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The tangential end fibration of an aspherical Poincaré complex

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Abstract

We construct a sphere fibration over a finite aspherical Poincaré complex X, which we call the tangential end fibration, under the condition that the universal cover of X is forward tame and simply connected at infinity. We show that it is tangent to X if the formal dimension of X is even or, when the formal dimension is odd, if the diagonal $X \to X \times X$ admits a Poincaré embedding structure.

MSC: 53C05; 53C07

Keywords: Aspherical Poincaré complex; End space; Tangent fibration

1. Introduction

Throughout the paper, X will denote a finite aspherical Poincaré complex of formal dimension n with a universal cover $\tilde{X}$ which is forward tame and simply connected at infinity. That X is aspherical means that $\tilde{X}$ is contractible. The definition for 'forward tame' is given in 2.3 below. The main goal of this paper is to show that an $(n-1)$-sphere fibration over X, which is defined in 1.2 below in terms of the end space of $\tilde{X}$, is tangent to X in the following sense:

Definition 1.1. Let P be a Poincaré complex of formal dimension n, and ν its Spivak normal fibration. A *tangent fibration* ξ of P is an $(n-1)$-sphere fibration over P satisfying the following conditions:

(a) ξ represents the stable inverse of ν.

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(b) If n is even, the Euler characteristic of ξ, with suitable choices of orientations, is the Euler–Poincaré number of P.
(c) If n is odd, the b-invariant of ξ, with respect to a suitable choice of the trivialization $\xi + \nu$, is the $\mathbb{Z}_2$-semi-characteristic of P.

The stable tangential properties of Poincaré complexes, for which the Spivak normal fibration is central, have been known from the work of Spivak [10]. The unstable ones, that is, the existence and the uniqueness results for the tangent fibration of a Poincaré complex in the sense of the above have been provided by the author [4]. A brief account for the b-invariant appears in the beginning of Section 7 below. The $\mathbb{Z}_2$-semi-characteristic $\chi_{\frac{1}{2}}(P)$ of a Poincaré complex P is defined when n is odd as

$$\chi_{\frac{1}{2}}(P) = \sum_{i=0}^{\frac{n-1}{2}} \operatorname{rank} H_i(P, \mathbb{Z}_2), \text{ mod } 2.$$

We write $\varepsilon(Y) = \{\sigma : [0,1) \to Y \mid \sigma \text{ is proper }\}$ for any space Y, which we consider with the compact-open topology. We call $\varepsilon(Y)$ the *end space* (cf. [6]). Let $p : \tilde{X} \to X$ denote the covering projection and write $\pi_1(X) = \Gamma$. Let Γ act on $\tilde{X}$ from the right. This induces an action of Γ on $\varepsilon(\tilde{X})$. We will write

$$\tau_X = \varepsilon(\tilde{X})/\Gamma$$

and let $q : \tau_X \to X$ be the map defined by: $q([\sigma]) = p\sigma(0)$ for any $[\sigma] \in \tau_X$. In fact, the map $q : \tau_X \to X$ defines a fibration (see 2.1 below).

Definition 1.2. We will call the fibration, $q : \tau_X \to X$, the *tangential end fibration of* X.

Main Theorem. *Let X be a finite aspherical Poincaré complex of formal dimension n with a universal cover which is forward tame and simply connected at infinity. Then, if n is even, the tangential end fibration τ_X is tangent to X. If n is odd, τ_X is tangent to X under the condition that the diagonal $X \to X \times X$ admits a Poincaré embedding structure of finite type.*

Main Theorem is a direct consequence of Theorems 2.1, 4.1, 6.1 and 7.2 below.

A definition of *Poincaré embedding structure of finite type* appears in Section 7.

We do not know whether every finite Poincaré complex P admits a Poincaré embedding structure of finite type on the diagonal $P \to P \times P$ (cf. [3]). This is not known even when we restrict ourselves to aspherical Poincaré complexes. However every smooth manifold clearly admits a Poincaré embedding structure on the diagonal. Therefore, if a finite aspherical Poincaré complex has a universal cover which is forward tame and simply connected at infinity but the tangential end fibration fails to be tangent to the complex, which is possible only in odd dimensions, then the complex will not be the homotopy type of a smooth manifold.

To find other examples of the end spaces being used to study the tangential properties of spaces, one may refer to 'Appendix B' of the book 'Ends of complexes' [6]. In fact, the book helped us to simplify the presentation of the paper remarkably. The author would like to express his special thanks to its authors, B. Hughes and A. Ranicki. He also would like to express a deep gratitude to Professor Frank Connolly at University of Notre Dame who introduced the author to the problem and showed many ideas which were indispensable in the shaping of the paper.

2. The homotopy type of the fiber

In this section, we prove the following:

Theorem 2.1. *$q : \tau_X \to X$ is an $(n-1)$-sphere fibration.*

By a fibration, we mean one in the sense of Hurewicz that it has the homotopy lifting property.

Refer to a subspace W of a space Y as *cocompact* if it is closed and the closure of $Y - W$ is compact. For any space Y, let $e : \varepsilon(Y) \to Y$ be the evaluation map defined by $e(\sigma) = \sigma(0)$ for any $\sigma \in \varepsilon(Y)$. Then first of all, we observe:

Proposition 2.2. *Let Y be a connected CW complex with a sequence $Y = Y_0 \hookleftarrow Y_1 \hookleftarrow Y_2 \hookleftarrow \cdots$ of co-compact subspaces such that $\bigcap_i Y_i = \emptyset$. Then the evaluation map $e : \varepsilon(Y) \to Y$ is a fibration on Y.*

Note that the sequence $Y_0, Y_1, Y_2, \ldots$ in the above can be used to construct a proper path $\sigma : [0, 1) \to Y$ (see 2.14, p. 18, [6]). Once it is shown that $\varepsilon(Y)$ is not empty, a complete, detailed proof of 2.2 is a standard one, which we omit.

Definition 2.3. A locally compact Hausdorff space Y is referred to as being *forward tame* if there is a co-compact subset U of Y for which there is a proper map $H : U \times [0, 1) \to Y$ such that $H_0 : U \to Y$ is the inclusion.

The definition is essentially due to Quinn [7].

Let $\tilde{H}_*(Y; G)$ denote the reduced homology with coefficient G for any Abelian group G. Also, let $H_*^{lf}(Y; G)$ denote the homology of the chain complex of the locally finite infinite chains. A CW complex is *strongly locally finite* if it is the union of a countable locally finite collection of finite subcomplexes. Then, we again quote Hughes and Ranicki (p. 137, [6]):

Proposition 2.4. *Let Y be a strongly locally finite CW complex with one end which is forward tame. Let G be any Abelian group. Then, there is a long exact sequence:*

$$\to \tilde{H}_{r+1}(Y; G) \to H_{r+1}^{lf}(Y; G) \to \tilde{H}_r(\varepsilon(Y); G) \to \tilde{H}_r(Y; G) \to .$$

Proposition 2.4 expresses the homology groups of $\varepsilon(Y)$ in terms of the homology groups of Y. We will need the following as well (7.10, [6]):

Lemma 2.5. *Let Y be as in 2.4 above, and let ε denote the end of Y. Then, we have:*

$$\pi_1(\varepsilon) = \pi_1(\varepsilon(Y)).$$

Then by combining 2.4 and 2.5 above, we prove that:

Proposition 2.6. *$\varepsilon(\tilde{X})$ is the homotopy type of S^{n-1}.*

Proof. Note that $\tilde{X}$ is a contractible strongly locally finite CW complex. By assumption, $\tilde{X}$ is forward tame and simply connected at infinity, which in particular means that there is only one end.

We apply 2.4 to conclude:

$$H_{r+1}^{lf}(\tilde{X}; \mathbb{Z}) \cong \tilde{H}_r(\varepsilon(\tilde{X}); \mathbb{Z}).$$

And it is well known that $H_*^{lf}(\tilde{X};\mathbb{Z}) \cong H_*(\Gamma;\mathbb{Z}\Gamma) \cong \tilde{H}_*(S^n;\mathbb{Z})$ (cf. [2]). Thus, $\varepsilon(\tilde{X})$ is a homology $(n-1)$-sphere. Furthermore, applying 2.5 together with the assumption that the end of $\tilde{X}$ is simply connected, $\varepsilon(\tilde{X})$ is simply connected. Note also that $\varepsilon(\tilde{X})$ is a homotopy type of a CW complex (7.6, [6]). Thus $\varepsilon(\tilde{X})$ is the homotopy type of S^{n-1}. □

Here we prove the main theorem of the section:

Proof of 2.1. We have a pull-back diagram,

$$\begin{array}{ccc} \varepsilon(\tilde{X}) & \longrightarrow & \tau(X) \\ {\scriptstyle e}\downarrow & & {\scriptstyle q}\downarrow \\ \tilde{X} & \xrightarrow{p} & X, \end{array}$$

where $e : \varepsilon(\tilde{X}) \to \tilde{X}$ is a fibration. Since the horizontal arrows are covering projections of the universal cover, it is straightforward to see that $q : \tau_X \to X$ has the homotopy lifting property.

Furthermore, for any $x \in X$, $\tilde{x} \in \tilde{X}$, $p(\tilde{x}) = x$, the fiber $q^{-1}(x)$ is homeomorphic to $e^{-1}(\tilde{x})$. Now, since $\tilde{X}$ is contractible, $e^{-1}(\tilde{x})$ is homotopy equivalent to $\varepsilon(\tilde{X})$. But the latter is a homotopy $(n-1)$-sphere by 2.6 above. This completes the proof of 2.1. □

3. The exponential map

The main purpose of this section is to prove 3.4 below. We start by recalling the well known notion of *fiberwise one-point compactification*, which is defined as follows: Let Y, Z be *locally compact Hausdorff* spaces and $f : Y \to Z$ be a continuous map. Let $+$ be a symbol such that $Y \cap Z \times \{+\} = \emptyset$, and write $Y_f = Y \cup Z \times \{+\}$. We make Y_f a topological space by choosing a topology determined by the basis whose elements are (a) the open sets of Y, and (b) the sets of the form $W \cup U \times \{+\}$ where U is an open set of Z, and $W = f^{-1}U - K$ where K is a compact subset of Y.

Then it is not difficult to see that Y_f is again locally compact Hausdorff space, and that the inclusions:

$$Y \hookrightarrow Y_f, \qquad Z \equiv Z \times \{+\} \hookrightarrow Y_f, \qquad f^{-1}(z) \cup \{(z,+)\} \hookrightarrow Y_f$$

are embeddings, where $f^{-1}(z) \cup \{(z,+)\}$ is the one-point compactification of the subspace $f^{-1}(z)$ of Y. Also it is clear that the map, $f_+ : Y_f \to Z$, defined by $f_+(y) = f(y)$ if $y \in Y$, $f_+(z,+) = z$ for any $z \in Z$ is continuous.

Definition 3.1. For any locally compact Hausdorff spaces Y, Z and any continuous map $f : Y \to Z$, the space Y_f, topologized as above, will be referred to as the *fiberwise one-point compactification of* Y, and $f_+ : Y_f \to Z$ as the *fiberwise one-point compactification of* f.

Let $\tilde{X} \times_\Gamma \tilde{X}$ denote the orbit space of the diagonal action of Γ on $\tilde{X} \times \tilde{X}$ and $\pi : \tilde{X} \times_\Gamma \tilde{X} \to X$, the map defined by: $\pi([\tilde{x},\tilde{y}]) = p(\tilde{x})$ for any $(\tilde{x},\tilde{y}) \in \tilde{X} \times \tilde{X}$ with $[\tilde{x},\tilde{y}]$ denoting the points in $\tilde{X} \times_\Gamma \tilde{X}$ represented by $(\tilde{x},\tilde{y})$. Note that both $\tilde{X} \times_\Gamma \tilde{X}$ and X are locally compact and Hausdorff. Let $\pi_+ : (\tilde{X} \times_\Gamma \tilde{X})_\pi \to X$ be the fiberwise one-point compactification. Since $\pi : \tilde{X} \times_\Gamma \tilde{X} \to X$ is a fiber bundle, we have:

Lemma 3.2. *$\pi_+ : (\tilde{X} \times_\Gamma \tilde{X})_\pi \to X$ is a fiber bundle.*

For any maps $f : Y \to Y'$ and $g : Y \to Y''$, let $\mathcal{M}(f, g)$ denote the double mapping cylinder which is the quotient space of $Y' \sqcup Y \times [0, 1] \sqcup Y''$. Consider the evaluation map $e : \varepsilon(Y) \to Y$ defined by $e(\sigma) = \sigma(0)$ for any $\sigma \in \varepsilon(Y)$, and the constant map $k : \varepsilon(Y) \to \{+\}$. To prove 3.4 below, we will need the following which is the assertion (ii) of 12.5, [6]:

Proposition 3.3. *Let Y be a σ-compact metric space. Assume Y is forward tame. Then the map $\mathcal{M}(e, k) \to Y_+$, defined by $(\sigma, t) \to \sigma(t)$ for any $(\sigma, t) \in \varepsilon(Y) \times [0, 1)$, $(\sigma, 1) \to +$ for any $\sigma \in \varepsilon(Y)$, $+ \to +$ and $y \to y$ for any $y \in Y$, is a homotopy equivalence* rel$\{+\}$.

Consider the trivial 0-sphere fibration $\epsilon^1 = X \times \{\pm 1\}$. Note that $\tau_X + \epsilon^1 = D_{-1}(\tau_X) \cup D_1(\tau_X)$, in which $D_{-1}(\tau_X) = \tau_X \times [-1, 0]/\sim$, $D_1(\tau_X) = \tau_X \times [0, 1]/\sim$ are the mapping cylinders where the equivalence relation, $\sim$, is generated by: for each $i = \pm 1$ '$([\sigma], i) \sim ([\sigma'], i)$ if and only if $q([\sigma]) = p\sigma(0) = p\sigma'(0) = q([\sigma'])$' (see the paragraphs preceding 4.2 below). Thus, $\tau_X + \epsilon^1$ can be viewed as a 'fiberwise unreduced suspension'.

We define a map

$$\exp : \tau_X + \epsilon^1 \to (\tilde{X} \times_\Gamma \tilde{X})_\pi$$

by the rule: $\exp[[\sigma], t] = [\sigma(0), \sigma(0)]$ if $-1 \le t \le 0$, $\exp[[\sigma], t] = [\sigma(0), \sigma(t)]$ if $0 \le t < 1$, and $\exp[[\sigma], t] = (p\sigma(0), +)$ if $t = 1$. It is straightforward to check the well-definedness and the continuity of exp. We will call this map *the exponential map*.

Now we state the main result of the section.

Theorem 3.4. $\exp : \tau_X + \epsilon^1 \to (\tilde{X} \times_\Gamma \tilde{X})_\pi$ *is a fiberwise homotopy equivalence.*

Proof. Clearly, $\exp : \tau_X + \epsilon^1 \to (\tilde{X} \times_\Gamma \tilde{X})_\pi$ is a fiber preserving map covering the identity. Therefore, it is enough to show that the restriction of exp,

$$\exp_x : (\tau_X + \epsilon^1)_x \to \pi_+^{-1} x$$

is a homotopy equivalence for each $x \in X$.

Let $\tilde{x} \in \tilde{X}$ be such that $p(\tilde{x}) = x$. Let $h : \pi_+^{-1} x \to \tilde{X}_+$ be the homeomorphism defined by $h([\tilde{x}, \tilde{y}]) = \tilde{y}$ for any $\tilde{y} \in \tilde{X}$ and $h(x, +) = +$. Consider the evaluation map $e : \varepsilon(\tilde{X}) \to \tilde{X}$. Let $\bar{\Sigma}(e^{-1}\tilde{x})$ denote the unreduced suspension and let the map $\exp' : \bar{\Sigma}(e^{-1}\tilde{x}) \to \tilde{X}_+$ be defined by $\exp'([\sigma, t]) = \sigma(t)$ if $0 \le t < 1$ and $\exp'([\sigma, 1]) = +$. Then there is a commutative diagram:

$$\begin{array}{ccc} (\tau_X + \epsilon^1)_x & \xrightarrow{\exp_x} & \pi_+^{-1} x \\ f \big\downarrow & & h \big\downarrow \\ \bar{\Sigma}(e^{-1}\tilde{x}) & \xrightarrow{\exp'} & \tilde{X}_+ \end{array}$$

in which $f : (\tau_X + \epsilon^1)_x \to \bar{\Sigma}(e^{-1}\tilde{x})$ is the collapse map:

$$\begin{aligned} (\tau_X + \epsilon^1)_x &\cong D_1(e^{-1}\tilde{x}) \cup D_{-1}(e^{-1}\tilde{x}) \\ &\to D_1(e^{-1}\tilde{x}) \cup D_{-1}(e^{-1}\tilde{x})/D_{-1}(e^{-1}\tilde{x}) = \bar{\Sigma}(e^{-1}\tilde{x}). \end{aligned}$$

Clearly, f is a homotopy equivalence. Note that $\bar{\Sigma}e^{-1}\tilde{x}$ can be regarded as a subspace of the double mapping cylinder $\mathcal{M}(e, k)$ and then $\exp' : \bar{\Sigma}(e^{-1}\tilde{x}) \to \tilde{X}_+$ is the restriction of the homotopy equivalence in 3.3. Since $\tilde{X}$ is contractible, the inclusion of $\bar{\Sigma}(e^{-1}\tilde{x})$ to $\mathcal{M}(e, k)$ is a homotopy equivalence, and therefore $\exp : \bar{\Sigma}(e^{-1}\tilde{x}) \to \tilde{X}_+$ is a homotopy equivalence. This completes the proof. □

The map $\pi_+ : (\tilde{X} \times_\Gamma \tilde{X})_\pi \to X$ defines an n-sphere fibration by 3.2 and 3.4 above. In particular, it admits a natural section:

$$s_+ : X \to (\tilde{X} \times_\Gamma \tilde{X})_\pi, \qquad s_+(x) = (x, +).$$

In general, consider the pairs (ξ, s) consisting of k-sphere fibrations ξ over a fixed space Y and a section $s : Y \to \xi$. We will say two such pairs (ξ, s), (η, t) are equivalent if there is a fiberwise homotopy equivalence $\theta : \xi \to \eta$ such that $\theta s = t$. Denote by $s_1 : Y \to \xi + \epsilon^1$ the section defined by $s_1(y) = [e, 1]$ where $e \in \xi_y$.

Theorem 3.5 (*Dupont, [5]*). *Let Y be an n-dimensional CW complex. Let (ξ, s) be a pair consisting of an n-sphere fibration ξ over Y and its section $s : Y \to \xi$. Then there is a unique $(n-1)$-sphere fibration η up to fiberwise homotopy equivalence such that $(\eta + \epsilon^1, s_1)$ is equivalent to (ξ, s).*

A finite Poincaré complex of formal dimension n is indeed homotopy equivalent to a CW complex of dimension n if $n \geq 4$ (2.9, p. 30, [13]). Therefore we have:

Theorem 3.6. *Assume $n \geq 4$. Then τ_X is the unique $(n-1)$-sphere fibration such that $(\tau_X + \epsilon^1, s_1)$ is equivalent to $((\tilde{X} \times_\Gamma \tilde{X})_\pi, s_+)$.*

4. A trivialization of $\tau_X + \nu$

The main purpose of this section is to show that:

Theorem 4.1. *Let ν denote the Spivak normal fibration of X. Then τ_X represents the stable inverse of ν.*

We begin by recalling some standard operations of sphere fibrations: Let ξ be a sphere fibration over a space W. We will denote the total space by the same notation ξ, and the mapping cylinder of the projection $\xi \to W$ by $D\xi$. We consider $D\xi$ with the obvious projection $D\xi \to W$. Furthermore, let $S\xi$ denote $\xi \times \{0\} \subset D\xi$. Then the Thom space $T(\xi)$ is defined as the quotient space $D\xi/S\xi$. Let η be another sphere fibration over W. We understand the Whitney sum $\xi + \eta$ as $D\xi \times_F S\eta \cup S\xi \times_F D\eta$, where '$\times_F$' denotes the fiberwise product. We topologize $\xi + \eta$ regarding it as the subspace of $D\xi \times D\eta$, which lies over the diagonal $\Delta W \subset W \times W$.

Note that there is a natural identification

$$D(\xi + \eta) = D\xi \times_F D\eta$$

so that $S(\xi + \eta)$ is identified with $\xi + \eta = D\xi \times_F S\eta \cup S\xi \times_F D\eta \subset D\xi \times_F D\eta$. Therefore, we have the identification

$$T(\xi + \eta) = (D\xi \times_F D\eta)/(D\xi \times_F S\eta \cup S\xi \times_F D\eta).$$

Let ϵ^1 denote the trivial sphere fibration $W \times S^0$. Then we have the homeomorphisms:

$$\begin{aligned} \xi + \epsilon^1 &\cong (D\xi \times \{-1, 1\}) \cup (S\xi \times [-1, 1]) \\ &= (D\xi \times \{-1\} \cup S\xi \times [-1, 1]) \cup (D\xi \times \{1\} \cup S\xi \times [-1, 1]) \\ &\cong D_{-1}\xi \cup D_1\xi, \end{aligned}$$

where $D_{-1}\xi = \xi \times [-1,0]/\sim$, $D_1\xi = \xi \times [0,1]/\sim$ are the mapping cylinders such that $D_{-1}\xi \cap D_1\xi = \xi \times \{0\} = S\xi$. In this sense, we identify $\xi + \epsilon^1$ with $D_{-1}\xi \cup D_1\xi$.

Now let $\rho : \xi + \epsilon^1 \to W$ be the projection.

Lemma 4.2. *$T(\xi + \eta)$ is homeomorphic to the quotient space*

$$D(\rho^*\eta)/(S(\rho^*\eta) \cup D(\rho^*\eta)|_{s_1(W)}).$$

The proof of the above is elementary and we leave it as an exercise for the reader. The following is the main step to prove 4.1.

Theorem 4.3. *Assume the Spivak fibration ν is an $(i-1)$-sphere fibration and $i > n$. Let N denote a regular neighborhood of $\tilde{X} \times_\Gamma \tilde{X}$ for a proper embedding $\tilde{X} \times_\Gamma \tilde{X} \hookrightarrow \mathbb{R}^{2(n+i)}$. Then we have a homotopy equivalence: $T(\tau_X + \nu + \nu) \simeq N_+/\partial N_+$.*

Proof. By 3.6 above, the fibration $\rho : \tau_X + \epsilon^1 \to X$ is fiberwise homotopy equivalent to the fibration $\pi_+ : (\tilde{X} \times_\Gamma \tilde{X})_\pi \to X$ by a fiberwise homotopy equivalence which maps $s_1(X) \subset \tau_X + \epsilon^1$ to $s_+(X) \subset (\tilde{X} \times_\Gamma \tilde{X})_\pi$. By applying 4.2 above, we have:

$$\begin{aligned} T(\tau_X + \nu + \nu) &\cong D(\rho^*(\nu + \nu))/(S(\rho^*(\nu + \nu)) \cup D(\rho^*(\nu + \nu))|_{s_1(X)}) \\ &\simeq D(\pi_+^*(\nu + \nu))/(S(\pi_+^*(\nu + \nu)) \cup D(\pi_+^*(\nu + \nu))|_{s_+(X)}). \end{aligned}$$

For any sphere fibration ξ, let ξ' denote a 'quasi-fibration' with a compact fiber which is fiberwise homotopy equivalent to ξ. For instance one may choose ξ' so that each fiber is homeomorphic to the sphere. Then we have

$$\begin{aligned} &D(\pi_+^*(\nu + \nu))/(S(\pi_+^*(\nu + \nu)) \cup D(\pi_+^*(\nu + \nu))|_{s_+(X)}) \\ &\quad \simeq D(\pi_+^*(\nu + \nu)')/(S(\pi_+^*(\nu + \nu)') \cup D(\pi_+^*(\nu + \nu)')|_{s_+(X)}) \\ &\quad = D(\pi^*(\nu + \nu)')_+/S(\pi^*(\nu + \nu)')_+. \end{aligned}$$

Let $r : \tilde{X} \times_\Gamma \tilde{X} \to X \times X$ denote the covering projection. Then $r^*(\nu \times \nu)$ is fiberwise homotopy equivalent to $\pi^*(\nu + \nu)$. Note that $s : X \to \tilde{X} \times_\Gamma \tilde{X}$, $s(x) = [x, x]$ is a homotopy equivalence. Also, we have that

$$s^*(r^*(\nu \times \nu)) = \nu + \nu = s^*(\pi^*(\nu + \nu)).$$

Thus we have the proper homotopy equivalence between the pairs:

$$(D(\pi^*(\nu + \nu)'), S(\pi^*(\nu + \nu)')) \simeq (D(r^*(\nu \times \nu)'), S(r^*(\nu \times \nu)')).$$

In particular, it is proper since it comes from a fiberwise map, $\pi^*(\nu + \nu)' \to r^*(\nu \times \nu)'$ over the identity on $\tilde{X} \times_\Gamma \tilde{X}$. Therefore, we have

$$D(\pi^*(\nu + \nu)')_+/S(\pi^*(\nu + \nu)')_+ \simeq D(r^*(\nu \times \nu)')_+/S(r^*(\nu \times \nu)')_+.$$

Now let L be a regular neighborhood of $X \times X$ in $\mathbb{R}^k$. Then we have $(D(\nu' \times \nu'), S(\nu' \times \nu')) \simeq (L, \partial L)$, which lifts to a proper homotopy equivalence:

$$(D(r^*(\nu \times \nu)'), S(r^*(\nu \times \nu)')) \simeq (\tilde{L}, \partial \tilde{L}),$$

where $\tilde{L}$ is the covering space of L such that $\tilde{X} \times_\Gamma \tilde{X} \subset \tilde{L}$.

Furthermore, by a usual argument we may assume that $\tilde{L} = N$ if i is large enough. Thus we have that $T(\tau_X + \nu + \nu) \simeq N_+/\partial N_+$. $\square$

Now, we provide:

Proof of 4.1. Assume the Spivak fibration ν is an $(i-1)$-sphere fibration and $c : S^{n+i} \to T(\nu)$ be a degree 1 map.

Let $N \subset \mathbb{R}^{2(n+i)}$ be as in 4.3. Note that $T(\tau_X + \nu + \nu) \simeq N_+/\partial N_+$ and consider the collapse map,

$$\hat{c} : \mathbb{R}_+^{2(n+i)} \to N_+/\partial N_+,$$

which is a degree 1 map. Thus we have a degree 1 map, say, $\bar{c} : S^{2(n+i)} \to T(\tau_X + \nu + \nu)$, which is the composite:

$$S^{2(n+i)} \cong \mathbb{R}_+^{2(n+i)} \xrightarrow{\hat{c}} N_+/\partial N_+ \simeq T(\tau_X + \nu + \nu),$$

where the homotopy equivalence is given by 4.3. According to Wall (3.5, [12]), assuming i is large enough, there is a fiberwise homotopy equivalence

$$\bar{\theta} : \epsilon^{n+i} + \nu \to \tau_X + \nu + \nu,$$

unique up to homotopy through the fiberwise homotopy equivalence so that the composite

$$S^{2(n+i)} \cong \Sigma^{n+i} S^{n+i} \xrightarrow{\Sigma^{n+i} c} \Sigma^{n+i} T(\nu) \cong T(\epsilon^{n+i} + \nu) \xrightarrow{T(\bar{\theta})} T(\tau_X + \nu + \nu)$$

is homotopic to $\bar{c} : S^{2(n+i)} \to T(\tau_X + \nu + \nu)$.

Now it is well known that $\bar{\theta}$ is homotopic to $\theta + 1 : \epsilon^{n+i} + \nu \to (\tau_X + \nu) + \nu$ through a fiberwise homotopy equivalences for some fiberwise homotopy equivalence $\theta : \epsilon^{n+i} \to \tau_X + \nu$ (see pp. 22–3, [1]). Such a θ is unique up to homotopy through fiberwise homotopy equivalences [12]. This completes the proof of 4.1. □

5. An identification of the homology groups of $(\tau_X + \epsilon_1, s_1(X))$ with those of $X \times X$

We begin the section by briefly summarizing the equivariant homology theory, which is used in the current and the next sections. Even if the homology groups only of spaces are discussed, the arguments can easily be adopted to pairs of spaces.

Given any group G, let $\mathbb{Z}G$ denote the integral group ring. We consider $\mathbb{Z}G$ with the involution given by the anti-automorphism of G which maps g to its inverse g^{-1} for any $g \in G$.

Let Y be a path-connected space with a universal cover $\tilde{Y} \to Y$ specified. Consider the singular simplicial chain complex $S_*(\tilde{Y})$ with the right $\mathbb{Z}\pi_1(Y)$-module structure coming from the right action of $\pi_1(Y)$ on $\tilde{Y}$. Let i be an integer. Then, for any right $\mathbb{Z}\pi_1(Y)$-module B, we define:

$$H_i(Y; B) = H_i(S_*(\tilde{Y}) \otimes_{\mathbb{Z}\pi_1(Y)} B), \qquad H^i(Y; B) = H_{-i}(\mathrm{Hom}_{\mathbb{Z}\pi_1(Y)}(S_*(\tilde{Y}), B)).$$

For the tensor product in the above, we use the left $\mathbb{Z}\pi_1(Y)$-module structure of B coming from the involution of $\mathbb{Z}\pi_1(Y)$. As usual we understand the Hom functor applied to a chain complex reverses the signs of the indices of the chain groups.

Let B' be another right $\mathbb{Z}\pi_1(Y)$-module, and let $B \otimes B' = B \otimes_{\mathbb{Z}} B'$ be given a $\mathbb{Z}\pi_1(Y)$-module structure by the diagonal action. Let B'' be another $\mathbb{Z}\pi_1(Y)$-module and a homomorphism $B \otimes B' \to B''$ between modules be given. Let p, q be integers. Then the cap product is defined:

$$\cap : H^p(Y; B) \otimes H_q(Y; B') \to H_{q-p}(Y; B'').$$

Let Z be another path-connected space with a universal cover $\tilde{Z} \to Z$ specified, and let C denote a right $\mathbb{Z}\pi_1(Z)$-module. A continuous map $f : Y \to Z$ will be understood as coming with a lifting $\tilde{f} : \tilde{Y} \to \tilde{Z}$, which will often be implicit. Assume there is an $f_\sharp$-homomorphism $\alpha : B \to C$ in the sense that $\alpha(bg) = \alpha(b) f_\sharp(g)$ for any $b \in B$ and $g \in \pi_1(Y)$. Then, there is a well-defined homomorphism

$$(f, \alpha)_* : H_p(Y; B) \to H_p(Z; C),$$

which is defined from the chain level. Likewise, if there is a co-$f_\sharp$-homomorphism $\beta : C \to B$, in the sense that $\beta(c f_\sharp(g)) = \beta(c)g$ for any $c \in C$ and any $g \in \pi_1(Y)$, then there is a well-defined homomorphism

$$(f, \beta)^* : H^p(Z; C) \to H^p(Y; B).$$

In case $Z = Y$ and $f = 1_Y$, we will write α_*, β^* respectively for $(f, \alpha)_*$ and $(f, \beta)^*$. On the other hand, if $B = C$ as Abelian groups and α, β are the identities, then we will write simply f_* and f^* omitting the references to the coefficient homomorphisms.

Assume that Y is a CW-complex with the universal cover $\tilde{;Y}$ is given the CW-structure compatible with the covering projection $\tilde{Y} \to Y$. Then one may proceed as in the singular case, starting from the cellular chain complex $C_*^{\mathrm{cell}}(\tilde{Y})$ to obtain the homology group $H_*^{\mathrm{cell}}(Y; B)$ or the cohomology group $H^*_{\mathrm{cell}}(Y; B)$. For the cap or cup product, we will need to choose a cellular approximation $d : Y \to Y \times Y$ of the diagonal. We have the natural isomorphisms

$$H_*^{\mathrm{cell}}(Y; B) \cong H_*(Y; B), \qquad H^*_{\mathrm{cell}}(Y; B) \cong H^*(Y; B).$$

If $w : \pi_1(Y) \to \{\pm 1\}$ is a homomorphism, let B^w denote the $\mathbb{Z}\pi_1(Y)$-module B with the action of $\mathbb{Z}\pi_1(Y)$ modified by the rule $b \cdot g = w(g)bg$ for any $b \in B$ and any $g \in \pi_1(Y)$. Let $\mathbb{Z}$ denote the group of all integers which is given the $\mathbb{Z}\pi_1(Y)$-module structure induced by the trivial action of $\pi_1(Y)$. Note that there is a natural isomorphism, $B \otimes_{\mathbb{Z}} \mathbb{Z}^w \to B^w$, between the $\mathbb{Z}\pi_1(Y)$-modules.

Assume, further, that Y is a finite CW-complex. If there is an integer n and a class $[Y] \in H_n(Y; \mathbb{Z}^w)$ so that the homomorphism

$$\cap [Y] : H^*(Y; B) \to H_{n-*}(Y; B^w)$$

is an isomorphism for any $\mathbb{Z}\pi_1(Y)$-module B, then we say Y is a Poincaré complex. In this case, w is referred to as the orientation character, n, as the formal dimension and $[Y]$, as a fundamental class.

Now let w denote the orientation character of X. Let $s_{-1} : X \to \tau_X + \epsilon^1$ be the section defined by: $s_{-1}(x) = [[\sigma], -1] \in \tau_X + \epsilon^1 = D_{-1}(\tau_X) \cup D_1(\tau_X)$, where $\sigma \in \varepsilon(\tilde{X})$ is such that $p(\sigma(0)) = x$. Note that $s_{-1\sharp} : \pi_1(X) \to \pi_1(\tau_X + \epsilon^1)$ is an isomorphism; in fact the condition that $\tilde{X}$ is simply connected at infinity implies that $n \geq 3$. Let w denote also the homomorphism, $w s_{-1\sharp}^{-1} : \pi_1(\tau_X + \epsilon^1) \to \{\pm 1\}$. Then the identity map $1 : \mathbb{Z}^w \to \mathbb{Z}^w$ is a (co-)$s_{-1\sharp}$-homomorphism.

For simplicity, we write Λ and Λ^2 to denote respectively the integral group ring over $\Gamma = \pi_1(X)$ and the one over $\pi_1(X \times X) = \Gamma \times \Gamma = \Gamma^2$. Let $w^2 : \Gamma \times \Gamma \to \{\pm 1\}$ denote the homomorphism defined by $w^2(g, h) = w(g)w(h)$ for any $(g, h) \in \Gamma^2$. Then $X^2 = X \times X$ is a Poincaré complex with the orientation character w^2. Furthermore, let Δ denote the diagonal subgroup of Γ^2, and consider the free Abelian group on the set Γ^2/Δ of the right cosets, which we denote by Λ^2_Δ. We provide Λ^2_Δ with the obvious right Λ^2-module structure. Let $w_l : \Gamma^2 \to \{\pm 1\}$ be the homomorphisms defined by $w_l(g, h) = w(g)$ for any $(g, h) \in \Gamma^2$. Let $d : X \to X^2$ be a cellular approximation of the diagonal map.

Also let $\varepsilon : \Lambda^2_\Delta \to \mathbb{Z}$ be the homomorphism which maps an element of Λ^2_Δ to the sum of its coefficients. Then it can be easily checked that $\varepsilon : (\Lambda^2_\Delta)^{w_l} \to \mathbb{Z}^w$ is a co-$d_\sharp$-homomorphism. Furthermore, let $\beta : (\Lambda^2_\Delta)^{w_l} \to \Lambda$ be the homomorphism defined by $\beta(\Delta(e, g)) = g$. Let $\iota_x : X \to X^2$ be the map defined by $\iota_x(y) = (x, y)$ for any $x \in X$. Then β is a co-$\iota_{x\sharp}$-homomorphism.

Lemma 5.1. *There is an isomorphism*

$$\Psi : H^*(X^2; (\Lambda^2_\Delta)^{w_l}) \to H^*(\tau_X + \epsilon^1, s_1(X); \mathbb{Z}^w)$$

so that the diagrams commute:

$$\begin{tikzcd} H^*(X^2; (\Lambda^2_\Delta)^{w_l}) \arrow[r, "\Psi"] \arrow[d, "(d,\varepsilon)^*"'] & H^*(\tau_X + \epsilon^1, s_1(X); \mathbb{Z}^w) \arrow[d, "s^*_{-1}"] \\ H^*(X; \mathbb{Z}^w) \arrow[r, equal] & H^*(X; \mathbb{Z}^w) \end{tikzcd}$$

and, with Ψ' below being another isomorphism,

$$\begin{tikzcd} H^*(X^2; (\Lambda^2_\Delta)^{w_l}) \arrow[r, "\Psi"] \arrow[d, "(\iota_x,\beta)^*"'] & H^*(\tau_X + \epsilon^1, s_1(X); \mathbb{Z}^w) \arrow[d, "\iota_x^*"] \\ H^*(X; \Lambda) \arrow[r, "\Psi'"] & H^*((\tau_X + \epsilon^1)_x, \{s_1(x)\}; \mathbb{Z}). \end{tikzcd}$$

Proof. We divide the proof into a few steps.

Step 1. By 3.4 above, we have the commutative diagram

$$\begin{tikzcd} (\tau_X + \epsilon^1, s_1(X)) \arrow[r, "\exp"] & ((\tilde{X} \times_\Gamma \tilde{X})_\pi, s_+(X)) \\ ((\tau_X + \epsilon^1)_x, \{s_1(x)\}) \arrow[r, "\exp'"] \arrow[u, "\iota_x"] & (\tilde{X}_+, \{+\}) \arrow[u, "\iota_x"'] \end{tikzcd}$$

for any $x \in X$. Note that both exp and exp$'$ in the above are homotopy equivalences. Also note that the following diagram commutes:

$$\begin{tikzcd} \tau_X + \epsilon^1 \arrow[r, "\exp"] & (\tilde{X} \times_\Gamma \tilde{X})_\pi \\ X \arrow[r, equal] \arrow[u, "s_{-1}"] & X \arrow[u, "s"'] \end{tikzcd}$$

in which $s : X \to (\tilde{X} \times_\Gamma \tilde{X})_\pi$ is the map defined by $s(x) = [\tilde{x}, \tilde{x}]$, where $\tilde{x}$ is such that $p(\tilde{x}) = x$.

Step 2. Note that both $(\tilde{X} \times_\Gamma \tilde{X})_\pi, s_+(X)$ are locally contractible compact Hausdorff spaces. We have the isomorphism by continuity of cohomology theories (cf. Theorem 6, p. 318 and Corollary 6, p. 341 of [8]):

$$\varinjlim_k H^*((\tilde{X} \times_\Gamma \tilde{X})_\pi, W_k; \mathbb{Z}^w) \xrightarrow{\cong} H^*((\tilde{X} \times_\Gamma \tilde{X})_\pi, s_+(X); \mathbb{Z}^w)$$

in which $\{W_k\}_{k\in\mathbb{N}}$ is any sequence of closed subsets of $(\tilde{X}\times_\Gamma \tilde{X})_\pi$ such that $W_1 \supset W_2 \supset \cdots \supset s_+(X)$ and $\bigcap_{k\in\mathbb{N}} W_k = s_+(X)$.

Step 3. Let Γ_0 denote the kernel of $w : \Gamma \to \{\pm 1\}$. Let $\tilde{X}\times_{\Gamma_0}\tilde{X}$ be the orbit space of the diagonal action of Γ_0 on $\tilde{X}\times\tilde{X}$. Let $\tilde{X}\times_{\Gamma_0}\tilde{X}$ be given a right Γ-set structure induced by the diagonal action on $\tilde{X}\times\tilde{X}$.

One may take W_k in Step 2 so that $(\tilde{X}\times_\Gamma\tilde{X})_\pi - W_k$ is the union of finitely many cells of $\tilde{X}\times_\Gamma\tilde{X}$ for each $k\in\mathbb{N}$. Write $W_k' = W_k - s_+(X)$ and let $\tilde{W}_k' \subset \tilde{X}\times_{\Gamma_0}\tilde{X}$ denote the inverse image of W_k' with respect to the covering projection $\tilde{X}\times_{\Gamma_0}\tilde{X} \to \tilde{X}\times_\Gamma\tilde{X}$. Let i be an integer. Then we have that $H^i((\tilde{X}\times_\Gamma\tilde{X})_\pi, W_k; \mathbb{Z}^w) \cong H^i_{\text{cell}}((\tilde{X}\times_\Gamma\tilde{X})_\pi, W_k; \mathbb{Z}^w)$ for each k. Also we have the isomorphism

$$\varinjlim_k H^i_{\text{cell}}((\tilde{X}\times_\Gamma\tilde{X})_\pi, W_k; \mathbb{Z}^w) \cong \varinjlim_k H_{-i}(\operatorname{Hom}_\Lambda(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}, \tilde{W}_k'), \mathbb{Z}^w)),$$

which comes from isomorphisms at the chain level.

Now let $\operatorname{Hom}_{\Lambda,c}(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}), \mathbb{Z}^w)$ denote the chain complex, with c denoting the fact that the chain groups consist of only those homomorphisms each of which is zero at all but finitely many cells. Then we also have the isomorphism:

$$\varinjlim_k H_{-i}(\operatorname{Hom}_\Lambda(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}, \tilde{W}_k'), \mathbb{Z}^w)) \xrightarrow{\cong} H_{-i}(\operatorname{Hom}_{\Lambda,c}(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}), \mathbb{Z}^w)),$$

which comes from the isomorphism,

$$\varinjlim_k \operatorname{Hom}_\Lambda(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}, \tilde{W}_k'), \mathbb{Z}^w) \xrightarrow{\cong} \operatorname{Hom}_{\Lambda,c}(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}), \mathbb{Z}^w),$$

together with the interchangeability between the homology functor and the direct limit.

We conclude that

$$H^i((\tilde{X}\times_\Gamma\tilde{X})_\pi, s_+(X); \mathbb{Z}^w) \cong H_{-i}(\operatorname{Hom}_{\Lambda,c}(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}), \mathbb{Z}^w)).$$

Step 4. Let $\delta : \Lambda^2_\Delta \to \mathbb{Z}$ be the homomorphism which reads the coefficient of Δ. Then we have a map

$$\rho : \operatorname{Hom}_\Lambda(C_*^{\text{cell}}(\tilde{X}\times\tilde{X}); (\Lambda^2_\Delta)^{w_l}) \to \operatorname{Hom}_{\Lambda,c}(C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}), \mathbb{Z}^w)$$

defined by

$$\rho(\alpha)(\sigma) = \delta(\alpha(\tilde{\sigma}))$$

for any $\alpha \in \operatorname{Hom}_\Lambda(C_*^{\text{cell}}(\tilde{X}\times\tilde{X}); (\Lambda^2_\Delta)^{w_l})$, and any cell σ of $\tilde{X}\times_{\Gamma_0}\tilde{X}$ where $\tilde{\sigma}$ is a cell of $\tilde{X}\times\tilde{X}$ which lifts σ (cf. 7.4, p. 208, [2]).

Claim. *ρ is well defined and is a chain isomorphism.*

Proof. First, we show that $\rho(\alpha)$ is well defined: Any cell of $\tilde{X}\times\tilde{X}$ which lifts a cell σ of $\tilde{X}\times_{\Gamma_0}\tilde{X}$ is $\tilde{\sigma}(g,g)$ for some $g\in\Gamma_0$, once a lifting $\tilde{\sigma}$ is given. Then we have

$$\delta(\alpha(\tilde{\sigma}(g,g))) = \delta(w(g)\alpha(\tilde{\sigma})(g,g)) = \delta(\alpha(\tilde{\sigma})).$$

Furthermore, $\rho(\alpha) : C_*^{\text{cell}}(\tilde{X}\times_{\Gamma_0}\tilde{X}) \to \mathbb{Z}^w$ is a Λ-homomorphism: for any $g\in\Gamma$ we have that

$$\rho(\alpha)(\sigma g) = \delta(\alpha(\tilde{\sigma}(g,g))) = \delta(w(g)\alpha(\tilde{\sigma})(g,g)) = w(g)\delta(\alpha(\tilde{\sigma})) = (\rho(\alpha)(\sigma))g.$$

Also, we note that $\rho(\alpha)$ vanishes except for finitely many cells of $\tilde{X} \times_{\Gamma_0} \tilde{X}$: let $\{\sigma_i\}_{i=1}^k$ be a cellular basis for $C_*^{\mathrm{cell}}(\tilde{X})$, which is regarded as a Λ-module. And choose an $h \in \Gamma$ as such that $h \notin \Gamma_0$ if $\Gamma_0 \neq \Gamma$. Then the cells of $\tilde{X} \times_{\Gamma_0} \tilde{X}$ are either $[(\sigma_i g) \times \sigma_j]$ or $[(\sigma_i hg) \times \sigma_j h]$ for some $i, j = 1, 2, \ldots, k$ and some $g \in \Gamma_0$. Note that, for any $\alpha \in \mathrm{Hom}_\Lambda(C_*^{\mathrm{cell}}(\tilde{X} \times \tilde{X}); (\Lambda_\Delta^2)^{w_l})$, and for any $i, j = 1, 2, \ldots, k$, we may write $\alpha(\sigma_i \times \sigma_j) = \sum_{g \in \Gamma} n_g \Delta(g, 1)$, where $n_g \in \mathbb{Z}$ are zeros except for all but finitely many g's. Since we have $\delta(\alpha((\sigma_i g) \times \sigma_j)) \neq 0$ if and only if $n_{g^{-1}} \neq 0$, we conclude that $\rho(\alpha)$ maps the cells of $\tilde{X} \times_{\Gamma_0} \tilde{X}$ to zero except for finitely many of them. Thus ρ is well defined.

It is clear by definition that ρ is a homomorphism between the Abelian groups. It is also straightforward to see that ρ is a chain map.

Let $p : \tilde{X} \times \tilde{X} \to \tilde{X} \times_{\Gamma_0} \tilde{X}$ denote the covering projection. Then one may define the inverse $\varrho : \mathrm{Hom}_{\Lambda,c}(C_*^{\mathrm{cell}}(\tilde{X} \times_{\Gamma_0} \tilde{X}), \mathbb{Z}^w) \to \mathrm{Hom}_\Lambda(C_*^{\mathrm{cell}}(\tilde{X} \times \tilde{X}); (\Lambda_\Delta^2)^{w_l})$ of ρ by

$$\varrho(a)(\tilde{\sigma}) = \sum_{g \in \Gamma} w(g) a(p(\tilde{\sigma}(g^{-1}, e)))\Delta(g, e),$$

for any $a \in \mathrm{Hom}_{\Lambda,c}(C_*^{\mathrm{cell}}(\tilde{X} \times_{\Gamma_0} \tilde{X}), \mathbb{Z}^w)$, where $\tilde{\sigma}$ is a cell of $\tilde{X} \times \tilde{X}$. Note that $\varrho(a) : C_*^{\mathrm{cell}}(\tilde{X} \times \tilde{X}) \to (\Lambda_\Delta^2)^{w_l}$ is well defined in particular, since $a : C_*^{\mathrm{cell}}(\tilde{X} \times_{\Gamma_0} \tilde{X}) \to \mathbb{Z}^w$ vanishes except for finitely many cells. It is straightforward to see that ϱ is indeed the inverse of ρ. This completes the proof of the claim.

Let $\rho_* : H_{\mathrm{cell}}^i(X \times X; (\Lambda_\Delta^2)^{w_l}) \to H_{-i}(\mathrm{Hom}_{\Lambda,c}(C_*^{\mathrm{cell}}(\tilde{X} \times_{\Gamma_0} \tilde{X}), \mathbb{Z}^w))$ denote the isomorphism induced by ρ. Since $H^i(X \times X; (\Lambda_\Delta^2)^{w_l}) \cong H_{\mathrm{cell}}^i(X \times X; (\Lambda_\Delta^2)^{w_l})$, we have:

$$H^i(X \times X; (\Lambda_\Delta^2)^{w_l}) \xrightarrow{\cong} H_{-i}(\mathrm{Hom}_{\Lambda,c}(C_*^{\mathrm{cell}}(\tilde{X} \times_{\Gamma_0} \tilde{X}), \mathbb{Z}^w)).$$

Step 5. By composing the four isomorphisms respectively in Steps 1–4, we have the isomorphism:

$$\Psi : H^*(X^2; (\Lambda_\Delta^2)^{w_l}) \to H^*(\tau_X + \epsilon^1, s_1(X); \mathbb{Z}^w).$$

Also, in addition to the commutative squares in Step 1, one may add appropriate commutative diagrams in each of the Steps 2–4 to prove the existence of the commutative squares of the lemma. □

6. The Euler characteristic

We retain the notations of the previous section. Let $[X] \in H_n(X; \mathbb{Z}^w)$ denote a fundamental class.

Theorem 6.1. *There is a Thom class $U \in H^n(\tau_X + \epsilon^1, s_1(X); \mathbb{Z}^w)$ so that we have, for the Kronecker index,*

$$\langle s_{-1}^* U, [X] \rangle = \chi(X).$$

Recall the inclusion ι_x of the fiber at $x \in X$ of $\tau_X + \epsilon^1$. That U is a Thom class means by definition that $\iota_x^* U$ is a generator of $H^n((\tau_X + \epsilon^1)_x, \{s_1(x)\}; \mathbb{Z})$ for any $x \in X$. The existence of a Thom class for any sphere fibration is well known. For an even dimension n, 6.1 above is enough to show that τ_X is the tangent fibration of X, since τ_X has been shown by 4.1 to be stably inverse to the Spivak normal fibration.

In addition to the notations introduced in the previous section above 5.1, let $w_r : \Gamma^2 \to \{\pm 1\}$ be the homomorphism defined by $w_r(g, h) = w(h)$ for any $(g, h) \in \Gamma^2$. Then note that the homomorphism,

$\alpha : \mathbb{Z}^w \to (\Lambda^2_\Delta)^{w_r}$, $\alpha(n) = n\Delta$, is a $d_\sharp$-homomorphism. Recall $\varepsilon : (\Lambda^2_\Delta)^{w_l} \to \mathbb{Z}^w$ of the previous section. The homomorphism $\varepsilon : (\Lambda^2_\Delta)^{w_l} \to \mathbb{Z}^{w_l}$, which also maps an element of $(\Lambda^2_\Delta)^{w_l}$ to the sum of its coefficient, is a (co-)$1_\sharp$-homomorphism where 1 denotes the identity on X^2. One may also consider $\varepsilon : (\Lambda^2_\Delta)^{w_r} \to \mathbb{Z}^{w_r}$, which is a (co-)$1_\sharp$-homomorphism. Note that $H^n(X;\Lambda) \cong \mathbb{Z}$ (see [2]).

Lemma 6.2. *Let $U' \in H^n(X^2; (\Lambda^2_\Delta)^{w_l})$ denote the Poincaré dual of $(d,\alpha)_*[X] \in H_n(X^2; (\Lambda^2_\Delta)^{w_r})$.*

(i) *We have that $\langle (d,\varepsilon)^* U', [X]\rangle = \chi(X)$.*
(ii) *Furthermore, assume that $w = 0$. Then $(\iota_x, \beta)^* U'$ is a generator of the infinite cyclic group $H^n(X;\Lambda)$.*

Proof. (i) Consider the Poincaré dual $u \in H^n(X^2; \mathbb{Z}^{w_l})$ of $d_*[X] \in H_n(X^2; \mathbb{Z}^{w_r})$. Then we have that $\langle d^*u, [X]\rangle = \chi(X)$ (cf. [4]). Note that the following diagram commutes:

$$\begin{array}{ccccccc}
H^n(X;\mathbb{Z}^w) & \xleftarrow{d^*} & H^n(X^2;\mathbb{Z}^{w_l}) & \xrightarrow{D} & H_n(X^2;\mathbb{Z}^{w_r}) & \xleftarrow{d_*} & H_n(X;\mathbb{Z}^w) \\
\| & & \uparrow{\scriptstyle \varepsilon_*} & & \uparrow{\scriptstyle \varepsilon_*} & & \| \\
H^n(X;\mathbb{Z}^w) & \xleftarrow{(d,\varepsilon)^*} & H^n(X^2;(\Lambda^2_\Delta)^{w_l}) & \xrightarrow{D} & H_n(X^2;(\Lambda^2_\Delta)^{w_r}) & \xleftarrow{(d,\alpha)_*} & H_n(X;\mathbb{Z}^w)
\end{array}$$

where the D's denote the Poincaré duality isomorphisms. By chasing the diagram, it is easy to see that $d^*u = (d,\varepsilon)^*U'$.

(ii) The diagram commutes:

$$\begin{array}{ccc}
H^n(X^2;\Lambda^2_\Delta) & \xrightarrow{(\iota_x,\beta)^*} & H^n(X;\Lambda) \\
{\scriptstyle \varepsilon_*}\downarrow & & {\scriptstyle \varepsilon_*}\downarrow \\
H^n(X^2;\mathbb{Z}) & \xrightarrow{\iota_x{}^*} & H^n(X;\mathbb{Z}).
\end{array}$$

On the other hand, the central square of the commutative diagram in the proof of (i) shows that $\varepsilon_* U' = u$. We conclude that $\iota_x^* u = \varepsilon_*(\iota_x,\beta)^*U'$. Therefore, to show that $(\iota_x,\beta)^*U'$ is a generator of $H^n(X;\Lambda) \cong \mathbb{Z}$, it is sufficient to show the following.

Claim. *$\iota_x^* u$ is a generator of $H^n(X;\mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Consider the equalities:

$$\begin{aligned}
\iota_{x*}((\iota_x{}^*u)\cap[X]) &= u \cap \iota_{x*}[X] \\
&= u \cap ([x]\times[X]) \\
&= u \cap ((a\cap[X])\times(b\cap[X]))
\end{aligned}$$

where $[x] \in H_0(X;\mathbb{Z})$ is the canonical generator and $a \in H^n(X;\mathbb{Z})$ and $b \in H^0(X;\mathbb{Z})$ are the Poincaré duals respectively of $[x]$ and $[X]$. Thus

$$\begin{aligned}
\iota_{x*}((\iota_x{}^*u)\cap[X]) &= u \cap ((a\times b)\cap[X\times X]) \\
&= (-1)^n (a\times b)\cap(u\cap[X\times X]) \\
&= (-1)^n (p_1^* a)\cap d_*[X]
\end{aligned}$$

where $p_1 : X^2 \to X$ is the projection to the first coordinate. Now apply $p_{1*} : H_0(X^2; \mathbb{Z}) \to H_0(X; \mathbb{Z})$ to have

$$p_{1*}(\iota_{x*}((\iota_x{}^*u) \cap [X])) = (-1)^n a \cap [X] \in H_0(X; \mathbb{Z}).$$

Thus $(\iota_x{}^*u) \cap [X]$ is a generator of $H_0(X; \mathbb{Z})$, which means that $\iota_x{}^*u$ is a generator of $H^n(X; \mathbb{Z})$. $\square$

Let Ψ be as in 5.1 above. Then, as an immediate consequence of 6.2, we have:

Corollary 6.3. *Assume $w = 0$. Let $U \in H^n(\tau_X + \epsilon^1, s_1(X); \mathbb{Z})$ be a Thom class. Then we have that $\Psi^{-1}(U) = \pm U'$. Furthermore, if U is chosen so that $\Psi^{-1}(U) = U'$, then $\langle s_{-1}^* U, [X]\rangle = \chi(X)$.*

Proof. Since $H^n(X^2; (\Lambda^2_\Delta)^{w_l}) \cong \mathbb{Z}$, we have $k\Psi^{-1}(U) = U'$ for some $k \in \mathbb{Z}$. Recall Ψ' introduced in 5.1 above. Then both $(\iota_x, \beta)^* U'$ and $\Psi'^{-1}(\iota_x^* U)$ are generators of $H^n(X; \Lambda) \cong \mathbb{Z}$ respectively by (ii), 6.2, and by 5.1. By the commutativity of the second square of 5.1, we have that $k = \pm 1$.

Note that $s_{-1}^* U = (d, \varepsilon)^* \Psi^{-1} U = (d, \varepsilon)^* U'$ by the commutativity of the first square of 5.1. Therefore by (i), 6.2 above, we have $\langle s_{-1}^* U, [X]\rangle = \chi(X)$. $\square$

Here we provide:

Proof of Theorem 6.1. Assume $w \neq 0$. Let $q : \bar{X} \to X$ be the orientation cover of X. Note that for the Euler number, we have $\chi(\bar{X}) = 2\chi(X)$. Let $U \in H^n(\tau_X + \epsilon^1, s_1(X); \mathbb{Z}^w)$ be a Thom class. Let $\tau_{\bar{X}}$ denote the tangential end fibration of $\bar{X}$. Let $\bar{s}_{\pm 1} : \bar{X} \to \tau_{\bar{X}} + \epsilon^1$ be the natural sections. Then there is a fiber preserving map

$$\bar{q} : (\tau_{\bar{X}} + \epsilon^1, \bar{s}_1(\bar{X})) \to (\tau_X + \epsilon^1, s_1(X))$$

which cover $q : \bar{X} \to X$. Note that $\bar{q}^* U \in H^n(\tau_{\bar{X}} + \epsilon^1, s_1(\bar{X}); \mathbb{Z})$ is also a Thom class. By 6.3 above, we may choose U so that

$$\langle \bar{s}_{-1}^* \bar{q}^* U, [\bar{X}]\rangle = \chi(\bar{X}).$$

On the other hand, we have that $\bar{q}\bar{s}_{-1} = s_{-1} q : \bar{X} \to \tau_X + \epsilon^1$. It follows that

$$\begin{aligned}\langle \bar{s}_{-1}^* \bar{q}^* U, [\bar{X}]\rangle &= \langle q^* s_{-1}^* U, [\bar{X}]\rangle \\ &= \langle s_{-1}^* U, q_*[\bar{X}]\rangle = \langle s_{-1}^* U, 2[X]\rangle = 2\langle s_{-1}^* U, [X]\rangle.\end{aligned}$$

Thus we have: $\langle s_{-1}^* U, [X]\rangle = \frac{1}{2}\chi(\bar{X}) = \chi(X)$. This completes the proof of the theorem. $\square$

7. The *b*-invariant

Assume the formal dimension n of X is odd. We begin by reviewing how the b-invariant is defined (cf. [4,5,11]). Consider the Spivak normal fibration (ν, c) of X, where $c : S^{n+i} \to T(\nu)$, $i \gg n$, is a degree 1 map. In fact, we fix c for the section as the composite:

$$S^{n+i} \cong \mathbb{R}^{n+i}_+ \to N/\partial N \simeq T(\nu)$$

where N is a regular neighborhood for an embedding of X in $\mathbb{R}^{n+i}$, ν is the fibration equivalent to the map $\partial N \to X$, and $\mathbb{R}^{n+i}_+ \to N/\partial N$ is the collapse map. Let (ξ, θ) be a pair consisting of an $(n-1)$-sphere fibration ξ and a trivialization $\theta : \epsilon^{n+i} \to \xi + \nu$. Let $\bar{\gamma}_{2i}$ denote the universal sphere fibration of $(2i-1)$-sphere fibrations whose $(n+1)$-st Wu classes vanish. It is known that there is a map $a : \nu + \nu \to \bar{\gamma}_{2i}$ which is a fiber map, that is, a map which preserves the fibers and is a homotopy equivalence at each fiber. In fact a needs to be chosen so that it factors through a 'symmetric lifting', $A : \nu \times \nu \to \bar{\gamma}_{2i}$ by the natural map $\nu + \nu \to \nu \times \nu$, which exploits the condition that n is odd [11]. Choose a dual Y of (a finite skeleton) of $T(\bar{\gamma}_{2i})$ together with the duality. We consider the S-dual

$$\alpha : Y \to \Sigma^l T(\xi)$$

of the map

$$T(a) : T(\nu + \nu) \to T(\bar{\gamma}_{2i})$$

with respect to the duality $S^{2n+2i+l} \to \Sigma^l T(\xi) \wedge T(\nu + \nu)$ given as the composite

$$\begin{aligned} S^{2n+2i+l} &\cong \Sigma^{n+i+l} S^{n+i} \xrightarrow{\Sigma^{n+i+l} c} \Sigma^{n+i+l} T(\nu) \cong T(\epsilon^{n+i+l} + \nu) \\ &\xrightarrow{T(1+\theta+1)} T(\epsilon^l + \xi + \nu + \nu) \xrightarrow{T(\bar{\Delta})} T(\epsilon^l + \xi) \wedge T(\nu + \nu). \end{aligned}$$

Here, $\bar{\Delta} : (\epsilon^l + \xi) + (\nu + \nu) \to (\epsilon^l + \xi) \times (\nu + \nu)$ denotes the natural fiber map which covers the diagonal $\Delta : X \to X \times X$. Note that the duality is determined by the trivialization $\theta : \epsilon^{n+i} \to \xi + \nu$.

Now let K_n denote the Eilenberg Maclain space $K(\mathbb{Z}_2, n)$ and $U : T(\xi) \to K_n$, the Thom class. And consider the map $g = (\Sigma^l U)\alpha : Y \to \Sigma^l K_n$.

Definition 7.1. The b-invariant $b(\xi, \theta)$ is defined as the functional Steenrod square:

$$b(\xi, \theta) = Sq_g^{n+1} \Sigma^l \iota \in H^{2n+l}(Y; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

where ι is the generator of $H^n(K_n; \mathbb{Z}_2)$.

Let $\theta_\tau : \epsilon^{n+i} \to \tau_X + \nu$ denote the trivialization given by the proof of 4.1 above. Then we will prove that:

Theorem 7.2. *Assume there is a Poincaré embedding structure of finite type on the diagonal $\Delta : X \to X \times X$. Then we have that*

$$b(\tau_X, \theta_\tau) = \chi_{\frac{1}{2}}(X).$$

We do not know at the moment whether the equality $b(\tau_X, \theta_\tau) = \chi_{\frac{1}{2}}(X)$ holds in general, that is, whether the condition is essential that the diagonal $\Delta : X \to X \times X$ should admit a Poincaré embedding structure of finite type in the following sense (see p. 113, [13]):

Definition 7.3. Let A, P be finite Poincaré complexes of respective formal dimensions k, $k + l$ and let $f : A \to P$ be a continuous map. A Poincare embedding structure of finite type on f consists of finite Poincaré pairs,

$$(Y, Z),\ (W, Z),$$

such that $Y \cap W = Z$, and a homotopy equivalence, $h : Y \cup W \to P$, subject to the following conditions:

(I) A is a subspace of Y which is a strong deformation retraction, and the restriction $Z \to A$ of the retraction $Y \to A$, when replaced by a fibration, is an $(l-1)$-sphere fibration which we denote by ν_f.

(II) the following diagram commutes up to homotopy:

$$\begin{array}{ccc} A & \xrightarrow{f} & P \\ \downarrow & & \uparrow h \\ Y & \longrightarrow & Y \cup W \end{array}$$

where the unlabelled arrows denote the inclusions.

Throughout the rest of the section, let the Poincaré embedding structure of finite type on $\Delta : X \to X \times X$ be given by the finite Poincaré pairs (Y, Z), (W, Z) and a homotopy equivalence $h : Y \cup W \to X \times X$ so that $(D\nu_\Delta, S\nu_\Delta) \simeq (Y, Z)$. Then a degree 1 map, $S^{2n+2i} \cong S^{n+i} \wedge S^{n+i} \to T(\nu_\Delta + \nu + \nu)$, is given as the composite,

$$T(\nu_\Delta + \nu + \nu), \quad \begin{array}{l} S^{n+i} \wedge S^{n+i} \xrightarrow{c \wedge c} T(\nu) \wedge T(\nu) \cong T(\nu \times \nu) \simeq T(h^*(\nu \times \nu)) \\ \qquad \to T(h^*(\nu \times \nu)|_Y)/T(h^*(\nu \times \nu)|_Z) \simeq T(\nu_\Delta + \nu + \nu), \end{array}$$

where $T(h^*(\nu \times \nu)) \to T(h^*(\nu \times \nu)|_Y)/T(h^*(\nu \times \nu)|_Z)$ is the collapse map. A trivialization $\theta_\Delta : \epsilon^{n+i} \to \nu_\Delta + \nu$ results from this degree 1 map (see the proof of 4.1). Then we have ([4] or [11]):

Lemma 7.4. $b(\nu_\Delta, \theta_\Delta) = \chi_{\frac{1}{2}}(X)$.

Now write for simplicity $Y \cup W = P$, and let $\iota : X \to P$ denote the inclusion. Let $\tilde{P} \to P$ be the covering space of P such that there is a lifting $\tilde{h} : \tilde{P} \to \tilde{X} \times_\Gamma \tilde{X}$ of $h : P \to X \times X$ and $\tilde{h}$ is a proper homotopy equivalence. Then, we have a homotopy equivalence

$$\tilde{h}_+ : \tilde{P}_+ \to (\tilde{X} \times_\Gamma \tilde{X})_+ \simeq T(\tau_X).$$

Also fix an embedding $Y \to \tilde{P}$ of Y as a lifting of the inclusion $Y \to P$. Then there is a collapse map $c : \tilde{P}_+ \to Y/Z$, and therefore we have the map $f : T(\tau_X) \to T(\nu_\Delta)$, which is the composite

$$T(\tau_X) \simeq (\tilde{X} \times_\Gamma \tilde{X})_+ \simeq \tilde{P}_+ \xrightarrow{c} Y/Z \simeq T(\nu_\Delta).$$

Lemma 7.5. *The map $f : T(\tau_X) \to T(\nu_\Delta)$ represents the S-dual of the identity of $T(\nu+\nu)$ with respect to the duality between $T(\tau_X)$ and $T(\nu + \nu)$ given by the trivialization θ_τ, and the duality between $T(\nu_\Delta)$ and $T(\nu + \nu)$ given by the trivialization θ_Δ.*

Proof. It is enough to see that the diagram

$$\begin{array}{ccc} S^{2n+2i} & \longrightarrow & T(\tau_X) \wedge T(\nu + \nu) \\ \downarrow & & \downarrow f \wedge 1 \\ T(\nu_\Delta) \wedge T(\nu + \nu) & \xrightarrow{1 \wedge 1} & T(\nu_\Delta) \wedge T(\nu + \nu) \end{array}$$

commutes up to stable homotopy [9], in which the maps from the sphere are the dualities of the lemma.

Replace $(P; Y, W; Z)$ with a simplicial complex. We may start from a (piecewise linear) embedding of Z into $\mathbb{R}^{2n+2i-1} \times \{0\} \subset \mathbb{R}^{2n+2i}$, and extend the embedding to $(Y, Z) \to (\mathbb{H}_+^{2n+2i}, \mathbb{R}^{2n+2i-1} \times \{0\})$ and to $(W, Z) \to (\mathbb{H}_-^{2n+2i}, \mathbb{R}^{2n+2i-1} \times \{0\})$, where $\mathbb{H}_\pm^{2n+2i}$ denote respectively the upper and the lower half Euclidean space. Thus, we may choose a regular neighborhood N of $P = Y \cup W$ in $\mathbb{R}^{2n+2i}$ such that there is a decomposition $N = N_Y \cup N_W$ which, writing $N_Z = N_Y \cap N_W$, satisfies the following conditions:

(i) N_Y, N_W are compact $(2n+2i)$-manifolds and N_Z, a $(2n+2i-1)$-manifold.
(ii) (N_Y, N_Z), (N_W, N_Z) respectively contain (Y, Z), (W, Z) as a strong deformation retract.
(iii) Writing $\partial_0 N_Y = \partial N \cap N_Y$, we have that $(N_Y, \partial_0 N_Y) \simeq (D(\nu+\nu), S(\nu+\nu))$.

The assertion (iii) follows, since $\partial_0 N_Y \hookrightarrow N_Y$ gives rise to the Spivak normal fibration for the pair $(Y, Z) \simeq (D\nu_\Delta, S\nu_\Delta)$ (see [10] or [12]). Note also that ν_Δ is the tangent fibration of X [4].

Let $\tilde{N} \to N$ be the covering space such that the strong deformation retraction $N \to P$ lifts to another strong deformation retraction $\tilde{N} \to \tilde{P}$ (we presume $\tilde{P}$ is a subspace of $\tilde{N}$). Also, fix a lifting $N_Y \to \tilde{N}$ of the inclusion $N_Y \to N$ and consider N_Y as a subspace of $\tilde{N}$. Furthermore, choose a proper embedding of $\tilde{N}$ in $\mathbb{R}^{2n+2i}$ so that it is the same as the inclusion $N \hookrightarrow \mathbb{R}^{2n+2i}$ when restricted to N_Y.

We may identify

$$\nu + \nu \equiv \{\sigma : [0,1] \to N \mid \sigma \text{ is continuous, } \sigma(0) \in X, \sigma(1) \in \partial N\}.$$

The right hand side of the above is homeomorphic to

$$\{\sigma : [0,1] \to \tilde{N} \mid \sigma \text{ is continuous}, \sigma(0) \in X, \sigma(1) \in \partial\tilde{N}\}.$$

Thus we have that

$$(D(\nu+\nu), S(\nu+\nu)) \simeq (\tilde{N}, \partial\tilde{N}) \simeq (N_Y, \partial_0 N_Y).$$

Note that the last homotopy equivalence is given by the inclusion. Let int N_Y denote $N_Y - N_Z$, the point-set theoretic interior of N_Y in $\tilde{N}$.

Claim. *The inclusion* $(N_Z, \partial N_Z) \to (\tilde{N} - \operatorname{int} N_Y, \partial\tilde{N} - \operatorname{int} N_Y)$ *is a homotopy equivalence.*

Proof. It is not difficult to see that the inclusion $N_Z \to \tilde{N} - \operatorname{int} N_Y$ induces an isomorphism between the fundamental groups by exploiting the van Kampen theorem. Then consider the universal covers and observe that the relative homology groups vanish using the excision argument, which means that the inclusion between the universal covers is a homotopy equivalence. It follows that $N_Z \to \tilde{N} - \operatorname{int} N_Y$ is a homotopy equivalence. Similarly with the inclusion $\partial N_Z \to \partial\tilde{N} - \operatorname{int} N_Y$. Now one may refer for instance to 7.18, p.185 and 3.8, p. 222 of [14] to complete the proof of the claim.

Therefore, one may choose a deformation retraction $r : (\tilde{N}, \partial\tilde{N}) \to (N_Y, \partial_0 N_Y)$ such that $r(\tilde{N} - N_Y) \subset N_Z$. Let the map $\hat{r} : \tilde{N}/\partial\tilde{N} \to N_Y/\partial_0 N_Y$ be induced by r. It follows that the diagram

$$\begin{tikzcd}
\tilde{N}_+/\partial\tilde{N}_+ \arrow[r, "\Delta"] \arrow[d, "c"] & \tilde{N}_+ \wedge \tilde{N}/\partial\tilde{N} \arrow[r, "c\wedge 1"] & N_Y/N_Z \wedge \tilde{N}/\partial\tilde{N} \arrow[d, "1\wedge\hat{r}"] \\
N_Y/\partial N_Y \arrow[r, "\Delta"] & N_Y/N_Z \wedge N_Y/\partial_0 N_Y \arrow[r, equal] & N_Y/N_Z \wedge N_Y/\partial_0 N_Y ,
\end{tikzcd}$$

is commutative (on the nose), where the c's denote the collapse maps, and the Δ's those coming from the diagonal. Replace $\tilde{N}/\partial\tilde{N}$ in the above with $N_Y/\partial_0 N_Y$ using the homotopy equivalence $\hat{r}$ and its homotopy inverse $N_Y/\partial_0 N_Y \to \tilde{N}/\partial\tilde{N}$ induced by the inclusion, to obtain the diagram which is commutative now only up to homotopy:

$$\begin{array}{ccc}
\tilde{N}_+/\partial\tilde{N}_+ & \xrightarrow{(1\wedge\hat{r})\Delta} & \tilde{N}_+\wedge N_Y/\partial_0 N_Y \\
\big\downarrow{\scriptstyle \Delta c} & & \big\downarrow{\scriptstyle c\wedge 1} \\
N_Y/N_Z\wedge N_Y/\partial_0 N_Y & \xrightarrow{1\wedge 1} & N_Y/N_Z\wedge N_Y/\partial_0 N_Y.
\end{array}$$

Note that $\tilde{N}_+ \simeq (\tilde{X}\times_\Gamma \tilde{X})_+ \simeq T(\tau_X)$, $N_Y/N_Z \simeq Y/Z \simeq T(\nu_\Delta)$ and $N_Y/\partial_0 N_Y \simeq T(\nu+\nu)$. Exploit these relations to replace $\tilde{N}_+$, N_Y/N_Z and $N_Y/\partial_0 N_Y$ respectively with Thom spaces in the above diagram. Also, observe that the dualities of the lemma are precisely the composites:

$$S^{2n+2i} \cong \mathbb{R}_+^{2n+2i} \xrightarrow{c} \tilde{N}_+/(\partial\tilde{N})_+ \xrightarrow{\Delta} \tilde{N}_+\wedge\tilde{N}/\partial\tilde{N} \simeq T(\tau_X)\wedge T(\nu+\nu),$$
$$S^{2n+2i} \cong \mathbb{R}_+^{2n+2i} \xrightarrow{c} N_Y/\partial N_Y \xrightarrow{\Delta} N_Y/N_Z\wedge N_Y/\partial_0 N_Y \simeq T(\nu_\Delta)\wedge T(\nu+\nu).$$

Therefore, let the diagram start from S^{2n+2i} using the collapse map $\mathbb{R}_+^{2n+2i} \to \tilde{N}_+/\partial\tilde{N}_+$. These lead to the diagram introduced in the beginning of the proof. □

Now we provide:

Proof of Theorem 7.2. Let $f : T(\tau_X) \to T(\nu_\Delta)$ be as in 7.5 above. Let $U' \in H^n(T(\nu_\Delta);\mathbb{Z}_2)$ denote the Thom class. Then $U = f^*U' \in H^n(T(\tau_X);\mathbb{Z}_2)$ is the Thom class, since f is a stable homotopy equivalence between $T(\tau_X)$ and $T(\nu_\Delta)$.

Therefore, we have a homotopy commutative diagram:

$$\begin{array}{ccccc}
Y & \longrightarrow & \Sigma^l T(\tau_X) & \xrightarrow{\Sigma^l U} & \Sigma^l K_n \\
\| & & {\scriptstyle \Sigma^l f}\big\downarrow & & \| \\
Y & \longrightarrow & \Sigma^l T(\nu_\Delta) & \xrightarrow{\Sigma^l U'} & \Sigma^l K_n.
\end{array}$$

It follows that $b(\tau_X,\theta_\tau) = b(\nu_\Delta,\theta_\Delta) = \chi_{\frac{1}{2}}(X)$ by Definition 7.1 above. □

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