligned}$$

for all positive integers m, n. Therefore, we get from (i) and (2.30) that

$$\left\| b - \frac{b}{m} g(y_0) \right\|_B \leqslant \theta \left(\frac{n}{m} \right)^s \|x_0\|_A^s \|y_0\|_A^s$$

for all positive integers m, n. Letting $n \to \infty$ in the last inequality, we get that

$$b = \frac{b}{m} g(y_0)$$

for all positive integers m. So $b = 0$. Hence we get from (2.34) that

$$f(x) = g(x) \tag{2.35}$$

for all $x \in A \setminus \{0\}$.

Letting $y = -x$ in (2.29), we get from (i) and (2.35) that

$$\| f(0) \|_B \leqslant \theta \|x\|_A^{2r} \tag{2.36}$$

for all $x \in A \setminus \{0\}$. So $f(0) = 0$ and (ii) follows from (i) and (2.35).

Case II. $x_0 y_0 = 0$. In this case, it follows from (i) and (2.30) that

$$\| f(0) - nf(x_0)g(y_0) \|_B \leqslant \theta n^s \|x_0\|_A^s \|y_0\|_A^s$$

for all positive integers n. So $\lim_{n\to\infty} nf(x_0)g(y_0) = f(0)$ and hence $f(x_0)g(y_0) = 0$. Therefore $f(0) = 0$. Replacing x and y by nx and $-nx$ in (2.29), respectively, we get from (i) that

$$\| g(nx) - f(nx) \|_B \leqslant \theta n^{2r} \|x\|_A^{2r}$$

for all $x \in A \setminus \{0\}$ and all positive integers n. So $\lim_{n\to\infty}[f(nx) - g(nx)] = 0$ for all $x \in A \setminus \{0\}$. Since $f(0) = 0$, (2.32) implies that $f = g$. □

Corollary 2.13. *Let θ be a positive real number and let $r, s < 0$ be real numbers. Assume that $f : A \to B$ is a mapping such that*

$$\| f(x+y) - f(x) - f(y) \|_B \leqslant \theta \|x\|_A^r \|y\|_A^r, \tag{2.37}$$

$$\| f(xy) - f(x)f(y) \|_B \leqslant \theta \|x\|_A^s \|y\|_A^s \tag{2.38}$$

for all $x, y \in A \setminus \{0\}$. If the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to B is continuous at zero for each fixed $x \in A$, then $f : A \to B$ is a homomorphism.

Theorem 2.14. *Let θ be a positive real number and let $r, s < 0$ be real numbers. Assume that $f, g, h : A \to B$ are mappings such that*

$$\| f(x+y) - g(x) - h(y) \|_B \leqslant \theta \|x\|_A^r \|y\|_A^r, \tag{2.39}$$

$$\| f(xy) - h(x)g(y) \|_B \leqslant \theta \|x\|_A^s \|y\|_A^s \tag{2.40}$$

for all $x, y \in A \setminus \{0\}$. Let f, h be odd mappings and let the mappings $t \mapsto f(tx)$ and $t \mapsto g(tx)$ from $\mathbb{R}$ to B be continuous at zero for each fixed $x \in A$. Then $f, g, h : A \to B$ are homomorphisms and moreover $f = g = h$.

Proof. Similar to the proof of Theorem 2.12, we have

$$\lim_{n\to\infty}\big[h(nx)-g(nx)\big]=h(y)-g(y) \tag{2.41}$$

for all $x, y \in A\setminus\{0\}$. Since f, h are odd mappings, letting $y=-x$ in (2.39), we get that

$$\big\|h(x)-g(x)\big\|_B \leqslant \theta\|x\|_A^{2r}$$

for all $x\in A\setminus\{0\}$. Therefore

$$\lim_{n\to\infty}\big[h(nx)-g(nx)\big]=0 \tag{2.42}$$

for all $x\in A\setminus\{0\}$. It follows from (2.41) and (2.42) that

$$h(x)=g(x) \tag{2.43}$$

for all $x\in A\setminus\{0\}$. Replacing y by $-y-x$ in (2.39), we get from (2.43) that

$$\big\|h(x+y)-h(x)-f(y)\big\|_B \leqslant \theta\|x\|_A^r\|x+y\|_A^r \tag{2.44}$$

for all $x\in A\setminus\{0\}$ and all $y\in A\setminus\{-x\}$. Therefore

$$\lim_{n\to\infty}\big[h(nx+y)-h(nx)\big]=f(y),$$
$$\lim_{n\to\infty}\big[h(nx+y)-f(nx)\big]=h(y)$$

for all $x, y\in A\setminus\{0\}$. So

$$\lim_{n\to\infty}\big[h(nx)-f(nx)\big]=h(y)-f(y) \tag{2.45}$$

for all $x, y\in A\setminus\{0\}$. Fix $y_0\in A\setminus\{0\}$ and let $a=h(y_0)-f(y_0)$. It follows from (2.45) that

$$h(x)=f(x)+a \tag{2.46}$$

for all $x\in A\setminus\{0\}$. Since f, h are odd mappings, we have from (2.46)

$$0=h(x)+h(-x)=f(x)+f(-x)+2a=2a$$

for all $x\in A\setminus\{0\}$. Therefore $a=0$. Since $f(0)=h(0)=0$, (2.46) implies that $f(x)=h(x)$ for all $x\in A$. Hence one can obtain the functional inequalities (2.29) and (2.30) from (2.39) and (2.40). Therefore, the results follow from Theorem 2.12. □

3. Homomorphisms between unital quasi-Banach algebras

Throughout this section, assume that A is a quasi-Banach algebra with quasi-norm $\|.\|_A$ and unit e and that B is a p-Banach algebra with p-norm $\|.\|_B$ and unit e'. Let K be the modulus of concavity of $\|.\|_B$.

We investigate homomorphisms between unital quasi-Banach algebras, associated to the Pexiderized Cauchy functional equation. We generalize the results of [15].

Theorem 3.1. *Let θ, r, s be positive real numbers with $r>\frac{1}{2}$ and $s>1$, and let $f, g, h: A\to B$ be mappings satisfying* (2.5) *and* (2.6). *If at least one of the mappings $t\mapsto f(tx)$, $t\mapsto g(tx)$ and $t\mapsto h(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$ and $\lim_{n\to\infty}2^n[f(\frac{e}{2^n})-f(0)]=e'$, then the mappings $f, g, h: A\to B$ are homomorphisms. Moreover, $f=g=h$.*

Proof. Using the proof of Theorem 2.5, we get that

$$f(x) - f(0) = g(x) - g(0) = h(x) - h(0) \tag{3.1}$$

for all $x \in A$. By Theorems 2.1 and 2.5, there exists a homomorphism $T : A \to B$ defined by

$$T(x) = \lim_{n\to\infty} 2^n\left[f\left(\frac{x}{2^n}\right) - f(0)\right]$$

for all $x \in A$. Let $H(x) = f(x) - f(0)$ for all $x \in A$. It follows from the proof of Theorem 2.5 that

$$H(x+y) - H(x) - H(y) = f(x+y) - g(x) - h(y),$$
$$H(xy) - H(x)H(y) = f(xy) - g(x)h(y)$$

for all $x, y \in A$. Therefore, (2.6) implies that

$$\begin{aligned}\left\|T(x) - H(x)\right\|_B &= \lim_{n\to\infty}\left\|2^n H\left(\frac{x}{2^n}\right) - H(x)\right\|_B \\ &= \lim_{n\to\infty}\left\|2^n H\left(\frac{ex}{2^n}\right) - e'H(x)\right\|_B \\ &= \lim_{n\to\infty}\left\|2^n H\left(\frac{ex}{2^n}\right) - 2^n H\left(\frac{e}{2^n}\right)H(x)\right\|_B \\ &\leqslant \theta \lim_{n\to\infty}\left(\frac{2}{2^s}\right)^n \|e\|_A^s\|x\|_A^s = 0\end{aligned}$$

for all $x \in A$. So $H = T$ and (3.1) imply that $f(e) - f(0) = g(e) - g(0) = h(e) - h(0) = e'$. Since $g(0)h(x) = g(x)h(0) = f(0)$ for all $x \in A$,

$$h(0) = e'h(0) = \left[g(e) - g(0)\right]h(0) = 0.$$

So $f(0) = g(0)h(0) = 0$. Since $f(0) = g(0) + h(0)$, $g(0) = 0$. Hence $f = g = h = T$. □

Corollary 3.2. *Let θ, r, s be positive real numbers with $r > \frac{1}{2}$ and $s > 1$, and let $f : A \to B$ be a mapping satisfying* (2.3) *and* (2.4). *If the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} 2^n f(\frac{e}{2^n}) = e'$, then the mapping $f : A \to B$ is a homomorphism.*

Theorem 3.3. *Let θ, r, s be positive real numbers with $r < \frac{1}{2}$ and $s < 1$, and let $f, g, h : A \to B$ be mappings satisfying* (2.5) *and* (2.6). *If at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ and $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mappings $f, g, h : A \to B$ are homomorphisms. Moreover, $f = g = h$.*

Proof. Using the proof of Theorem 2.5, we get (3.1). By Theorems 2.2 and 2.6, there exists a homomorphism $T : A \to B$ defined by

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. Similar to the proof of Theorem 3.1, let $H(x) = f(x) - f(0)$ for all $x \in A$. Then we get that $H = T$.

The rest of the proof is the same as the proof of Theorem 3.1. □

Corollary 3.4. *Let θ, r, s be positive real numbers with $r < \frac{1}{2}$ and $s < 1$, and let $f : A \to B$ be mappings satisfying* (2.3) *and* (2.4). *If the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is a homomorphism.*

Remark 3.5. In Theorems 3.1 and 3.3, one can obtain the result if e and e' are left (right) units for A and B, respectively.

Theorem 3.6. *Let θ be a positive real number and let $f, g, h : A \to B$ be mappings satisfying* (2.9) *and* (2.10). *If at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ and $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mappings $g, h : A \to B$ are homomorphisms. Moreover, $g = h$ and*

$$\left\| f(x) - g(x) \right\|_B \leqslant \theta \tag{3.2}$$

for all $x \in A$.

Proof. By Theorem 2.7 and its proof, there exists a unique homomorphism $T : A \to B$ satisfying (2.12). Then we have

$$\begin{aligned} \left\| T(x) - h(x) \right\|_B &= \lim_{n\to\infty} \left\| \frac{1}{2^n} f\left(2^n x\right) - h(x) \right\|_B \\ &= \lim_{n\to\infty} \left\| \frac{1}{2^n} f\left(2^n ex\right) - e'h(x) \right\|_B \\ &= \lim_{n\to\infty} \frac{1}{2^n} \left\| f\left(2^n ex\right) - g\left(2^n e\right)h(x) \right\|_B \\ &\leqslant \lim_{n\to\infty} \frac{\theta}{2^n} = 0 \end{aligned}$$

for all $x \in A$. Therefore $T = h$.

Similarly, we get that $T = g$. Hence (3.2) follows from (2.9). □

Corollary 3.7. *Let θ be a positive real number and let $f : A \to B$ be a mapping satisfying*

$$\left\| f(x+y) - f(x) - f(y) \right\|_B \leqslant \theta, \qquad \left\| f(xy) - f(x)f(y) \right\|_B \leqslant \theta$$

for all $x, y \in A$. If the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is a homomorphism.

Remark 3.8. In Theorem 3.6, we cannot infer that f is a homomorphism. Let A be a unital algebra with unit e, and let $f, g, h : A \to A$ be mappings defined by

$$f(x) = x + \frac{\theta}{2\|e\|} e, \qquad g(x) = h(x) = x$$

for all $x \in A$. It is clear that the conditions of Theorem 3.6 hold (with $A = B$), but the mapping $f : A \to A$ is not a homomorphism.

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