

Ampere's law and energy loss in AdS/CFT duality

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Abstract

We note that the energy loss in $\mathcal{N} = 4$ SYM measures directly the spatial string tension $\sigma_S = \pi\sqrt{\lambda}T^2/2$ which is at the origin of the area law for large spatial Wilson loops. We show that the latter reflects on the nonperturbative nature of Ampere's law in $\mathcal{N} = 4$ SYM both in vacuum and at finite temperature.

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1. Introduction

Recent relativistic heavy ion collisions at RHIC have suggested that the partonic matter is unleashed in the form of a strongly coupled quark–gluon plasma sQGP which is likely in the liquid phase [1]. The key characteristics of this phase reside in its transport properties and not in its bulk properties.

The transport properties of the sQGP are unknown from first principles, since most current lattice formulations are restricted to the Euclidean domain. Time-like properties of the sQGP are elusive. QCD just above the critical temperature is in a strongly coupled Coulomb phase. Recently, it was suggested [2] that this phase may share similarities with finite temperature $\mathcal{N} = 4$ SYM in the dual limit of strong coupling and large N_c [3].

A number of transport properties of the $\mathcal{N} = 4$ SYM theory at finite temperature have been recently reported [4] all consistent with a hydrodynamical limit in the strongly coupled limit. In [5] we have emphasized the concept of the energy loss in hot $\mathcal{N} = 4$ SYM theory and its importance for current RHIC experiments. Indeed, we have shown that any color triggered wave is absorbed on a distance of $1/\pi T$ irrespective of its frequency/energy content in hot $\mathcal{N} = 4$ SYM. The color opacity length is $1/\pi T$ and short.

Recently, there has been a flurry of activities related to the issue of absorption or energy loss in hot $\mathcal{N} = 4$ SYM using heavy probes [6–11] and light dipoles [12,13]. One of the central result is the presence of a dragging force ($\gamma = 1/\sqrt{1-v^2}$)

$$F_x = \frac{dE}{dx} = -\gamma v m_T^2, \quad (1)$$

proportional to the velocity v . Stoke's law hold in hot $\mathcal{N} = 4$ SYM at strong coupling as first suggested in [14] (Section Vb).

From the 'single string solution' of [6,8] it is clear that the quark–antiquark $Q\bar{Q}$ configuration with large separation is the mirror reflected solution of single heavy probe as shown in Fig. 1. This is in fact supported by the numerical solution given in [6]. In light of this and the well established fact [15] that the static color force is screened in hot SYM (see Fig. 1), it is tempting to conclude that *the relevant force in the medium is color magnetic rather than color electric*. Time- and space-like $Q\bar{Q}$ configurations have been originally analyzed in the context of scattering at high energy in [16–18].

In this short Letter we argue that this is indeed the case by discussing Ampere's law and the nature of the magnetic force of hot $N = 4$ SYM both in the perturbative and AdS/CFT context. Throughout we will use exchangeably Euclidean and Minkowski descriptions for the analysis of the Wilson lines. Which is which will be clear by the notation and discussion.

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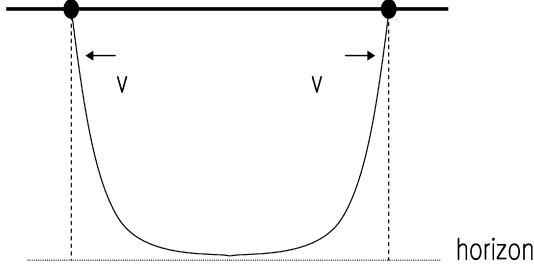


Fig. 1. The string connecting a receding quark–antiquark with velocity v . The dashed lines are the string configuration for the static case $v = 0$. They show the screened nature of the color electro-static force.

2. Spatial Wilson loops

At finite temperature *parallel* space-like Wilson loops confine in the AdS/CFT correspondence [19,20]. Spatial Wilson loops map onto two stationary current wires and reflect on Ampere’s law in the gauge theory. Finite temperature YM theory as well as QCD are known to confine space-like [21] with the string tension proportional to the magnetic mass $m_M \approx \alpha_s T$ asymptotically. Across T_c the spatial string tension remains about unchanged from its zero temperature value.

To probe space-like $Q\bar{Q}$ configurations at finite temperature in AdS/CFT we consider the Euclidean set up in [16,17] with a black hole at $z_0 = 1/\pi T$. The black-hole causes the Euclidean time direction to be compact with length $\beta = 1/T$. The AdS black-hole metric is

$$ds^2 = \frac{R^2}{z^2} \left(-f dt^2 + \frac{dz^2}{f} + \sum_i dx_i^2 \right) \quad (2)$$

with $f = 1 - z^4/z_0^4$ and $i = 1, 2, 3$. For large spatial Wilson loops, the relevant part is near the black hole and the metric simplifies to be $ds^2 = (R^2/z_0^2)(\sum_i dx_i^2)$. Consider the space-like Wilson lines with one support of length x running along the x^1 -direction and the other support running at a distance y and at an angle θ away from the x^1 -direction in the Euclidean x^1x^3 -plane. For small x, y the minimal surface is in general involved, but for $x, y \gg \beta$ it simplifies into the helicoidal surface [17]

$$\begin{aligned} x^1 &= \tau \cos(\theta\sigma/y), & x^2 &= \sigma, \\ x^3 &= \tau \sin(\theta\sigma/y) \end{aligned} \quad (3)$$

at the bottom attached by two parallel falling surfaces. Here τ, σ are affine parameters of the 2-dimensional surface, with $0 \leq \tau \leq x$ and $0 \leq \sigma \leq y$. For $x \gg y \gg \beta$, the area swept by the helicoid grazes the black-hole surface. The straight-down parts are infinite. They are *self-energy* like insertions on the probe lines and will be subtracted out. In this sense our space-like Wilson lines are understood as renormalized *heavy quarks* or just *massless* light quarks. In both cases, the minimal surface contribution is the helicoidal surface in the approximately flat metric near the horizon.

The Nambu–Goto action associated to (3) is

$$S(x, y, \theta) = \sigma_S \int_0^x d\tau \int_0^y d\sigma \sqrt{1 + \theta^2 \frac{\tau^2}{y^2}} \quad (4)$$

with σ_S the spatial string tension,

$$\sigma_S = \frac{1}{2\pi\alpha'} \frac{R^2}{z_0^2} = \frac{\sqrt{\lambda}}{2\pi} (\pi T)^2 = \frac{\pi\sqrt{\lambda}}{2} T^2 \quad (5)$$

and the AdS radius $R^2 = \sqrt{\lambda}\alpha'$. The AdS/CFT duality implies that

$$\text{Tr}(W^*(x, y, \theta) e^{-\beta(H-F_0)}) = e^{-\beta F(x, y, \theta)} = e^{-S(x, y, \theta)} \quad (6)$$

with the normalization $\text{Tr} \mathbf{1} = 1$, and where W^* is the renormalized spatial Wilson loop and F is the free energy. The last equality follows from the AdS/CFT duality. $F(x, y, \theta)$ is the free energy of two stationary current carrying *wires* of length x , at a distance y , sloped at an angle θ . Thus

$$\begin{aligned} F(x, y, \theta) &= \sigma_S T \left(xy \sqrt{1 + \theta^2 \frac{x^2}{y^2}} + \frac{y^2}{2\theta} \ln \frac{+\theta \frac{x}{y} + \sqrt{1 + \theta^2 \frac{x^2}{y^2}}}{-\theta \frac{x}{y} + \sqrt{1 + \theta^2 \frac{x^2}{y^2}}} \right). \end{aligned} \quad (7)$$

Which gives rise to both a longitudinal and transverse force.

For $x \gg y$, Eq. (7) simplifies

$$F(x \gg y, \theta) = \sigma_S T \theta x^2 \quad (8)$$

which is the area of the surface $x \times \theta x$. For $\theta = 0$, Eq. (7) reduces to

$$F(x, y, 0) = \sigma_S T xy \quad (9)$$

which can be interpreted as the *energy* of a fixed surface $x \times y$ with *spatial* surface tension $\sigma_S T$, the natural scale in [12]. Eq. (9) when reinterpreted in 2 + 1 dimensions as an action βF , yields an area law with string tension σ_S as discussed in [19,20]. The forces on the support of the spatial Wilson loop are

$$\begin{aligned} F_x/x &= -\sigma_S T y/x, \\ F_y/x &= -\sigma_S T. \end{aligned} \quad (10)$$

The former vanishes for $x \gg y$, while the latter is *constant* and *attractive* in hot $\mathcal{N} = 4$ SYM.¹

3. Ampere’s law

The constant force between two infinitely long wires is a consequence of the fact that Ampere’s law confines in hot $\mathcal{N} = 4$ SYM. The relation of (10) to Ampere’s law can be seen in perturbation theory.

The leading cumulant to (6) is

$$\begin{aligned} \text{Tr}(W^*(x, y, \theta) e^{-\beta(H-F_0)}) &\approx \exp\left(\frac{\lambda}{2} \cos \theta x \int dr \Delta(\sqrt{r^2 + y^2})\right) \end{aligned} \quad (11)$$

¹ We are not using the Gibbs relation $E = F - T dF/dT$ for the spatial Wilson loop as it gives $E = -2F$ nonperturbatively, and $E = -F$ perturbatively (see below) implying that the entropy is dominant, a rather unintuitive result. Our interpretation follows the standard lore for finite temperature Wilson loops.

with $\text{Tr}(T^a T^b) = \delta^{ab}/2N_C$ (the Casimir is $N_C/2$) and

$$\Delta(r) = \frac{T}{4\pi r} \coth(\pi T r) \quad (12)$$

the thermal Wightman function for the gluon propagator in covariant Feynman gauge. In general,²

$$\begin{aligned} \text{Tr}(e^{-\beta(H-F_0)} A_\mu^a(x) A_\nu^b(0)) \\ = \delta^{ab}(g_{44}\Delta_E(x) + g_{ij}\Delta_M(x)). \end{aligned} \quad (13)$$

At large temperature $\Delta(r) \approx T/4\pi r$, which is checked to be gauge independent. Eq. (11) yields the free energy to leading order in perturbation theory ($x \gg y \gg \beta$)

$$F(x, y, \theta) = -\cos\theta \frac{\lambda T^2}{8\pi} x \ln(R/y) \quad (14)$$

where R is a finite cutoff in the relative separation. The forces stemming from (14) are

$$\begin{aligned} \frac{F_x}{x} &= -\cos\theta \frac{\lambda T^2}{8\pi} \frac{\ln(y/R)}{x}, \\ \frac{F_y}{x} &= -\cos\theta \frac{\lambda T^2}{8\pi} \frac{1}{y} \end{aligned} \quad (15)$$

which are to be compared with (10). For infinite wires $x \gg y$, only the transverse force survives which is reminiscent of Ampere's law,

$$\frac{F_y}{x} = -\frac{N_c/2}{4\pi} \frac{I_1 \cdot I_2}{y} \quad (16)$$

where $I = g/\beta$ plays the role of a current in the heat bath. Again, $N_c/2$ is the Casimir.

4. Non-static Wilson loop

These arguments generalize to finite temperature the arguments presented in [22]. Indeed, 2 charged particles moving with velocity $\dot{x}_{1,2}$ are described by the following *non-static Wilson loop*

$$\begin{aligned} \text{Tr} \left[e^{-\beta(H-F_0)} \mathbf{P} \exp \left(ig \int_{-T/2}^{+T/2} dt_1 \dot{x}_1 \cdot A(x_1) \right) \right. \\ \left. \times \exp \left(-ig \int_{-T/2}^{+T/2} dt_2 \dot{x}_2 \cdot A(x_2) \right) \right] \end{aligned} \quad (17)$$

with T a large time (in the center of mass). The first cumulant expansion is driven by the kernel

$$\text{Tr} [e^{-\beta(H-F_0)} (ig\dot{x}_1 \cdot A(x_1)) \times (-ig\dot{x}_2 \cdot A(x_2))], \quad (18)$$

which is of the form

$$\frac{\lambda}{2} (\Delta_E(x_1 - x_2) + \dot{x}_1 \cdot \dot{x}_2 \Delta_M(x_1 - x_2)) \quad (19)$$

² The ordering generates a T -independent c-number which is not relevant for the analysis below.

for the electric and magnetic contribution respectively.

At zero temperature, $\Delta_E = \Delta_M = 1/(4\pi^2 x^2)$ after rotation to Euclidean space for regularization, and (19) becomes

$$\frac{\lambda}{8\pi^2} \frac{1 + \dot{x}_1 \cdot \dot{x}_2}{(t_1 - t_2)^2 + (\vec{x}_1 - \vec{x}_2)^2}. \quad (20)$$

The velocity dependent part translates in Minkowski space to $(1 - \vec{v}_1 \cdot \vec{v}_2)$ which is the expected correction to Coulomb's law from charge motion. This part renormalizes the Coulomb strength through [22,23]

$$\lambda \rightarrow \lambda(1 - \vec{v}_1 \cdot \vec{v}_2). \quad (21)$$

For parallel moving particles with constant velocity $\dot{x}_1 = \dot{x}_2 = \dot{x}$, the potential $E(L)$ follows from the large time exponential behavior of (17)

$$\begin{aligned} E(L) &= -\frac{\lambda}{8\pi^2} \int dt \frac{1 + \dot{x}^2}{(1 + \dot{x}^2)t^2 + L^2} \\ &= -\frac{\lambda}{8\pi} \frac{1}{L} \gamma(1 + \dot{x}^2) \end{aligned} \quad (22)$$

which is the *boosted* form of (Ampere's corrected) Coulomb's law. Again the extra factor of $1/2$ relates to the Casimir $N_c/2$. The occurrence of $\gamma = 1/\sqrt{1 + \dot{x}^2}$ is guaranteed by relativity even in the interacting case, as the dependence in the integral is generically of the form $(1 + \dot{x}^2)t^2$ by Lorentz invariance. Thus the change of variable and the overall factor γ . So, the induced magnetic part is necessary even in Coulomb's law when motion is involved even between very heavy particles.

At finite temperature, the thermal but free Minkowski contributions to the Wightman functions (temporal and spatial) are equal $\Delta_E = \Delta_M = \Delta$ with (modulo regularization)

$$\begin{aligned} \Delta(t, |x|) \\ = \frac{T}{8\pi|x|} (\coth(\pi T(|x| + t)) + \coth(\pi T(|x| - t))). \end{aligned} \quad (23)$$

Here $t = t_1 - t_2$ and $|x| = |\vec{x}_1 - \vec{x}_2|$, and (23) reduces to (12) for $t = 0$. For two parallel particles, $|x| = \sqrt{L^2 + v^2 t^2}$ and at finite temperature the energy (22) is now

$$\begin{aligned} E(L) &= -\frac{\lambda T}{8\pi} \int dt \frac{(1 - v^2)}{\sqrt{1 + v^2 t^2}} \\ &\quad \times \coth(\pi T L(\sqrt{1 + v^2 t^2} - t)). \end{aligned} \quad (24)$$

Lorentz invariance is more subtle in this case, but can be checked to follow readily in the zero temperature limit of (24).

5. Ladder resummation

Eq. (24) receives electric and magnetic contributions through the Wightman functions

$$\begin{aligned} E_E(L) &= -\frac{\lambda}{2} \int dt \Delta_E(t, \sqrt{L^2 + v^2 t^2}), \\ E_M(L) &= +\frac{v^2 \lambda}{2} \int dt \Delta_M(t, \sqrt{L^2 + v^2 t^2}). \end{aligned} \quad (25)$$

In the interacting case $\Delta_E \neq \Delta_M$, and also higher cumulants contribute. At finite temperature, the pure Coulomb contribution Δ_E is screened in strong coupling [15], while the magnetic contribution is not as discussed in (9).

To see how this may develop we apply the ordered ladder resummation to the kernel (19), (23) for the unscreened magnetic contribution following the arguments developed in [24]. Indeed, a rerun of their arguments yields

$$E_{\text{ladd}}(L) = \sqrt{v^2 \sigma_S} L F(\pi T L) \quad (26)$$

with $F(x) = \sqrt{\coth(x)/2x^3}$. Only the repulsive contribution to the Bethe–Salpeter kernel for long times was retained. Noticeably, $v^2 \lambda$ transmutes to $\sqrt{v^2 \lambda}$, much like λ transmutes to $\sqrt{\lambda}$ in the electric part at strong coupling for ordered ladders [24]. At zero temperature, (26) reduces to

$$\frac{\sqrt{v^2} \sqrt{\lambda/2}}{\pi L} \quad (27)$$

which is the leading magnetic contribution to the strongly coupled version of Coulomb's law (22). Note that $F(x \gg 1) \approx 1/x^{3/2}$, which makes (26) not confining but weaker than Coulomb at high temperature.

6. String tension and energy loss

The energy loss reported recently in hot $\mathcal{N} = 4$ SYM in [6–11] may have its origin in the magnetic sector. To support this, consider an open string of length L near the black-hole horizon z_0 . When pulled with velocity v along the x -direction, the string action is

$$S = -\frac{L}{2\pi z_0^2} \int dt \sqrt{\lambda(1 - (z/z_0)^4 - v^2)}. \quad (28)$$

A number of remarks follows from this relation:

- The argument of the square root in (28) is the string analogue of (21) at finite temperature. As $z \rightarrow z_0$ the *electric part* $1 - (z/z_0)^4$ is completely screened, while the *magnetic part* is unchanged. As a result the string action is purely imaginary

$$S = i v \sigma_S L t. \quad (29)$$

- We interpret the *imaginary action* as the energy lost to the black-hole horizon as we pull the string. The energy loss per length L is

$$\frac{E}{L} = v \sigma_S. \quad (30)$$

The extra γ factor can only be obtained through a full analysis with the help of dz/dx (the tilting of the string in the z -direction for the minimal surface). Our simplified argument is only aimed at understanding the magnetic origin of the loss.

- The original analysis in [6] shows that the energy flows along the dragging string and is stored in the string tail infinitesimally above z_0 . Our qualitative description shows that the string loses energy to the black hole when it touches its horizon at z_0 .

- Finally, we note that if the above string connects a heavy $Q\bar{Q}$ configuration on the boundary (instead of sitting on the black-hole horizon), the string action (28) shows that the string will not touch the horizon z_0 , but instead will break at fixed height

$$\frac{z_{\text{max}}}{z_0} = (1 - v^2)^{1/4}. \quad (31)$$

This height translates to a maximum separation $l_* = c_1 z_{\text{max}}$, for some constant c_1 . For $v = 0$, $z_{\text{max}} = z_0$ and c_1 was determined in [15]. The breaking occurs when the screened electric force is about to compensate the magnetic force.

Eq. (30) for small v is Stoke's law $F = -6\pi r \eta v$ with a vanishing *effective* particle size $r \approx \sigma_S / \eta \approx \sqrt{\lambda} / (N_c^2 T)$. This may suggest that the Ohmic trail behind a dragged heavy quark is parametrically thin in strong coupling. Also the Einstein formulae implies that the diffusion constant of slow particles is $D = T / \sigma_S$ as in [7]. The relevance of Stoke's law and the Einstein formulae for moving particles in AdS/CFT were first suggested in [14] (Section Vb) to (re)derive bounds on the drag viscosity and the diffusion constant both at finite temperature and density.

At strong coupling, a measurement of the energy loss is a measurement of the spatial string tension in the underlying gauge theory. Some of these observations maybe pertinent for QCD and therefore RHIC. Indeed, QCD confines space-like at all temperatures with a spatial string tension $\sigma_S \approx \alpha_s^2 T^2$ at asymptotic temperatures.

7. Conclusions

We have discussed Ampere's law in hot $\mathcal{N} = 4$ SYM, for space-like Wilson loops with arbitrary angle θ . The loops show space-like confinement at all temperatures, a property shared by other gauge theories. The spatial string tension is $\sigma_S = \pi \sqrt{\lambda} T^2 / 2$. Coulomb's law is screened but Ampere's law is not.

All the results reported in [6–13] involve σ_S . The technical reason is due to the fact that in the stationary limit (long times, long distances) the subtracted AdS/CFT surface is mostly determined by the nearly flat metric near the horizon. In a way, the subtracted surfaces stemming from heavy-quarks or eikonalized quarks at the *top* of the AdS space, are just massless quarks at the *bottom*. Time is frozen on the black-hole surface, making most physics spatial. The physical reason is due to the fact that at least at high temperature, the gluinos (energy πT) together with the scalars (energy $\sqrt{\lambda} T$) decouple and the electric fields are screened. So very hot $\mathcal{N} = 4$ is mostly a theory of magnetic gluons.

In pure lattice YM the spatial string tension has been measured [21]. The spatial string tension across T_c shows very little change from its zero temperature value, in contrast to the temporal string tension. In zero temperature QCD the temporal string tension breaks due to $q\bar{q}$ production. In matter, the *static* temporal string tension is Debye-like screened, however the moving temporal string tension and the spatial string tension survives at high temperature since the quarks decouple (energy

of order $n\pi T$) space-like. It is of the order of the magnetic mass squared $m_M^2 \approx \alpha_s^2 T^2$. In the sQGP at RHIC with $T \approx 1.5T_c$, we have $\sigma_S \approx 1$ GeV/fm. Jet quenching at RHIC may measure this number and its slow dependence on temperature as suggested by lattice measurements [21] across T_c .

Finally and rather remarkably, the little bang at RHIC maybe directly mappable onto a cosmological-like big-bang by the AdS/CFT duality in hot $\mathcal{N} = 4$ SYM [25], bringing more insights to issues of transport and dynamics of relevance to the sQGP at RHIC.

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